Definition 1. Let $X_1, X_2, \ldots, X_n$ be a random sample. Let $Y_1, Y_2, \ldots, Y_n$ be the same sample arranged in increasing order. The random variable Y_k is the k^{th} *order statistic*.

1. Let Y_n be the n^{th} order statistic of a random sample of size n. Let the population have probability density function f and distribution function F. Use the distribution function method to find the density of Y_n. Hint: Y_n is the largest number in the sample and this helps to interpret $Y_n \leq y$ in terms of the relationships between $X_1, X_2, \ldots, X_n$ and y.

For the remaining problems, the population is uniformly distributed on $[0, \beta]$. This makes $f(x) = \frac{1}{\beta}$ for $0 < x < \beta$.

2. Calculate $E(Y_n)$ and then find an unbiased estimator for β based on Y_n.

3. Show that $2\overline{X}$ is also an unbiased estimator for β.

Definition 2. The *relative efficiency* of unbiased estimators $\hat{\Theta}_1$ to $\hat{\Theta}_2$ is the ratio

$$\frac{\operatorname{var}\left(\hat{\Theta}_1\right)}{\operatorname{var}\left(\hat{\Theta}_2\right)}.$$

An estimator with a smaller variance will be more likely to be close to θ and, thus is a better estimator for θ. Relative efficiency is a way of comparing the variances of two unbiased estimators. If the relative efficiency of $\hat{\Theta}_1$ to $\hat{\Theta}_2$ is $\frac{1}{4}$ then this can be interpreted as saying that the estimator $\hat{\Theta}_1$ requires only $\frac{1}{4}$ as many observations to achieve the same accuracy at $\hat{\Theta}_2$.

4. Calculate the relative efficiency of your estimators for β to $2\overline{X}$ (your answer should depend on the sample size).

5. (Challenge for anyone who finishes early) Does the Cramér-Rao inequality allow for estimators with a smaller variance than yours? Compare the variance of your estimator with

$$\frac{1}{nE\left[\left(\frac{\partial \ln f(X)}{\partial \beta}\right)^2\right]}.$$

6. (Challenge for anyone who finishes early) Calculate the density function of the k^{th} order statistic Y_k.