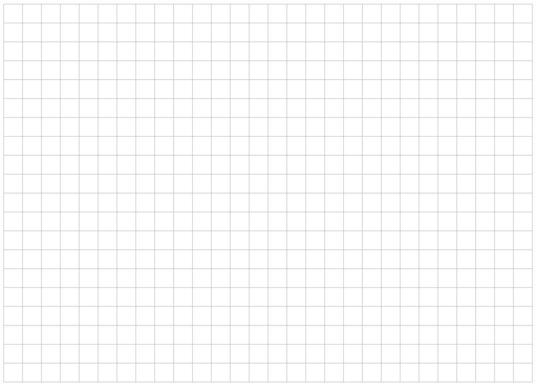
Name(s):

Math 421 Expected Values and Variances October 11, 2017

1. Let X_1 be a continuous random variable with density $f(x) = \frac{3}{2}x^2$ for $-1 < x < 1$. Let X_2 be a continuous random variable with density $g(x) = \frac{3}{4}\left(1 - x^2\right)$ for $-1 < x < 1$.

a) Sketch graphs of the two densities on the same axes.



b) Calculate $E(X_1)$ and $E(X_2)$ (observing features of the graphs may suffice).

Definition. The *variance* of a random variable X is $\sigma^2 = V(X) = E\left[(X - \mu)^2\right]$ (where $\mu = E(X)$).

The variance measures the tendency of the random variable to spread away from its mean. A larger variance indicates a greater likelihood of the random variable taking values far from its mean.

2. Prove that $V(X) = E(X^2) - [E(X)]^2$ for any random variable X. Hint: expand $(X - \mu)^2$ and apply theorem 4.2. Keep in mind that $\mu = E(X)$ is constant.

3. Consider again the two random variables of problem 1.

a) Which of X_1 and X_2 should have a higher variance and why?

b) Calculate $V(X_1)$ and $V(X_2)$.

Challenge. A game, which we'll call St. Petersburg, starts with \$1 in the pot. A fair coin is then flipped until the first heads appears, at which point you win the pot. Each time the coin comes up tails the pot is doubled (so you win \$1 if the first flip is heads, \$2 for tails then heads, \$4 for tails, tails, heads, ...).

a) What is the expected value of the game?

b) How much would you pay to play? What is the probability of a profit if you pay that price?

Challenge. Petrograd is a new version of St. Petersburg in which the game stops after 20 flips if heads has not appeared, in which case you win nothing. Otherwise the rules are the same.

a) What is the expected value of this game? This is called the fair price for the game.

b) What is the probability of a profit if you pay the fair price?

Challenge. (The Gambler's Ruin). Alice and Bob are gambling on flips of a fair coin: Alice gives Bob \$1 if the flip is heads, otherwise Bob gives Alice \$1. Alice starts with \$$a$ and Bob starts with \$$b$ and they'll play until one of them runs out of money. The game has two possible outcomes: Alice has all the money or Alice has none of the money. The game is also fair: Alice and Bob each expects to walk away with the same amount of money he/she started with (that is, the expected value of Alice's winnings is exactly a and the expected value of Bob's winnings is exactly b). Calculate the probability of Alice winning all the money.