Math Club **February 28, 2018**

This first page consists of warm-up problems. Find some that look interesting and solve them.

1. Assuming convergence, find $x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \dots}}}$

2. What real numbers x satisfy the inequality $x - \frac{3}{x} > 2$?

3 (KöMaL)**.** Strings are made out of 6 letters A and 7 letters B. How many such strings can be made that are palindromes, i.e. that read the same either from the beginning or from the end?

4 (KöMaL)**.** Use the digits 9, 8, 8, 7, 7, 7 and one more digit of your choice to write down the largest seven-digit number divisible by 36.

5 (KöMaL)**.** A large cube is built out of small white cubes, and then the faces of the large cube are painted blue. The large cube is then taken apart again. What is the size of the large cube if the number of small cubes with an even number of blue faces is the same as those with an odd number of blue faces? Remember the cubes with 0 blue faces.

6 (GRE)**.** What is the greatest possible area of a triangular region with one vertex at the center of a circle of radius 1 and the other two vertices on the circle?

7 (GRE)**.**

$$
\begin{aligned}
J &= \int_0^1 \sqrt{1 - x^4} dx \\
K &= \int_0^1 \sqrt{1 + x^4} dx \\
L &= \int_0^1 \sqrt{1 - x^8} dx \\
M &= 1
\end{aligned}
$$

Place the numbers J, K, L, and M in increasing order.

8 (GRE)**.** What is the units digit in the standard decimal expansion of 7^{25}?

9 (GRE)**.** Determine the minimal distance between any point on the sphere $(x-2)^2 + (y-1)^2 + (z-3)^2 = 1$ and any point on the sphere $(x+3)^2 + (y-2)^2 + (z-4)^2 = 4$.

The problems on this page are real Putnam problems. Links will take you to solutions.

10 (2010 A1)**.** Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

11 (2010 A2)**.** Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

12 (2012 B3)**.** A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

13 (2013 A2)**.** Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.