QUICK RECAP OF CALCULUS

In Calc 1, we focus on the fact that limits are important, defined based on the ideas of being "close enough" and we use them to define everything else. Derivatives are the limits of slopes of secant lines and tell us quite a bit about the behavior of our functions (increasing/decreasing, concavity, etc.). Because of this, we can apply them to many situations and obtain information about extrema from those computations. There are two main integration notions. The first is the idea of the anti-derivative (undoing derivatives via indefinite integrals). The second is the idea of finding areas via limits of Riemann sums (via definite integrals). The two notions come together with the Fundamental Theorem of Calculus.

In Calc 2, there are three larger portion of the class and they are frequently mostly disconnected. The first is the idea of integration applications and using the notion of a definite integral of a function of one variable to compute various things. The second is the collection of tools we build in order to compute indefinite integrals. This collection of tools can largely be thought of as high-school algebra tricks and trigonometry identities that you forgot and/or never learned. The third portion of Calc 2 is the sequences and series portion. This starts in a fairly abstract way and culminates in the notion of Taylor series where we have series that represent functions and these series allow us to approximate tricky things and find anti-derivatives for functions we've failed to find them for before.

In Calc 3, we mostly start over and do Calc 1 again, but now for functions of multiple variables. The course (ideally) includes vectors and functions whose outputs are vectors (i.e. functions $\mathbb{R} \to \mathbb{R}^n$). We also explore what functions of multiple variables look like in space (these are mostly functions $\mathbb{R}^2 \to \mathbb{R}$). We then explore derivatives and integrals of these functions. As we go to integrate functions of several variables, we introduce polar, cylindrical, and spherical coordinate systems in order to have more options of how we describe regions (in one dimension, the only possible regions are intervals and those don't need choices). Then, and this is the part that you all missed here, the course should culminate in doing line integrals, flux integrals, and surface integrals. Combining all those notions are Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

"DETAILED" CALCULUS II TOPICS LIST

Integration Techniques Here's a quick recap of the most used techniques:

Integration by parts Consider $u(x)v(x)$ and differentiate using product rule to obtain

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x).$$

Subtracting $u'(x)v(x)$ from both sides and then integrating yields

$$u(x)v(x) - \int u'(x)v(x)\,dx = \int u(x)v'(x)\,dx.$$

Suppressing function notation, swapping sides of the equality, and writing $u'(x)\,dx$ as du and similarly $v'(x)\,dx = dv$, we obtain the usual rule:

$$\int u\,dv = uv - \int v\,du.$$

Partial Fraction Decomposition This is the main technique used for rational function integration (fractions of polynomials). The rules and algebra here get messy quickly, and honestly the best strategy on the field test is probably guess-and-check for such a problem.

Trig Substitution Here, we substitute $x = a\tan\theta$ when we see $a^2 + x^2$ in the integrand. When we see $a^2 - x^2$, we use $x = a\sin\theta$ and when we see $x^2 - a^2$, we use $x = a\sec\theta$.

Trigonometric Integrals This is the strategies we develop for dealing with products of powers of trig functions. There are two types. First, if the function looks like $\cos^{2n}(\theta)\sin^{2m}(\theta)$, then you'll need to use the appropriate half-angle formulas:

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \qquad \text{and} \qquad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}.$$

For any other product, we want to use some sort of u-substitution along with the pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{and} \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \text{and} \qquad 1 + \cot^2\theta = \csc^2\theta,$$

and we need to be careful to ensure that we've got the correct du lying around. For $u = \sin\theta$, we need to leave a $du = \cos\theta\, d\theta$ and want to turn all other trig functions into $\sin\theta$ so that we can make those u's. The case for $u = \cos\theta$ is similar. For $u = \tan\theta$, we have $du = \sec^2\theta\, d\theta$, so we're looking for everything to be put into $\tan\theta$, except for two copies of $\sec\theta$. For $u = \sec\theta$, we have $du = \tan\theta\sec\theta\, d\theta$, so we're looking for everything to be put into $\sec\theta$, except for one $\sec\theta$ and one $\tan\theta$. Rarely will you see these involving csc and cot, but the ideas there are similar.

Integration Applications Here's a list of some things you can do with integrals of one variable.

Volumes of revolution These can be computed via washers (or disks) or shells.

The formula for the volume of a **washer** is $\pi(R^2 - r^2)\Delta h$ where R is the radius of the outside of the washer, r is the inner radius, and Δh is the height of the washer. Here, if we're rotating around the y-axis, we'll have Δh become dy in our integral and it'll become dx if we're rotating about the x-axis. Note that a **disk** is just a washer without a hole, i.e. with $r = 0$.

The formula for the volume of a **cylindrical shell** is $2\pi rh\Delta r$ where r is the radius of the shell, h is the height of the shell, and Δr is the width of the shell. If we're rotating about a line parallel to the y-axis, then Δr becomes dx in our integral and if we're rotating about a line parallel to the x-axis, it becomes dy. Also, r is the distance from the shell to the axis of rotation, and so typically of the form $x - c$ or $y - c$ for some constant c related to the axis of rotation.

Arc Length The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by the formula

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Surface area of revolution The simplest version of the surface area formula is $\int 2\pi r\, ds$ where r is the radius to the axis of revolution and ds represents a small piece of arc length. In particular, if $x = g(y)$ on the interval $c \le y \le d$ is rotated about the y-axis, then the integral will have bounds c and d and be evaluated with respect to y. Additionally, we'll have $r = x = g(y)$ and $ds = \sqrt{1 + (g'(y))^2}\, dy$. All told, the surface area will be $\int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy$. The three other formulas are no harder to derive from this than the example.

Physics Applications - Work, Pumping, Pressure Each of these applications is a bit different, and you probably didn't cover all of them in the course you took. They key to all integral applications is to split an interval into small pieces, compute the value of the desired output on that small piece, add them up and take a limit, resulting in an integral. Memorizing a bunch of formulas probably won't help if you don't understand what they mean or when to use them.

Series Basics A **sequence** is an infinite list: $a_1, a_2, \ldots$. While we use this notion only with $a_i \in \mathbb{R}$ in calculus, the notion is useful in other contexts with a_i in other sets. The **limit** of a sequence is what the sequence of numbers approaches as n heads towards infinity. Assuming we have a closed formula for a_n, we can think of the limit $\lim_{n\to\infty} a_n$ just like we would the limit of $f(x)$ as x heads towards infinity. The **partial sums** of a sequence are the sums

$$S_n = \sum_{i=1}^{n} a_i$$

and this sequence is the **series** denoted $\sum_{i=1}^{\infty} a_i$. Note that it's totally okay to start a series/sequence at $i = 0$, or $i = 3$, or anywhere else. This will not change whether or not the sequence or series converges; however, it frequently changes what a series converges *to*. Recall also that we say a convergent series $\sum a_i$ **converges absolutely** if $\sum |a_i|$ converges and **converges conditionally** if $\sum |a_i|$ diverges. Here's a quick list of series you should be familiar with off hand:

Geometric Series These have the form $\sum_{i=0}^{\infty} ar^i$. When $|r| < 1$, these converge absolutely, and when $|r| \geq 1$, they diverge. Such convergent series converge to $\frac{a}{1-r}$. Note that this has been set up so that a is the first term of the series and r is the ratio. If the indexing is different, things will look different!

p-Series These have the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$. This series converges when $p > 1$ and diverges for $p \leq 1$. When $p = 1$, we call this the **harmonic** series.

Alternating Harmonic Series This is the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ and it converges conditionally.

Series Tests Here's a quick recap of each of the series tests, phrased as a theorem. If not explicitly given, you should assume $\sum_{i=1}^{\infty} a_i$ is a series of real numbers.

Divergence test If $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{i\to\infty} a_i = 0$. The contrapositive of this is what we use to make this useful.

Ratio Test Let $L = \lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$. If $L < 1$, the series converges absolutely. If $L > 1$, the series diverges. If $L = 1$ or the limit doesn't exist, the test fails.

Comparison Test This requires a second series $\sum_{i=1}^{\infty} b_i$ with $a_i, b_i \geq 0$ for all i. Suppose $a_i \leq b_i$ for all i. If $\sum a_i$ diverges, then so does $\sum b_i$. If $\sum b_i$ converges, then so does $\sum a_i$.

Limit Comparison Test This requires a second series $\sum_{i=1}^{\infty} b_i$ with $a_i \geq 0$ and $b_i > 0$ for all i. Let $c = \lim_{i\to\infty} \frac{a_i}{b_i}$. If $0 < c < \infty$ then both series converge or both series diverge.

Alternating Series Test This requires the series to look like $\sum (-1)^k a_k$ with $a_k \geq 0$. If $a_{k+1} \leq a_k$ for all k and $\lim_{k\to\infty} a_k = 0$, then $\sum (-1)^k a_k$ converges.

Integral Test This requires a function $f(x)$ where $f(x) \geq 0$ for $x \geq c$ and $f(k) = a_k$ for all integers $k \geq c$. Then $\int_c^{\infty} f(x)\,dx$ and $\sum_{k=c}^{\infty} a_k$ either both converge or both diverge.

Root Test Let $L = \lim_{i\to\infty} \sqrt[i]{|a_i|}$. If $L < 1$, the series converges absolutely. If $L > 1$, the series diverges. If $L = 1$ or the limit doesn't exist, the test fails.

Power Series Such series look like

$$\sum_{k=1}^{\infty} a_k (x-c)^k$$

where c is a constant and $a_k \in \mathbb{R}$. We say the series is **centered at** c. One can frequently use the ratio test to identify a value $p \in \mathbb{R}$ such that $p|x-c|$ is the limit from the ratio test. Then, the power series converges (absolutely) as long as $|x-c| < 1/p$ and we say $1/p$ is the **radius of convergence**, and it will converge for the values of x which make $|x-c| < 1/p$ true, i.e. for $c-1/p < x < c+1/p$. To identify whether or not the series converges for $x = c \pm 1/p$, we must plug in those values of x to the original series and use other tools (these give $L = 1$ for the ratio test, so that test fails). If the limit from the ratio test is 0, then noting that $0 < 1$ is always true, we know that the series converges for all values of x and say that the **radius of convergence** is infinity. If the limit from the ratio test is ∞, then noting that $1 < \infty$ is always true, we know that the series diverges for all values of x and say that the **radius of convergence** is zero.

Taylor Series These are special kinds of power series and are series designed to represent functions. The **Taylor Series of** $f(x)$ **centered at** $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

where $f^{(k)}(a)$ is the k^{th} derivative of f evaluated at a. When $a = 0$, we sometimes call this the **Maclaurin Series** for $f(x)$. Some important Maclaurin series are given below.

Function	Maclaurin Series	Radius of Convergence
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	1
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	∞
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	∞
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	∞
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	1
$\ln(x+1)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$	1
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	1