QUICK RECAP OF CALCULUS

In Calc 1, we focus on the fact that limits are important, defined based on the ideas of being "close enough" and we use them to define everything else. Derivatives are the limits of slopes of secant lines and tell us quite a bit about the behavior of our functions (increasing/decreasing, concavity, etc.). Because of this, we can apply them to many situations and obtain information about extrema from those computations. There are two main integration notions. The first is the idea of the anti-derivative (undoing derivatives via indefinite integrals). The second is the idea of finding areas via limits of Riemann sums (via definite integrals). The two notions come together with the Fundamental Theorem of Calculus.

In Calc 2, there are three larger portion of the class and they are frequently mostly disconnected. The first is the idea of integration applications and using the notion of a definite integral of a function of one variable to compute various things. The second is the collection of tools we build in order to compute indefinite integrals. This collection of tools can largely be thought of as high-school algebra tricks and trigonometry identities that you forgot and/or never learned. The third portion of Calc 2 is the sequences and series portion. This starts in a fairly abstract way and culminates in the notion of Taylor series where we have series that represent functions and these series allow us to approximate tricky things and find anti-derivatives for functions we've failed to find them for before.

In Calc 3, we mostly start over and do Calc 1 again, but now for functions of multiple variables. The course (ideally) includes vectors and functions whose outputs are vectors (i.e. functions $\mathbb{R} \to \mathbb{R}^n$). We also explore what functions of multiple variables look like in space (these are mostly functions $\mathbb{R}^2 \to \mathbb{R}$). We then explore derivatives and integrals of these functions. As we go to integrate functions of several variables, we introduce polar, cylindrical, and spherical coordinate systems in order to have more options of how we describe regions (in one dimension, the only possible regions are intervals and those don't need choices). Then, and this is the part that you all missed here, the course should culminate in doing line integrals, flux integrals, and surface integrals. Combining all those notions are Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

BIG TOPICS IN CALC 3

Vectors In $\mathbb{R}^2$ or $\mathbb{R}^3$, vectors have a length and a direction. We can find that length/magnitude via the distance formula: for $\vec{v} = \langle a, b, c\rangle \in \mathbb{R}^3$, $|\vec{v}| = \sqrt{a^2+b^2+c^2}$. The direction of the vector can be given as a unit vector by $\vec{u} = \frac{1}{|\vec{v}|}\vec{v}$. As none of you took the version of Calc 3 with vector calculus in it, the main purpose of vectors was to understand $\mathbb{R}^3$ better and to manipulate them to compute a variety of things. Here's some more details:

Dot Product The **dot product** is formally defined for vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ based on their magnitudes and the angle between them:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta.$$

Due to this definition, we know that the dot product of orthogonal vectors is always zero. However, as is frequently the case in calculus, we prefer easier computational strategies. So, we also have the formula

$$\langle a,b,c\rangle \cdot \langle d,e,f\rangle = ad+be+cf \qquad \text{or} \qquad \langle a,b\rangle \cdot \langle d,e\rangle = ad+be.$$

Note that the dot product yields a *scalar* and that $|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}}$.

Cross Product The **cross product** of vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ (only) is formally defined to be the *vector* with magnitude given by $|\vec{v}||\vec{w}|\sin\theta$ where again θ is the angle between the vectors and direction given by the right hand rule. Again, this is great, but not what we frequently prefer to work with, which is given by a determinant:

$$\langle a,b,c\rangle \times \langle d,e,f\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \langle bf-ce, -(af-cd), ae-bd\rangle.$$

Note that the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$ is $|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\theta$.

Projections The **projection** of the vector $\vec{w}$ onto the vector $\vec{v}$ is given by

$$\text{proj}_{\vec{v}}(\vec{w}) = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v}.$$

The denominator in the formula scales everything appropriately. In particular, as $\vec{v}\cdot\vec{v} = |\vec{v}|^2$, we have that $\dfrac{\vec{w}\cdot\vec{v}}{|\vec{v}|}$ is the length of the desired vector and $\dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector in the desired direction.

Vector-Valued Functions We can define function $\mathbb{R}\to\mathbb{R}^2$ or $\mathbb{R}\to\mathbb{R}^3$ which we can think about as functions of position in space as a function of time. Then, taking derivatives of these (component-wise) gives velocity and acceleration functions.

Functions of Multiple Variables We can think of functions $z = f(x,y)$ as surfaces in $\mathbb{R}^3$ where z represents the height of the function at the point (x,y) in the $xy-$plane. We can do similar things with other iterations of these variables. With such functions, we can discuss their partial derivatives and use those partial derivatives to find local and global extrema, and also rates of change and tangent planes. Here's some of the most ubiquitous stuff.

Planes Just like the general form of a line in $\mathbb{R}^2$ is $ax+by=c$, one can think of the general form of a plane in $\mathbb{R}^3$ as $ax+by+cz=d$. However, as is the case for lines, a more useful form of the equation is in point-slope form. Planes are a bit more complex than lines though. To define the "slope" of a plane, we need a vector perpendicular to the plane, say $\vec{n} = \langle a,b,c\rangle$. Then, for any fixed point $P_0 = (x_0,y_0,z_0)$ and an arbitrary point $Q = (x,y,z)$ on the plane, the vectors $\vec{n}$ and $\overrightarrow{P_0Q} = \langle x-x_0, y-y_0, z-z_0\rangle$ are perpendicular, so that their dot product must be zero:

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0.$$

Partial Derivatives Suppose you have a function $f(x,y,z)$ or another function of multiple variables. The **partial derivatives** of the function are found by holding all but one of the variables constant and taking the derivative (as in Calc 1) of the remaining variable. We denote such things in the form

$$\frac{\partial f}{\partial x} = f_x$$

and use f_{xy} for the partial of f first taken with respect to x and then taken with respect to y. Note that as long as things are sufficiently nice, $f_{xy} = f_{yx}$. Also, the **gradient** of $f(x,y,z)$ is the vector-valued function $\nabla f = \langle f_x, f_y, f_z\rangle$. Note that the gradient at a given point is in the direction of maximum increase of f at that point and the magnitude is that maximum rate of increase.

Directional Derivatives Let $\vec{u}$ be a vector of length 1. Then, the derivative of f in the direction of $\vec{u}$ at the point (a,b) is

$$f_{\vec{u}}(a,b) = \nabla f(a,b)\cdot\vec{u}.$$

This (as with almost everything) can be done in $\mathbb{R}^3$ by extending in the "obvious" way.

Tangent Planes The plane tangent to the surface $F(x,y,z) = 0$ at the point (a,b,c) is

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

assuming of course that $F(a,b,c) = 0$ so that the proposed point is actually on the surface. If you're so inclined, you can sometimes solve for the surface to look like $z = f(x,y)$ and have $c = f(a,b)$ and the equation of the plane can then be written as

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) = z$$

Local Extrema The **critical points** of a function $f(x,y)$ are the points where $f_x = f_y = 0$. We can (as in Calc 1) use second derivatives to classify critical points. Let (a,b) be a critical point of $f(x,y)$ and define

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Then, if $D > 0$ we have a local extrema, with $f_{xx} > 0$ telling us the extrema is concave up locally and so a local minimum. If instead $f_{xx} < 0$, we have an extrema that is concave down locally and so must be a local maximum. If instead we have $D < 0$, we have a **saddle point** and there are points nearby (a,b) where the function is larger and points where it is smaller than the value of the function at (a,b). If $D = 0$, we obtain no information from the second derivative test.

Lagrange Multipliers This is algebraically messy, and unlikely to come up, but is the strategy used to identify local extrema along curves. As of this semester, not all Calc 3 sections at Gonzaga are covering this topic.

Multiple Integrals We can do double integrals in Cartesian or polar coordinates. For triple integrals, these can be done in Cartesian, cylindrical, or spherical coordinates. Here's a quick recap of those coordinate systems and setting up multiple integrals. Note that a double integral of a function $H(x,y)$ or $H(r,\theta)$ gives the volume of the region below $z = H$ and above the xy-plane over the region of integration. If $H = 1$, this is the area of the region. Also, a triple integral of a function δ gives the mass of the object with density given by δ over the region of $\mathbb{R}^3$ of integration. If $\delta = 1$, this gives the volume of the region. The setup here is typically the interesting part and pictures are key, but very hard to type, so I'll do any desired pictures only in class.

Cartesian Double Integrals Given a region R of the plane, we can hopefully describe the region as being bounded in one direction (x or y) by functions in the other variable and bounded in the remaining variable by constants. Then, we use those to set up double integrals. I find it helpful to think of a point in the region as running between the two function bounds to give me a line and then taking those lines to run across the region to give the whole region. Note that here $dA = dx\,dy$ or $dA = dy\,dx$ depending on the direction we're going first.

Polar Double Integrals In this coordinate system, we set $x = r\cos\theta$ and $y = r\sin\theta$. This is done so that r is the radius from the origin to the point in the plane and θ is the angle (from the positive x-axis) to the point. Note that $x^2 + y^2 = r^2$ in this coordinate system. Here, the context for double integrals is similar, but things are a bit more complicated as we need $dA = r\,dr\,d\theta$. I'm not sure I've ever seen an instance where we would want to integrate with respect to θ before we integrate with respect to r. So, we consider our point with a (temporarily) fixed angle and allow that point to range from the smallest radius (as a function of θ) to the largest radius possible. Then, we ask which angles we're allowed to use.

Cartesian Triple Integrals Now our integral has $dV = dx\,dy\,dz$ (or any of the other five orders of integration). Assuming this order though, we fix a point and temporarily think of y and z as fixed and ask which functions of y and z bound our options on x to obtain a line. Then, we take our line and ask which functions of z bound our options on where we can start and end those lines. Finally, we ask what the smallest and largest options for z are.

Cylindrical Triple Integrals This coordinate system is formed by taking polar coordinates and adding a z-axis. So, we have $dV = r\,dz\,dr\,d\theta$ and frequently integrate in that order, first asking what functions of r and θ bound our height in the z direction, and then finishing out the problem by thinking about the region remaining in the xy-plane thought of in polar coordinates.

Spherical Triple Integrals This coordinate system is formed by letting ρ be the radius from the origin to the point (in three dimensions), keeping the θ from polar/cylindrical coordinates intact, and using an angle down from vertical as the third coordinate (called ϕ). This yields the formulas $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$ along with $x^2 + y^2 + z^2 = \rho^2$. Here, our bounds are frequently constants since the sphere centered at the origin of radius a now has the form $\rho = a$ and cones have the form $\phi = c$ for some constant angle c - in particular the cone $z = \sqrt{x^2 + y^2}$ is $\phi = \pi/4$. We almost always insist that $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$. Moreover, now $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ and this is the most common order of integration.