The die-coin experiment consists of rolling a fair die and then flipping a fair coin the number of times shown on the die. A sample space for this experiment is

$$S = \{(1, H), (1, T), (2, HH), (2, HT), (2, TH), (2, TT), \ldots, (6, TTTTTT)\}$$

but what we're really interested in is the numbers showing up in the experiment. First there's the number rolled on the die, which we'll call A. Then there's the number of times we flip heads, which we'll call B. The variables A and B are called *random variables* because their values will be determined by a random process (in this case the die-coin experiment).

1. What are the possible values for A? What is the probability that A takes each of these values?

2. What are the possible values for B? In this case calculating the probability with which B takes each of those values is difficult. Instead conditional probabilities are easier. Calculate the conditional probability that $B = 1$ given that $A = 3$.

3. Calculate the probability that $B = 6$. Hint: The law of total probability says $P(B = 6) = P(B = 6|A = 6)P(A = 6) + P(B = 6|A \neq 6)P(A \neq 6)$.

4. Calculate $P(B = 5)$.

5. Suppose you know that your friend ran the die-coin experiment and flipped 5 heads. Calculate the conditional probabilities of her having rolled 1, 2, 3, 4, 5, and 6 on the die. Which was most likely to have been her roll?

The function $f(x) = P(A = x)$ is the *probability distribution function (pdf)* of the random variable A. You have already found that $f(x) = \begin{cases} \frac{1}{6} & \text{if } x = 1, 2, \ldots, 6 \\ 0 & \text{otherwise.} \end{cases}$

6. The function $g(x|y) = P(B = x|A = y)$ is the *conditional pdf of B given $A = y$*. Describe the function $g(x|3)$ (this means calculating $g(0|3)$, $g(1|3)$, $g(2|3)$, and $g(3|3)$ or giving a formula for calculating these values).

7. The function $f(x|y) = P(A = x|B = y)$ is the *conditional pdf of A given $B = y$*. Describe the function $f(x|5)$.

Challenge. The function $f(x, y) = P(A = x \text{ and } B = y)$ is the *joint probability distribution function of A and B*. Give a formula for $f(x, y)$.

Challenge. Give a formula for the pdf of B: $g(x) = P(B = x)$ (your formula may involve a sum–you do not need to simplify the sum).

Challenge. Give a general formula for $g(x|y)$ and for $f(x|y)$.