Let $X_1, X_2, \ldots, X_n$ be a sequence of independent, identically distributed (iid) Bernoulli random variables with probability of success $\theta$ (with $0 < \theta < 1$). The total number of successes in the $n$ trials is $T = X_1 + X_2 + \cdots + X_n$. The random variable $T$ has a binomial distribution and its probability distribution function is

$$b(x; n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} \text{ for } x = 0, 1, \ldots, n.$$  

Applying Theorem 4.14, we find that $E(T) = n\theta$ and $V(T) = n\theta(1 - \theta)$.

For the following two problems we’ll consider another (possibly infinite) sequence of iid Bernoulli random variables $X_1, X_2, X_3, \ldots$.

1. Let $X$ be the time of the first success (so the possible values for $X$ are $1, 2, 3, \ldots$). Find the probability distribution function of $X$. This is the geometric distribution function with parameter $\theta$, which the book calls $g(x; \theta)$.

2. Let $Y$ be the time of the $k^{th}$ success (so the possible values for $Y$ are $k, k+1, k+2, \ldots$). Such a random variable is said to have a neg**ative binomial** distribution with parameters $k$ and $\theta$.

   a) Find the probability distribution function of $Y$ (the book calls this $b^*(x; k, \theta)$).

   b) Calculate $E(Y)$. Hint: $\sum_{x=k}^{\infty} b^*(x + 1; k + 1, \theta) = 1$.

   c) Prove that $b^*(x; k, \theta) = \binom{k}{x} b(k; x, \theta)$.

3. A random variable with the probability distribution function

$$p(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x = 0, 1, 2, \ldots$$

is said to have a *Poisson* distribution (with parameter $\lambda > 0$). Find the moment-generating function and use it to calculate the mean and variance of the distribution.