Definition. A random variable has a Poisson distribution with parameter $\lambda > 0$ if its probability distribution function is 

$$p(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x = 0, 1, 2, \ldots.$$ 

Proposition 1. If $X$ has a Poisson distribution with parameter $\lambda > 0$, then $E(X) = \lambda$ and $Var(X) = \lambda$.

1. Suppose that a (radioactive) neutron source emits 3 neutrons every 1 millisecond on average. The actual number emitted is the result of complicated (quantum) physics: experience has shown that Poisson distributions can be useful here. Let $N_t$ be the number of neutrons emitted over $t$ milliseconds. We’ll assume that $N_t$ has a Poisson distribution.

a) What parameter $\lambda$ does $N_1$ have?

b) What parameter does $N_t$ have?

(technically, $N_t$ is a family of random variables known as a Poisson process).

Definition. A random variable has an exponential distribution with parameter $\theta > 0$ if its density is

$$g(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Proposition 2. If $X$ is exponentially distributed with parameter $\theta$, then $E(X) = \theta$ and $Var(X) = \theta^2$.

2. We’ll continue to work with the same neutron sources as in the first problem. Let $T$ be the duration between the emission of neutrons by the neutron source, that is the time $t$ at which $N_t$ changes from 0 to 1.

a) $T > t$ if and only if $N_t$ is equal to what?

b) Find the cumulative distribution function for $T$.

c) Differentiate the cdf for $T$ to find a probability density function for $T$. Name the distribution.

d) How long do you have to wait to be 99% sure than at least one neutron has been emitted?

e) Show that $P(T \geq t + t_0 | T \geq t_0) = P(T \geq t)$ for any $t, t_0 > 0$.

Challenge. Let $T_2$ be the time to the second emission from the neutron source. Find the probability density function of $T_2$. 