Math 422 **Order Statistics and Estimators** **January 29, 2018**

Definition. Let $X_1, X_2, \ldots, X_n$ be a random sample. Let $Y_1, Y_2, \ldots, Y_n$ be the same sample arranged in increasing order (so Y_1 is the smallest, Y_2 is the second smallest, ..., and Y_n is the largest). The random variable Y_k is the k^{th} *order statistic*.

1. Let Y_n be the n^{th} order statistic of a random sample of size n from a population with probability density function f and cumulative distribution function F. Use the distribution function method to find the density of Y_n, specifically:

a) Find the cumulative distribution function of Y_n in terms of F.

b) Differentiate to find the density of Y_n.

For the remaining problems, assume the population is uniformly distributed on $[0, \beta]$. This makes the probability density $f(x) = \frac{1}{\beta}$ for $0 < x < \beta$.

2. Is Y_n an unbiased estimator for β? It is asymptotically unbiased?

3. Find a number C such that $E(CY_n) = \beta$ (i.e. find an unbiased estimator for β based on Y_n).

4. Show that $2\overline{X}$ is also an unbiased estimator for β.

Definition. The *relative efficiency* of unbiased estimators $\hat{\Theta}_1$ to $\hat{\Theta}_2$ is the ratio $\dfrac{\text{var}\left(\hat{\Theta}_1\right)}{\text{var}\left(\hat{\Theta}_2\right)}$.

An estimator with a smaller variance will be more likely to be close to θ and, thus is a better estimator for θ. Relative efficiency is a way of comparing the variances of two unbiased estimators. If the relative efficiency of $\hat{\Theta}_1$ to $\hat{\Theta}_2$ is $\frac{1}{4}$ then this can be interpreted as saying that the estimator $\hat{\Theta}_1$ requires only $\frac{1}{4}$ as many observations to achieve the same accuracy at $\hat{\Theta}_2$.

5. Calculate the relative efficiency of your estimators for β (your answer should depend on the sample size). Evaluate at $n = 2, 3, 6,$ and 10. Which estimator is better?

Homework. Does the Cramér-Rao inequality allow for estimators with a smaller variance than yours? Compare the variance of your estimator with

$$\frac{1}{nE\left[\left(\frac{\partial \ln f(X)}{\partial \beta}\right)^2\right]}.$$

Homework. Calculate the density function of the k^{th} order statistic Y_k.