Name(s):
Math 421 Gamma, gamma, gamma October 27, 2017

Definition 1. The **gamma function** is defined as $\boxed{\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy}$ for $\alpha > 0$.

1. The goal of this problem is to prove (by induction) that for any positive integer n, $\boxed{\Gamma(n) = (n-1)!}$.

a) Show that $\Gamma(1) = 1$

b) Integrate by parts to show that for any $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$. Integration by parts: $\int udv = uv - \int vdu$.

c) Prove by induction that for any positive integer n, $\Gamma(n) = (n-1)!$

Definition 2. A random variable X has a **gamma distribution** with parameters $\alpha > 0$ and $\beta > 0$ if and only if the following function is a probability density for X: $\boxed{g(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}}$

2. Prove that $g(x;\alpha,\beta)$ is actually a probability density by observing that it is always non-negative and then showing that $\int_{-\infty}^{\infty} g(x;\alpha,\beta)dx = 1$. Hint: make the substitution $y = \frac{x}{\beta}$.

Challenge. Find the moment-generating function for a gamma distribution.

3. Calculate the mean and variance of a gamma distribution (if you have calculated the mgf, then I recommend using it here).

Some of the gamma distributions are special enough to get their own names.

Definition 3. A gamma distribution with $\alpha = 1$ and $\beta = \theta$ is an **exponential distribution** with parameter $\theta > 0$. A gamma distribution with $\alpha = \frac{\nu}{2}$ and $\beta = 2$ is a **chi-square distribution** with parameter $\nu > 0$.

4. Calculate the mean and variance of a chi-square distribution (use the previous problem's answers).