BIG TOPICS FROM LINEAR ALGEBRA

Vector Space A **real vector space** is a collection of vectors V such that for any $v, w \in V$ and any $a, b \in \mathbb{R}$ we have that $av + bw \in V$. Moreover, we want the existence of a zero vector and we want to have other nice algebraic properties such as associativity and distributivity. We can also have vector spaces over any other **field** such as the rational numbers or complex numbers; however, most linear algebra courses focus on real vector spaces and the most common example of such a vector space is $\mathbb{R}^n$. However, another useful vector space to keep in mind is the set of all polynomials with real coefficients.

Subspace A **subspace** of a vector space V is a set of vectors $W \subseteq V$ which forms a vector space in its own right. As a simple example, $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$. This can be realized in multiple ways, the most common of which is identifying the vector $\langle a, b\rangle$ in $\mathbb{R}^2$ with the vector $\langle a, b, 0\rangle$ in $\mathbb{R}^3$.

Linear Independence We say vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m$ are **linearly independent** if none of the v_i's can be written as a linear combination of the others. Equivalently, if any equation of the form $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$ must have all $a_i = 0$. Manually checking whether or not a set of vectors is linearly independent is best done by putting the (column) vectors into a matrix and row reducing. The vectors are linearly independent only if there's a pivot position in each column. Additionally, it may help to note that if $n > m$, the vectors cannot be linearly independent.

Span Let $\vec{v}_1, \ldots \vec{v}_n \in \mathbb{R}^m$. The **span** of these vectors is the subspace $V = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n | a_i \in \mathbb{R}\}$ of $\mathbb{R}^m$. We also say here that V is the subspace of $\mathbb{R}^m$ **generated** by the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$. We denote this by $V = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$.

Dimension Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$. The **dimension** of the subspace of $\mathbb{R}^m$ generated by the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is the maximum number of linearly independent vectors in $\{\vec{v}_1, \ldots, \vec{v}_n\}$. Identifying the dimension of the subspace generated by a set of vectors is best done by putting the vectors into a matrix and row reducing. The number of pivot positions is the number of linearly independent vectors. Alternately, it may be helpful to note that the dimension of the set above can be no more than m (i.e. the dimension of the larger space in question). Additionally, if all vectors have a zero in the same position, then that will reduce the possible dimension of the subspace by 1.

Basis If V is vector space of dimension n, we say a set $A = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a **basis** if the subspace of V generated by A is V, i.e. if $\text{Span}\, A = V$. Note here that the number of vectors in the generating set is the same as the dimension. If we have a set $B \subseteq V$ with $|B| > n$, then we know B is not a linearly independent set, and so cannot be a basis. Similarly, if we have a set $C \subseteq V$ with $|C| < n$, then we know that $\text{Span}\, C \subsetneq V$, and so C cannot be a basis.

Determinants Let A be an $n \times n$ matrix with entries a_{ij}. In order to find the determinant of A, we can expand along any row or column. To expand along the i^{th} row, we have

$$\det A = \sum_{j=1}^{n} \sigma_{ij} \det A_{ij}$$

where $\sigma_{ij} = (-1)^{i+j}$ and A_{ij} is the $(n-1) \times (n-1)$ matrix formed from A by by removing the i^{th} row and j^{th} column. Determinants play nicely with row and column operations. Swapping rows/columns changes the sign of the determinant. Scaling a row/column by a scales the determinant by a as well. Adding a multiple of a row/column to another does not change the determinant. Also, it may be helpful to note $\det(AB) = \det(A)\det(B)$.

Null Space Let A be a matrix. The **null space** of A is the set of all solutions to $A\vec{x} = \vec{0}$.

Column Space Let A be a matrix. The **column space** of A is the span of the columns of A.

Rank The **rank** of a matrix is the number of linearly independent rows (or columns) of the matrix. That is, the rank is the dimension of the column space. We can similarly define a row space of a matrix and its dimension will be the same. This is equivalent to the number of pivot positions, and so row reducing the matrix to find pivot positions is an easy way to compute rank.

Rank-Nullity If A is a matirx with n columns, then $\operatorname{rank} A + \dim \operatorname{Nul} A = n$.

Eigenvalues and Eigenvectors Consider an $n \times n$ matrix A. If $\lambda \in \mathbb{C}$ and $\vec{v} \in \mathbb{R}^n$ with $A\vec{v} = \lambda\vec{v}$, then we say λ is an **eigenvalue** of A and $\vec{v}$ is an **eigenvector** of A. Note that $A\vec{v} = \lambda\vec{v}$ is equivalent to $(A - \lambda I)\vec{v} = \vec{0}$ where I is the $n \times n$ identity matrix. In order for this to have nonzero solutions, $\vec{v}$, we must have that $\det(A - \lambda I) \neq 0$ and we find the eigenvalues by finding the zeros of the polynomial $\det(A - xI)$. Note that as roots of polynomials can be real or complex, we may have complex eigenvalues. Note also that any complex eigenvalues must come as complex conjugate pairs.

The Invertible Matrix Theorem Let A be an $n \times n$ matrix. There's about a thousand conditions equivalent to the condition that A is an invertible matrix. In fact, Lay's text letters the conditions, starts early, continues to add to the list, and gets up to at least condition x (see page 423 of the 5th edition). Here's some of the most important ones:

- A is invertible
- $\det A \neq 0$
- The columns of A form a basis for $\mathbb{R}^n$.
- The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

Note that this fourth condition could equivalently be phrased as saying that the null space of A is just the zero vector.

Inverse of a 2×2 I always forget the details of this formula, so here it is in case you're in that boat too!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$