Definition 1. Let X be a continuous random variable with density $f(x)$ and cumulative distribution function $F(x)$. The *expected value of* X is $E(X) = \int_{-\infty}^{\infty} xf(x)dx$. The *median of* X is $F^{-1}\left(\frac{1}{2}\right)$.

1. On our previous worksheet you found that if X was the distance from the center of a dartboard to a dart that hit the board uniformly at random, then X had pdf $f(x) = \frac{x}{200}$ for $0 < x < 20$ and cdf

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x^2}{400} & \text{if } 0 < x < 20 \\ 1 & \text{if } 20 \leq x \end{cases}$$

Calculate the expected value of X and the median of X. Do your answers make sense? It may help to sketch the graph $y = f(x)$.

2. Let R and G be the numbers rolled on a pair of dice, one red and the other green. Calculate $E[\min(R, G)]$.

Definition 2. If μ is the expected value of a random variable X, then $V(X) = E\left[(X-\mu)^2\right]$ is called the *variance of* X.

The variance measures the tendency of the random variable to spread away from its mean. A larger variance corresponds to a greater likelihood of the random variable taking values far from its mean.

3. Let X be a random variable with density $f(x) = |x|$ for $-1 < x < 1$. Let Y be a random variable with density $g(x) = 1 - |x|$ for $-1 < x < 1$.

a) Sketch graphs of the two densities.

b) Calculate $E(X)$ and $E(Y)$ (observing features of the graphs may suffice).

c) Which of X or Y should have a higher variance? Calculate $V(X)$ and $V(Y)$.

4. A game, which we'll call St. Petersburg, starts with \$1 in the pot. A fair coin is then flipped until the first heads appears, at which point you win the pot. Each time the coin comes up tails the pot is doubled (so you win \$1 if the first flip is heads, \$2 for tails then heads, \$4 for tails, tails, heads, and so on).

a) What is the expected value of the game?

b) How much would you pay to play? What is the probability of a profit if you pay that price?

5. Petrograd is a new version of St. Petersburg that is the same except that the game stops at 20 flips if heads has not appeared. If this happens, you win nothing.

a) What is the expected value of this game? This is called the fair price for the game.

b) What is the probability of a profit if you pay the fair price?

6. (The Gambler's Ruin). Suppose Alice and Bob are gambling on the flips of a fair coin: Alice gives Bob \$1 if the flip is heads, otherwise Bob gives Alice \$1. They play until one of them runs out of money. Alice stated with \$$a$ and Bob started with \$$b$. The game has two possible outcomes: Alice has all the money or Alice has none of the money. The game is also fair: Alice and Bob each expects to walk away with the same amount of money he/she started with (on average). Calculate the probability of Alice winning all the money.