Algorithmically random closed sets and probability

Logan Axon

January, 2009

University of Notre Dame

Outline.

1. Martin-Löf randomness.
2. Previous approaches to random closed sets.
3. The space of closed sets $\mathcal{F}(\mathbb{E})$.
4. Martin-Löf randomness for $\mathcal{F}(\mathbb{E})$.
5. Examples.

Cantor space.

Computably enumerate $2^{<\mathbb{N}} = \{\sigma_0, \sigma_1, \dots\}$.

Definition

- Let $\tau \in 2^{<\mathbb{N}}$.

$$[\tau] := \left\{ f \in 2^{\mathbb{N}} : \tau \preceq f \right\}.$$

- The sets $[\tau]$, $\tau \in 2^{<\mathbb{N}}$, form a basis for $2^{\mathbb{N}}$.
- $U \subseteq 2^{\mathbb{N}}$ is Σ^1_0 if there is a Σ^1_0 $f \in 2^{\mathbb{N}}$ such that

$$U = \bigcup_{f(i)=1} [\sigma_i].$$

- $2^{\mathbb{N}}$ has a probability measure, m, defined by

$$m([\tau]) = 2^{-|\tau|}.$$

Martin-Löf randomness.

The idea: $f \in 2^{\mathbb{N}}$ is random if it passes all "reasonable" statistical tests.

Definition

- A uniformly Σ^1_0 sequence of subsets of $2^{\mathbb{N}}$, $\{U_i\}_{i\in\mathbb{N}}$, is a **Martin-Löf test** if
$$m(U_i) \leq 2^{-i}.$$
- $f \in 2^{\mathbb{N}}$ passes Martin-Löf test $\{U_i\}_{i\in\mathbb{N}}$ if $f \notin \bigcap_{i\in\mathbb{N}} U_i$.
- $f \in 2^{\mathbb{N}}$ is **Martin-Löf random** if f passes every Martin-Löf test.

Other approaches to random closed sets (of $2^{\mathbb{N}}$).

Each non-empty closed set $E \subseteq 2^{\mathbb{N}}$ corresponds to a unique binary tree without dead ends. Barmpalias, Brodhead, Cenzer, Dashti, and Weber code such trees as ternary reals. This real is the **canonical code** for a (non-empty) closed set. They defined a closed set to be random if its canonical code is Martin-Löf random. This definition gives interesting results.

McLinden and Mauldin noticed that the previous approach was, in effect, looking at randomly constructed trees. This is one way of building random fractals. They applied theorems about random fractals to analyze the Hausdorff dimension and measure of these random closed sets.

The space $\mathcal{F}(\mathbb{E})$.

Let $\mathbb{E}$ be a locally compact, Hausdorff, second countable topological space (LCHS), e. g. $\mathbb{R}^n$, $2^{\mathbb{N}}$, $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}\dots$.

Definition

- $\mathcal{F}(\mathbb{E})$ is the space of closed (sub)-sets of $\mathbb{E}$.
- Let $A \subseteq \mathbb{E}$. Then

  $$\mathcal{F}_A(\mathbb{E}) := \{F \in \mathcal{F}(\mathbb{E}) : F \cap A \neq \emptyset\}$$

  and

  $$\mathcal{F}^A(\mathbb{E}) := \{F \in \mathcal{F}(\mathbb{E}) : F \cap A = \emptyset\}$$

  .
- If $\{E_i : i \in \mathbb{N}\}$ is a (countable) basis for $\mathbb{E}$ such that $\overline{E_i}$ is compact for each i, then a sub-basis for $\mathcal{F}(\mathbb{E})$ consists of the sets $\mathcal{F}_{E_i}$ and $\mathcal{F}^{\overline{E_i}}$ for $i \in \mathbb{N}$.

More about $\mathcal{F}(\mathbb{E})$.

The resultant topology is called the Fell topology or the hit-or-miss topology.

Proposition

1. *If* $\{A_i : i \in I\}$ *is a collection of subsets of* $\mathbb{E}$ *then:*
   - $\bigcup_{i \in I} \mathcal{F}_{A_i} = \mathcal{F}_{\bigcup_{i \in I} A_i}$;
   - $\bigcap_{i \in I} \mathcal{F}^{A_i} = \mathcal{F}^{\bigcup_{i \in I} A_i}$;
   - $\bigcap_{i \in I} \mathcal{F}_{A_i} \supseteq \mathcal{F}_{\bigcap_{i \in I} A_i}$;
   - $\bigcup_{i \in I} \mathcal{F}^{A_i} \subseteq \mathcal{F}^{\bigcap_{i \in I} A_i}$.
2. $\mathcal{F}$ *is compact, Hausdorff, and second countable.*
3. $\mathcal{F}$ *is the one-point compactification of* $\mathcal{F} \setminus \{\emptyset\}$.
4. *Let* $\{E_i : i \in \mathbb{N}\}$ *be a basis for* $\mathbb{E}$. *The Borel* σ*-algebra on* $\mathcal{F}(\mathbb{E})$ *is generated by the sets* $\mathcal{F}_{E_i}$, $i \in \mathbb{N}$.

Measures on $\mathcal{F}(\mathbb{E})$.

There is no canonical measure on $\mathcal{F}(\mathbb{E})$. However, there are two ways to put a measure on $\mathcal{F}(\mathbb{E})$.

1. Let $(\Omega, \mathfrak{A}, P)$ be a probability space. Let $\phi : \Omega \to \mathcal{F}(\mathbb{E})$ be a measurable map. Such maps are called **random closed sets** or **RACS**. Then ϕ induces a measure, μ_ϕ, on $\mathcal{F}(\mathbb{E})$:

$$\mu_\phi(\mathcal{H}) = P\left(\phi^{-1}(\mathcal{H})\right).$$

2. Choquet capacities. Let $\mathcal{K}(\mathbb{E})$ be the collection of compact subsets of $\mathbb{E}$. Assign measures to $\mathcal{F}_K$ for $K \in \mathcal{K}$ using a functional $T : \mathcal{K} \to [0, 1]$:

$$\mu_T(\mathcal{F}_K) = T(K).$$

This extends to a measure on the Borel σ-algebra of $\mathcal{F}(\mathbb{E})$ when T meets certain conditions.

The Choquet capacity theorem.

Theorem

Let $T : \mathcal{K} \to [0,1]$. Then T gives rise to a (necessarily unique) probability measure μ_T on $\mathcal{F}(\mathbb{E})$ such that $\mu_T(\mathcal{F}_K) = T(K)$ for $K \in \mathcal{K}$ if and only if T satisfies the following conditions:

1. $T(\emptyset) = 0$;
2. *T is* **upper semi-continuous** *on $\mathcal{K}$ (that is, if $K, K_1, K_2, K_3, \dots \in \mathcal{K}$ with $K_n \subseteq K_{n+1}$ for all n and $\bigcap_n K_n = K$, then $\lim_{n\to\infty} T(K_n) = T(K)$);*
3. *T is* **completely alternating** *on $\mathcal{K}$ (that is, $\forall K, K_1, \dots, K_n \in \mathcal{K}$, $\Delta_{K_n} \dots \Delta_{K_1} T(K) \leq 0$).*

Note: In the case of $\mathbb{E} = 2^{\mathbb{N}}$ it is sufficient to define $T : 2^{<\mathbb{N}} \to [0,1]$. Then, if T is computable, the measure μ_T will also be computable (in the sense that computing the measure of a subset of $\mathcal{F}(2^{\mathbb{N}})$ is exactly as complicated as the subset).

Martin-Löf tests for $\mathcal{F}(\mathbb{E})$.

Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$ be a computable enumeration of the basic open sets of $\mathcal{F}(\mathbb{E})$ and let μ be a probability measure on $\mathcal{F}(\mathbb{E})$.

Definition

- $\mathcal{U} \subseteq \mathcal{F}(\mathbb{E})$ is Σ^1_0 if there is a Σ^1_0 $f \in 2^{\mathbb{N}}$ such that

$$\mathcal{U} = \bigcup_{f(i)=1} \mathcal{B}_i.$$

- A uniformly Σ^1_0 sequence of subsets of $\mathcal{F}(\mathbb{E})$, $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$, is a **$\mu$-Martin-Löf test** if

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

- $F \in \mathcal{F}(\mathbb{E})$ is **μ-Martin-Löf random** if F passes every μ-Martin-Löf test.

Example 1: the canonical decoding.

Consider the canonical coding of Barmpalias, et al. Look at the **canonical decoding**:

$$d : 3^{\mathbb{N}} \to \mathcal{F}\left(2^{\mathbb{N}}\right).$$

Theorem

The spaces $3^{\mathbb{N}}$ and $\mathcal{F}(2^{\mathbb{N}}) \setminus \{\emptyset\}$ are homeomorphic via d. Moreover, $\{U_i\}_{i\in\mathbb{N}}$ is a uniformly Σ^1_0 sequence of subsets of $3^{\mathbb{N}}$ if and only if $\{d(U_i)\}_{i\in\mathbb{N}}$ is a uniformly Σ^1_0 sequence of subsets of $\mathcal{F}(2^{\mathbb{N}})$

Corollary

$F \in \mathcal{F}(2^{\mathbb{N}})$ is random in the sense of Barmpalias, et al. if and only if F is μ_d-Martin-Löf random.

Example 2: a generalized Poisson process.

Let $\mathbb{E} = \mathbb{R}$ and let m be the Lebesgue measure on $\mathbb{R}$. Let $T : \mathcal{K}(\mathbb{R}) \to [0, 1]$ be given by

$$T(K) = 1 - 2^{-m(K)}.$$

Proposition

T satisfies the Choquet capacity theorem and thus induces a measure, μ_T, on $\mathcal{F}(\mathbb{R})$.

What properties do μ_T-Martin-Löf random closed sets have?

Interjection: Robbins' theorem.

Theorem

If m is a σ-finite measure on $\mathbb{E}$ and $\phi : \Omega \to \mathcal{F}(\mathbb{E})$ is measurable (i. e. ϕ is a RACS), then $m \circ \phi : \Omega \to \mathbb{R}$ is a random variable and

$$\mathbf{E}(m \circ \phi) = \int_{\mathbb{E}} P(x \in \phi) dm(x).$$

Proof sketch.

Use the Fubini-Tonelli theorem to change the order of integration. □

Example 2, continued.

Theorem (A.)

If $F \in \mathcal{F}(\mathbb{R})$ is μ_T-Martin-Löf random then $m(F) = 0$.

By Robbins' theorem

$$\mathbf{E}(m(E)) = \int_{\mathbb{R}} \mu_T\left(\mathcal{F}_{\{x\}}\right) dx = 0.$$

Hence μ_T-almost every $E \in \mathcal{F}(\mathbb{R})$ has $m(E) = 0$.
Suppose $F \in \mathcal{F}(\mathbb{R})$ is such that $m(F \cap [0,1]) \geq 2^{-k}$ for some $k \in \mathbb{N}$. We know that $F \in \{E \in \mathcal{F}(\mathbb{R}) : m(E) > 0\}$ and that $\mu_T(\{E \in \mathcal{F}(\mathbb{R}) : m(E) > 0\}) = 0$. Unfortunately, this is a very complicated set. The solution is to approximate it.

Proof.

Define I_σ for $\sigma \in 2^{<\mathbb{N}}$ by induction. Let $I_\lambda = (0,1)$. Given I_σ define $I_{\sigma^\frown 0}$ and $I_{\sigma^\frown 1}$ to be the two open half intervals of I_σ.

Define, for $E \in \mathcal{F}(\mathbb{R})$,

$$S_n(E) := |\{\sigma \in 2^n : I_\sigma \cap E \neq \emptyset\}|.$$

Because $m(F \cap [0,1]) \geq 2^{-k}$ we must have $S_{n+k}(F) \geq 2^n$ for every $n \in \mathbb{N}$.

Define

$$\mathcal{U}_n := \{E \in \mathcal{F}(\mathbb{R}) : S_{n+k}(E) \geq 2^n\}.$$

Then $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a uniformly Σ^1_0 sequence, $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \ldots$, and $\mu_T(\mathcal{U}_n)$ is uniformly computable.

Proof, continued.

Furthermore,

$$\bigcap_n \mathcal{U}_n \subseteq \{E \in \mathcal{F}(\mathbb{R}) : m(E) > 0\}.$$

Hence

$$\mu_T\left(\bigcap_n \mathcal{U}_n\right) \leq \mu_T\left(\{E \in \mathcal{F}(\mathbb{R}) : m(E) > 0\}\right) = 0.$$

We can easily get a μ_T-Martin-Löf test by taking a subsequence of $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ (because $\mu_T(\mathcal{U}_n)$ is uniformly computable). But $F \in \bigcap_n \mathcal{U}_n$ and so F must not be μ_T-Martin-Löf random.

Further comments on example 2.

Proposition (A.)

If $F \in \mathcal{F}(\mathbb{R})$ is μ_T-Martin-Löf random and $x \in F$ then x is Martin-Löf random.

Question

Does the converse hold?

A positive answer would show that $x \in \mathbb{R}$ is Martin-Löf random if and only if there is some μ_T-Martin-L"of random closed set F such that $x \in F$.

Question

What is the expected fractal dimension of closed sets under μ_T?

More RACS.

$\phi : 2^{\mathbb{N}} \to \mathcal{F}\left(2^{\mathbb{N}}\right)$ given by

$$\phi(f) = \left\{ g \in 2^{\mathbb{N}} : g \leq_{lex} f \right\}.$$

In this case $F \in \mathcal{F}(2^{\mathbb{N}})$ is μ_ϕ-Martin-Löf random if and only if there is some Martin-Löf random $f \in 2^{\mathbb{N}}$ such that $F = \phi(f)$.

$\phi : C[0,1] \to [0,1]$ is given by

$$\phi(f) = \{x \in [0,1] : f(x) = 0\}.$$

This is a RACS when $C[0,1]$ is equipped with the Wiener measure. Note: Fouché developed, Kjos-Hanssen and Nerode studied algorithmic randomness for $C[0,1]$. The connections, if any, between algorithmically random Brownian motions and μ_ϕ-Martin-Löf random closed sets have not been examined.

Yet more.

$\phi : 2^{\mathbb{N}} \to \mathcal{F}\left(2^{\mathbb{N}}\right)$ given by

$$\phi(f) = \bigcap_{f(i)=1} [\sigma_i]^{\complement}.$$

In this case almost every $f \in 2^{\mathbb{N}}$ maps to $\emptyset$. Hence $\emptyset$ is the only μ_ϕ-Martin-Löf random closed set. In this case there are non-random $f \in 2^{\mathbb{N}}$ such that $\phi(f) = \emptyset$.

This example is another construction of a random fractal. It becomes more interesting when $2^{\mathbb{N}}$ is given the $(\frac{2}{3}, \frac{1}{3})$ Poisson distribution. In this case the probability of the next bit in a sequence being 0 is $\frac{2}{3}$ and the probability of a 1 is $\frac{1}{3}$. In this situation $P(\phi(f) = \emptyset) = \frac{1}{4}$ and with probability $\frac{3}{4}$ the Hausdorff and box dimension of $\phi(f)$ is $2 - \log_2(3)$.

References

Barmpalias, Brodhead, Cenzer, Dashti, and Weber. 2007. Algorithmic randomness of closed sets. *J Logic Comput* 17(6):1041-1062.

Matheron, G. 1975. *Random sets and integral geometry*. Wiley.

Molchanov, Ilya. 2005. *Theory of random sets*. Springer-Verlag.