NAME(S):

MATH 421 CONTINUOUS DISTRIBUTIONS OCTOBER 5, 2015

Let X be a continuous random variable.

Definition 1. The **cumulative distribution function (cdf)** of X is defined for all real numbers x by

$$F(x) = P(X \leq x)$$

Definition 2. The function f is a **probability density function (pdf)** for X if

$$P(a < X < b) = \int_a^b f(x)dx$$

for any real numbers $a < b$.

Theorem 1. *Let F be the cdf of X and f be a density of X. Then*

a) $F(x) = \int_{-\infty}^{x} f(t)dt$

b) $\frac{d}{dx}[F(x)] = f(x)$ *(except perhaps for a small number of points where the derivative may not exist or may not be equal to* $f(x)$*)*

For the following problems a dart is thrown so that it hits a dartboard at a totally random location. The dartboard has a radius of 20 cm. Define a random variable X as the distance from the center of the dartboard to the dart.

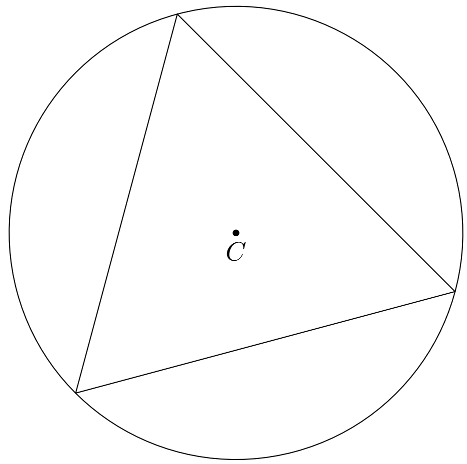
1. Determine the cdf of X. Hint: what proportion of the board is within x cm of the center?

2. Find a density function for X.

3. Sketch the graphs of the cdf and the density on the same set of axes.

4. French mathematician Joseph Bertrand (1822-1900) doubted that probabilities could be defined on infinite sample spaces. As an argument for his position he came up with the following problem:

> What is the probability that a *random chord* of a unit circle is longer than the side length of an equilateral triangle inscribed in the circle?



To solve this problem we must determine a way of finding a random chord of the circle. Three ways we might do this are described below. Let C be the center of the circle.

a) Choose a random point A on the circle and determine a random distance $D \in [0, 1]$. Place a point M on the line CA at a distance D from C. The random chord is then the line perpendicular to CA through the point M. The chord we have determined is longer than the length of a side of the triangle if and only if what is true about D? What is the probability that this happens?

b) Choose (independent) random points A and D on the circle and connect them. To compare this chord with an inscribed triangle we let B and E be the points on the circle such that ABE is an equilateral triangle. Then the random chord AD is longer than the sides of ABE if and only if D is on the segment of the circle between B and E. What is the probability that this happens?

c) Choose a random point M in the disk described by the circle. Our chord is the chord with midpoint M and perpendicular to the line segment CM. In this case the chord is longer than the side of the equilateral triangle if and only if M lies within a smaller circle inscribed in the triangle. What is the probability that this happens?

How would you resolve the paradox?