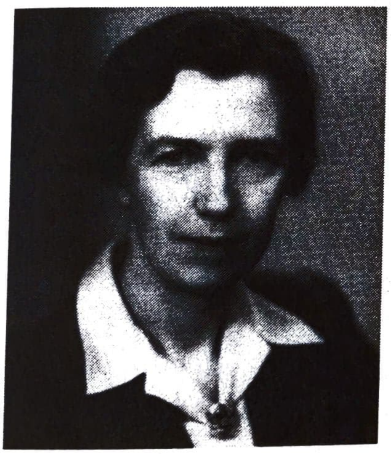
Mathematics is like checkers—it is suitable for the young, not too difficult, amusing, and without peril to the state.
—Plato (ca. 429–347 B.C.)

Mathematics and Thinking Mathematically

Mary Lucy Cartwright



Dame Mary Lucy Cartwright (1900–1998), D.B.E., F.R.S., to give her full name and honors, was born in Northamptonshire, the daughter of a rector. She was educated at home and subsequently at a "public" school in Salisbury. At this time the family was struggling to cope with the tragic death of Cartwright's two older brothers in World War I. She entered St. Hughes College, Oxford, in 1919 and, although she was engrossed in the study of history, she opted to "read" mathematics. She was one of only five women studying mathematics in Oxford at that time. Having devoted more time to history than to mathematics at school, the program in mathematics was a challenge but her natural mathematical endowments enabled her to earn a first class degree in 1923.

Not wishing to put a strain on the family finances, she taught school for four years and then returned to Oxford where she was awarded a D.Phil. in 1930 under the supervision of G. H. Hardy and E. C. Titchmarsh. In 1934 she was appointed to a faculty position at Girton College (a women's college in Cambridge University) where she was director of studies in mathematics.

Her mathematical research has ranged over a wide field of classical analysis and has been seminal in many areas, especially in integral equations.

In 1939 the Department of Scientific and Industrial Research asked Cartwright to help in solving "certain very objectionable looking equations occurring in connection with radar." In this enterprise, she collaborated with J. E. Littlewood who (tongue in cheek) described her as "the only woman in my life to whom I have written twice in a single day." He held her in high esteem as a colleague.

In 1949 she became mistress of Girton, a position that placed great demands upon her time, demands she cheerfully and willingly gave. It is said that she provided quiet, unassuming and clear-headed leadership of the college during a time of many challenges. Although shy by nature, she managed to interview all incoming students and to regale her younger faculty with tales of her foreign adventures.

Her earlier interest in history never left her entirely, for her work is permeated with historical perspectives that add interest and dimension to her work.

Her scientific work was recognized by election to the Royal Society in 1947. She also served as president of the London Mathematical Society from 1961 to 1963 as well as President of the Mathematical Association. She received honorary degrees from Cambridge, Edinburgh, Leeds, Hull, Wales, and Oxford. In addition, she was awarded the Sylvester Medal of the Royal Society in 1964 and the De Morgan Medal of the London Mathematical Society in 1968. In June 1969 she was made a Dame Commander of the British Empire for services to mathematics.

She died in 1998 and is revered for, among other things, having helped pave the way for more widespread recognition of women in mathematics.

Editor's Preface

In this essay, Mary Cartwright endeavors to throw some light on the difficult question of what constitutes "mathematical thinking," especially abstract thinking. The power to engage in complete abstraction, she notes, comes very slowly and, in the case of most individuals, comes not at all. Yet as we note below, surprising levels of understanding abstract ideas can be achieved. At present the mechanism of that achievement appears to be an impenetrable phenomenon of human thought. Present and future research in the neurological or psychological nature and basis of abstract thought may cast some light on this currently intractable problem.

Although abstract thinking arises in numerous contexts, its most conspicuous manifestation seems to be in mathematics. In the ongoing debate involving "nature" and "nurture," that is, whether a characteristic is genetically or culturally acquired, we suggest that abstract thinking is a *learned* attribute and in the case of mathematics, it seems to be intimately intertwined with the specific context.

Mary Cartwright points to another distinctive characteristic of mathematics, viz. that consumers of mathematics (engineers, physicists, etc.) invariably relate the mathematical construct and content to the individual physical phenomenon that interests them and that governs the phenomenon. This same process takes place at different levels of abstraction. It is said that Newton, for example, invariably associated differential equations with physical events. In a somewhat different direction, it is reported that a noted algebraist of the early part of the 20th century always thought of a linear transformation as a *matrix*. For early group theorists, a group was generally thought of as a set of *permutations*. In neither of these cases was the current abstract form used as a basis for deeper investigations. We hasten to add, however, that the less abstract point of view often led, nevertheless, to some deep and enduring discoveries.

To achieve a higher level of abstraction, it is often the case that a mathematical construct is disassembled into its component parts and reassembled, more nearly "to the heart's desire." The process may lead to a higher level of abstraction and, according to Cartwright, could thus lead to a greater understanding.

Returning to the matter of abstraction, it is a very interesting, curious and challenging question to account for the fact that in the historic evolution of mathematics, higher levels of abstraction are achieved and better understood, apparently quite simply *with the passage of time*. Each new generation is better able to cope with greater levels of abstraction. Since genetic differences do not occur in such short periods of time, we must account for this phenomenon by a change in the environment—teachers and writers of mathematics are better able to work with abstraction and are better able to articulate it. The passage of time has miraculously enhanced the clarity. Obviously this has been a consequence of evolving cultural changes. One of the most notable examples is the evolution of the Hindu-Arabic system of enumeration and the computational algorithms associated with it. Beginning from an arcane and almost mystical study in the 9th till the 12th century, to a system of computational algorithms that were, at first, limited in use to specially trained calculators in the 15th and later centuries, the system is now routinely taught to children in grade schools. It would seem that each generation contributed to a greater clarity and systematization of the underlying abstraction. It could now more easily be communicated at an early stage in a child's development.

Much has been written in recent years as people endeavor to understand the process of learning mathematics among children and to decipher the abstractions that underlie it. It is assumed that such investigations will cast light on the mathematical way of thinking. Among the leading investigators have been Piaget and Inhelder. They have unearthed many very interesting phenomena related to mathematical learning among children. However, as many elementary school teachers will testify, the enterprise has not been as effective as was originally hoped for and the most recent researches have not satisfactorily resolved the basic problems confronting teachers of mathematics, that is, the learner's ability to cope with abstraction.

Professor Cartwright however, has made some thoughtful observations.

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This year I find myself in the Division of Applied Mathematics at Brown University. Not every University or College has a separate department of applied mathematics, but for over thirty years I was classed as a pure mathematician in the University of Cambridge, England, and some applied mathematicians there prefer to call themselves theoretical physicists. Moreover, I once heard a geophysicist with a mainly mathematical training say that he used 'applied mathematics' as a term of abuse, meaning stuff which was not good mathematics and not really relevant to any physical problem. All these factors have made me think about the borderline between mathematics and its applications, not only to physical problems of a more or less traditional type, but also to statistical, economic, and industrial problems.

It is well known that the origins of some of the most abstract pure mathematics can be traced through the theory of Fourier series to a problem about vibrating strings, or through the theory of irrational numbers to Greek geometry and Egyptian devices for measurement of right angles, but the pure mathematicians of the last 100 or 150 years have been pursuing the mathematics for its own sake without any thought of vibrating strings. On the other hand many major new developments in pure mathematics were initiated quite specifically for the purpose of using them in some application. For instance this is certainly true of Newton's contributions to the calculus, and of probability theory, and this still seems to be happening in operations research and control theory. In distinguishing pure mathematics from applied two questions seem to arise. Is the work truly abstract and separated from all applications? And is it any more mathematical if it is truly abstract and pursued strictly for its own sake?

If we delve into the beginnings of mathematical thought in very young children or primitive peoples, there is plenty of evidence to show that the power of complete abstraction comes very slowly, and indeed to many people it probably only ever comes in a very restricted sense. A number of eminent people take the view that thought begins with the idea of actions performed in the mind only, that is to say operations. According to Piaget an ordinary child, by the time the child is two, can work out how he is going to do something before he does it, *provided* that the situation is simple and is familiar to him, but in order to understand abstract mathematical concepts such as 1, 2, 3, 4,..., the child has to move from perceptions arising from his environment and actions to the abstractions, a long and gradual process. Much work has been done by Piaget and Innhelder on the child's conception of space, and, for instance, its powers to distinguish between different kinds of figures such as a circle, a square, and a circle with a little one either inside or outside. Their experiments have thrown much light on the development of numerical, spatial, and physical concepts of a very elementary kind among young children, but it seems doubtful to me whether the abilities tested are always truly mathematical. For young blackbirds will gape at a piece of black cardboard consisting of one large circle and two small ones attached to it, but they only gape at the small circle whose size is a certain proportion of that of the large circle. This indicates that the ability to distinguish between certain shapes may have psychological foundations.

H. and H. A. Frankfort in an essay on myth and reality point out that ancient man could reason and work out the causes of things, but worked on very different hypotheses from ours. The primitive mind asks 'who' when it looks for a cause, and cannot withdraw far from perceptual reality. When the river does not rise, the river has *refused* to rise, and so the river or the gods intend to convey something to the people. At the same time primitive man used symbols much as we do, but he can no more conceive them as signifying, yet separate from, the gods or powers than he can consider a relationship—such as resemblance—as connecting, and yet separate from, the objects compared. Hence there is a coalescence of the symbol and what it signifies, as there is coalescence of two objects compared so that one may stand for the other.

Frankfort then gives an example of this coalescence in which pottery bowls with the names of hostile tribes were solemnly smashed at a ritual by the Egyptians in the belief that real harm was done to the enemies by the destruction of their names. It may seem a far cry from this to modern mathematics, but Bochner has drawn a parallel between mathematics and myth, and replaced myth by mathematics in some of Frankfort's sentences. I am not prepared to go as far as he does by replacing the word myth by mathematics in a sentence which then asserts that mathematics transcends reasoning in that it wants to bring about the truth it proclaims. However, in the ritual we have two fundamental features of mathematics, symbols representing something and operations on those symbols representing operations on the thing itself. Symbols and notation are part of the essential basis of mathematics, and I believe that the development and standardization of a good notation is an extremely important part of the development of mathematics.

If we turn to the extreme other end of the scale, we run into another kind of difficulty in separating the mathematics from its applications. Some pure mathematicians seem to do their mathematical thinking in terms of idealized physical and spatial ideas. The late G. H. Hardy, who taught me, was very much against applied mathematics, but in a footnote to a joint paper with J. E. Littlewood published in a Swedish periodical he wrote that a certain problem is most easily grasped in terms of cricket averages. Norbert Wiener would translate a mathematical problem into the language of Brownian motion, and I believe that his thinking was completely abstract although I do not know the theory, or remember what he said well enough to be quite sure. Hadamard has described his visualization of the proof that there is a prime greater than 11. To consider all prime numbers from 2 to 11, i.e., 2, 3, 5, 7, 11 he visualized a confused mass. Forming the product $2 \times 3 \times 7 \times 11 = N$, since N is large, he visualized a point remote from the mass. Increasing the product by 1 he saw another point a little beyond the first. $N + 1$, if not a prime, is divisible by a prime greater than 11; Hadamard saw a place between the mass and the first point. This seems to me to be a sort of mathematical shorthand and would certainly have to be translated back to numbers before it could be communicated to anyone else.

As I said earlier, I have until now always been classed as a pure mathematician, but Professor J. E. Littlewood and I did a lot of work on the theory of ordinary differential equations arising from problems of radio engineering. Littlewood is also a very pure mathematician in many ways, but he worked on antiaircraft gun fire in the First World War, and he translated our problems, which were suggested by radio values and oscillations, capacitance and inductance, etc., into dynamical problems and called all the solutions of our equations 'trajectories' as if they were the paths of missiles shot from a gun. In the radio problems there are oscillations with negative damping, and so we had periodic trajectories going up and down over and over again, and I am sure that the abstraction was complete although there was often a certain woolliness until the argument was complete, just as in Hadamard's visualization. Between these two extremes there are sonic users of complicated mathematics, physicists and engineers in particular, who are thinking all, or nearly all, the time in terms of the physics of the problem. Engineers have consulted me about a number of different types of problem, radio, control theory, oscillations of stretched wires; they usually come with some equations and very little explanation. I have to ask a lot of questions before they tell me everything relevant to the mathematical problem. It seems difficult for them to think in abstract mathematical terms, the symbols to them seem to mean the engineering concepts, currents and circuit constants such as impedance and inductance. This is important in two ways. The engineers have mental reservations and can check at every stage because they visualize how the physical system works. On the other hand they find it difficult to apply the mathematical processes used in one field to any other physical problem, even if they are just as relevant there. Some years ago at a conference for engineers I was asked to speak on Liapunov's method for stability problems. I described the basic principles as simply as I could, and after I spoke Professor Parks lectured on applications of the method. Many in the audience commented that the order of our lectures should have been reversed, and that they would have understood my lecture much better if they had understood that I was talking about the phase plane. It is possible that this was partly a question of notation and terminology, but I believe that they could do advanced mathematics best by thinking of it in terms of their particular engineering problems. The Liapunov method was developed mainly in connection with control engineering and by now has adopted much of its terminology, but the mathematics arising there need to be abstracted and put in a form which makes it available in connection with other applications. Problems of ordinary differential equations have arisen in connection with astronomy, ballistics, radio engineering, control theory, mechanical oscillations of machinery; each application has special features, and the theory of it was often developed in a correct logical form quite a long way before it was fitted into the general theory of ordinary differential equations as pure mathematics. The individual who formulated the equation and asked the question is, in the sense of my title, thinking mathematically, but he is not doing mathematics until he operates on his symbols. Please note that I do not say 'asked for a solution of the equation' because, although he may say that, he really wants to know something about the solutions in general. Is there a periodic solution? Is it stable? Will it remain stable if I change a certain parameter? Will the period be longer or shorter? He may find the methods which he needs in the literature and do the work himself. He may find a mathematician to help him. Although I myself have helped to develop the general theory and settle certain theoretical problems, I do not think that I have ever produced a result useful for any specific practical problem when it was needed. For soon after Littlewood and I began work on these problems, it was realized that the variations in individual thermionic valves was so great that precise mathematical results were not worth the trouble, and satisfactory experimental determinations could be more easily obtained. In recent times the person who formulates the mathematical statement of a physical or other real life problem usually does not do anything very original in the mathematical handling of it, although some interesting purely mathematical work on matrices appears in journals concerned with computing or applications to economics, detached from other pure mathematics.

To sum up so far I believe that the dividing line between strictly abstract thinking in mathematics and thinking in terms of the real world is by no means clearly defined and some of the major developments in mathematics such as the calculus were thought out more or less in terms of the real world. Further abstraction does not necessarily make the mathematics any better. For the Babylonian schoolmasters constructed sets of most complicated artificial formulae, perhaps 200 on one tablet, for their pupils to simplify. Their mathematics was sufficiently abstract for them to be indifferent whether they added the number of men to the number of days. In present circumstances this seems abstraction at its worst, but perhaps then it was a step forward. The Babylonians must have developed the laws of arithmetic a long way to set these complicated exercises, but mainly for practical purposes whether it was accounting or astronomy.

Now let us turn to those who do mathematics for its own sake. I should like to begin with the Hindu who in about 1200 B.C. wrote, "As crests on the heads of peacocks, as the gems on the hoods of snakes, so is ganita, mathematics, at the top of the sciences known as the Vedanga." Ganita is literally the science of calculation and in the early days it consisted of finger arithmetic, mental arithmetic, and higher arithmetic in general. At first it included astronomy, but geometry belonged elsewhere. At one stage higher mathematics was called 'dust work' because it was done in sand spread on the board or on the ground. We owe our so-called Arabic numerals to the Hindus, and they advanced a long way in algebra very early.

Most people consider that the Greeks were the first to do mathematics for its own sake and to realize the need for proof. The word 'mathema' meant originally a subject of instruction, but very early it was restricted to mathematical subjects among which Pythagoras included geometry, theory of numbers, sphaeric (or spherical trigonometry used for astronomy), and music. They classified numbers not only as odd and even, but as even-even, 2^m; even-odd, $2(2n+1)$; odd-even $2^{m+1}(2n+1)$, and also proved that there are an infinity of primes. I doubt whether they could calculate as well as the Babylonians, but probably that did not attract them, and also they lacked the incentives provided by the government of a far flung empire. I feel that I have to remind myself of the difficulties due to the absence of convenient symbols. Sir Thomas Heath writing of the arithmetic of Nicomachus said 'If the verbiage is eliminated, the mathematical content can be stated in quite a small compass,' but Heath used modern notation and Arabic numerals. In the Wasps of Aristophanes one of the characters tells his father to do an easy sum 'not with pebbles but with fingers,' and Herodotus says that, in reckoning with pebbles, Greeks move left to right, Egyptians right to left, which implies vertical columns facing the reckoner.

The Greeks also developed a theory of geometry which remained more important than any other for nearly 2,000 years, and was the first deliberate development of a logical system in mathematics. In the third century A.D. an unknown writer jokingly used words of Homer intended for something else to describe mathematics:

Small at her birth, but rising every hour.
She stalks on earth and shakes the world around.

For, says Anatolius, Bishop of Laodacia, who quoted it, mathematics begins with a point and a line and forthwith it takes in the heaven itself and all things within its compass. If this was the Greek viewpoint at such a late date, is it possible that their geometry was not truly abstract and that the symbols of point and line were still partly coalesced with the abstract point and line?

The position of geometry and more generally spatial concepts in mathematics is not completely clear to me. In recent times all types of geometry have been given an analytical basis and freed from the logical difficulties such as those which used to worry schoolmasters teaching about congruent triangles by the method of superposition. I therefore ask myself whether geometry and spatial concepts are really part of the basis of mathematics or a field of application similar to mechanics, both terrestrial and celestial, or to games of chance. The reason for the traditional special position of geometry may be that in geometry the symbols are the objects themselves; the abstract point, line, and triangle are represented by a point, line, and triangle; what is more, so long as the geometry is plane geometry they can be drawn on a flat surface by pen or pencil on paper or in sand on the ground. When Greek geometry was being developed there was no good notation for dealing with numbers, and even in the 15th Century the solution of a cubic equation was described in geometrical terms and illustrated by a figure for lack of a good algebraic notation. In mechanics a comparable real life representation of motion could not be used to explain the theory; written symbols or geometrical figures were needed for communication. But if we ask whether the contributions of spatial concepts to modern mathematics are greater than those of other real life problems it is difficult to answer. Spatial thinking has led to the highly abstract theory of irrational numbers of Cantor and Dedekind, and permeates mathematical thought in almost all fields; the physical sciences have given rise to the calculus (not without the help of geometry), and statistics and probability have their basis in multitudinous practical problems.

Pfeiffer explains the situation well in relation to probability. Some of the salient points in his account are as follows: The history of probability theory (as is true of most theories) is marked both by brilliant intuition and discovery and by confusion and controversy. Until certain patterns had emerged to form the basis of a clear-cut theoretical model, investigators could not formulate problems with precision, and reason about them with mathematical assurance.

From what some people say it sounds to me as if quantum theory had not yet reached this stage, but it is certainly beyond my competence to form a valid judgment.

Pfeiffer continues by saying that although long experience was needed to produce a satisfactory theory, we need not retrace and relive the fumblings which delayed the discovery of an appropriate mathematical model. That is, a mathematical system whose concepts and relationships correspond to the appropriate concepts and relationships of the real world. Once the model has been discovered, studied, and refined, it becomes possible for an ordinary mind to grasp, in a reasonably short time, a pattern which took decades of effort and the insight of genius to develop in the first place. I note that Pfeiffer asserts that the most successful model of probability theory known at present is characterized by considerable mathematical abstractness.

J. Willard Gibbs wrote 'One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity' and Bushaw says that one of the distinctive characteristics of modern mathematics is its way of taking old mathematical ideas apart like watches, studying the parts separately, and putting these parts together again in new and interesting combinations and studying these complications in turn. I believe that this process has contributed enormously to this simplification in mathematics itself, and so made it more readily available for applications. Mandelbrojt referring to the quotation from Willard Gibbs says 'Integration in function spaces provided such a point of view over and over again in widely scattered areas of knowledge and it gave us not only a new way of looking at problems but actually a new way of thinking about them.' Now one might call Fréchet the father of abstract spaces, and in the front of his book he puts a quotation from Hadamard's survey of functional analysis given in 1911. 'The functional continuum does not present any simple concept to our imagination. Geometrical intuition tells us nothing *a priori* about it. We are forced to remedy this ignorance and we can do it only analytically, by creating a chapter of the theory of sets for handling the functional continuum.' Elsewhere Hadamard wrote that the calculus of variations was nothing but the first chapter of functional analysis, and of his own work on the calculus of variations, hyperbolic partial differential equations, and certain other topics he said that he owed the greater part to his contacts with the physicist Duhem, through Duhem's book on hydrodynamics, elasticity, and acoustics and many conversations when they were both at Bordeaux. So we have a record here of the complete cycle from a physical basis through the calculus of variations to functional analysis and abstract spaces, and thence to a multitude of applications through the process of analyzing geometrical ideas and putting them together again in a most abstract new way to create function spaces.

A further variation on this pattern has become evident of recent years and that is the use of an auxiliary model consisting of various graphical, mechanical, and other aids to visualizing, remembering, and even discovering things about the mathematical model. The visual images of Hadamard, Hardy's cricket averages, and Littlewood's trajectories might be considered as auxiliary models, but of more universal significance are the analogue machines with electronic devices which simulate what happens in, for instance, fluid mechanics, or rather what corresponds in the mathematical model. We now have

(A) The real world of actual phenomena, known to us by various ways of experiencing these phenomena.

(B) The abstract world of the mathematical model which uses symbols to state relationships and facts with great precision and economy.

(C) The auxiliary model.

The transition from A to B is the formulation of real world phenomena in mathematical terms; the transition B to A is the interpretation of the deduction by pure mathematics from that formulation. Both these I consider to be thinking mathematically, but only the deductions inside B are mathematics. We may also think mathematically by moving from B to C which is a secondary interpretation, and then either back to B to confirm what C has suggested or from C direct to A.

As Pfeiffer points out, the value of both the mathematical model and the auxiliary model depends on how successfully the appropriate features of the model may be related to the 'real-life' situation. The models cannot be used to prove anything about the real world, although a study of it may help us to discover important facts about the real world. A model is not true or false; it fits or it does not fit. It is unsatisfactory if either (1) the solutions of the model problems have unrealistic interpretations, for instance, arbitrarily large quantities or arbitrarily fine differences, or (2) it is incomplete or inconsistent so that the mathematics produces contradictions. Many models fit amazingly well. Karl Pearson wrote 'The mathematician, carried along on his flood of symbols, dealing apparently with purely formal truths, may still reach results of endless importance for our description of the physical universe.

Until perhaps 100 years ago many scientists and mathematicians knew a bit of everything, and the mathematical formulation, as I said of Newton in particular, was done by someone who was a good enough mathematician to develop the mathematics to a considerable extent. This is particularly true of Sir Isaac Newton, but in these days of specialization the scientist or economist, or worker in close contact with the real world situation must do stage A → B. Sir Cyril Hinshelwood, former President of the Royal Society, said 'Scientists need to be taught mathematics as a language they can actually speak. It is of great importance for the scientist to be able to learn the art of formulating problems in mathematical terms which of course is a quite difficult job. You have to think very accurately and carefully about a problem before you can do it. You have to have practice in speaking the language of mathematics. It does not matter being an expert in differential equations. You can go to the expert for help in solving an equation. But you cannot expect the mathematician to do the translation into mathematics. There should be an early and rather intensive cultivation of the power of thinking about real things and the application of mathematical symbolism to physical ideas.' He went on to draw a parallel between learning simple French as a child and learning to express physical ideas in mathematics when the level of physics and mathematics reached are both elementary, so that the child becomes accustomed to the process by easy stages. Although he advocates, as I do, that the scientist should do the mathematical formulation, his words seem to imply an incomplete abstraction. In his mind the mathematical symbols were still representing their physical counterparts, not that this matters for a scientist who has access to an expert mathematician, but it is clear from Mandelbrojt's remarks on function spaces and Hadamard's remarks about the functional continuum that without complete abstraction on the part of some mathematicians we should lack some of the most expressive parts of the mathematical language used by scientists.

'Euclid's geometry was supposed to deal with real objects, whether in the physical world or in some ideal world. The definitions which preface several books in the *Elements* are supposed to communicate what object the author is talking about even though, like the famous definition of the point and the line, they may not be required in the sequel. The fundamental importance of the advent of non-Euclidean geometry is that by contradicting the axiom of parallels it denied the uniqueness of geometrical concepts and hence, their reality. By the end of the nineteenth century, the interpretation of the basic concepts of geometry had become irrelevant. This was the more important since geometry had been regarded for a long time as the ultimate foundation of all mathematics. However, it is likely that the independent development of the foundations of the number system which was sparked by the intricacies of analysis would have deprived geometry of its predominant position anyhow.'

Although it confirms my views on Euclidean geometry, it does not seem to recognize the geometrical origin of the theory of irrational numbers.

I also noticed that A. Aaboe in *Episodes from the Early History of Mathematics,* writes 'Even the oft repeated statement that the Egyptians knew the 3, 4, 5 right angle has no basis in available texts, but was invented about 80 years ago.'

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