The die-coin experiment consists of rolling a (normal, 6-sided) die and then flipping a fair coin the number of times shown on the die. Thus after rolling a 1 we flip the coin once, after rolling a 2 we flip twice, after rolling a 3 we flip thrice, et cetera. A sample space for this experiment is

$$S = \{(1, H), (1, T), (2, HH), (2, HT), (2, TH), (2, TT), \ldots, (6, TTTTTT)\}$$

but what we're really interested in is the numbers showing up in the experiment. First there's the number rolled on the die, which we'll call R. Then there's the number of times we flip heads, which we'll call F. The variables R and F are called *random variables* because their values will be determined by a random process (in this case the die-coin experiment).

1. What are the possible values for R? What is the probability that R takes each of these values?

2. What are the possible values for F?

Calculating the probabilities for F is difficult unless we are given information about the roll of the die. For example, if we know that $R = 1$, then we know that $F = 0$ with probability $\frac{1}{2}$. This is expressed symbolically as $P(F = 0|R = 1) = \frac{1}{2}$ (read as "the probability of $F = 0$ given $R = 1$").

3. Calculate $P(F = 2|R = 3)$.

Theorem (The Law of Total Probability)**.** *If event A has probability strictly between 0 and 1, then for any event B,* $\boxed{P(B) = P(B|A)P(A) + P(B|A')P(A')}$.

4. Calculate $P(F = 6)$.

5. Calculate $P(F = 5)$.

Probabilities like $P(F = 5|R = 6)$ are called *conditional probabilities*. Technically, conditional probabilities are defined as follows (don't lose whatever informal understanding you have at this point–it should agree with this definition).

Definition. If A is an event with non-zero probability and B is any event, then $\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}}$.

6. Prove the Law of Total Probability (using the definition of conditional probability).

7. Suppose you know that your friend ran the die-coin experiment and flipped 5 heads. Calculate the conditional probabilities of your friend having rolled 1, 2, 3, 4, 5, and 6 on the die. Which was most likely to have been her roll?