The ϵ-δ game

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April 20, 2018

A definition of the limit

From Stewart's **Essential Calculus**:

Definition

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x \to a} f(x) = L$, if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon$$

Similar definitions

Definitions of continuity are equally challenging:

Definition

A function f is continuous at $a \in \text{dom}(f)$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \epsilon$.

Definition

A function f is continuous at $a \in \text{dom}(f)$ if for every sequence (x_n) in $\text{dom}(f)$ converging to a, $\lim_{n\to\infty} f(x_n) = f(a)$.

All have quantifiers $\forall$ $\exists$

Related exercises/examples

From Stewart's **Essential Calculus**:

Example

Find the number δ such that if $|x-1| < \delta$, then $|x^2 - 1| < \frac{1}{2}$

Example

Use the ϵ-δ definition of a limit to prove that $\lim_{x \to 3}(2x - 1) = 5$.

Related exercises/examples

From Stewart's **Essential Calculus**:

Example

Find the number δ such that if $|x-1|<\delta$, then $|x^2-1|<\frac{1}{2}$

Example

Use the ϵ-δ definition of a limit to prove that $\lim_{x\to 3}(2x-1)=5$.

These problems build skills much faster than they build understanding.

The ϵ-δ game

After introducing the idea of a limit, play the ϵ-δ game.

Rules

Play with a function f and numbers a and L (all determined in advance).
To play:

1. Player 1 picks a distance: ϵ
2. Player 2 picks a distance: δ
3. Player 1 selects a number b less than distance δ from a, but not equal to a (in symbols: $0 < |a - b| < \delta$)

Player 2 wins if the distance from L to $f(b)$ is less than ϵ (in symbols: $|L - f(b)| < \epsilon$). Otherwise Player 1 wins.

Who wins?

Example

Play the ϵ-δ game with $f(x) = 2x - 1$, $a = 3$, and $L = 5$.

a) Who wins?

b) Can either player guarantee that he/she wins?

c) If so, what's the winning strategy?

Who wins?

Example

Play the ϵ-δ game with $f(x) = 2x - 1$, $a = 3$, and $L = 5$.

a) Who wins?

b) Can either player guarantee that he/she wins?

c) If so, what's the winning strategy?

Make a table of different choices of ϵ and δ and look for a pattern. Draw a graph.
Player 2 wins by selecting $\delta \leq \frac{\epsilon}{2}$.

Another round of the game

Have the students play with a couple of functions, including non-limits.

Example

Play the ϵ-δ game with $f(x) = \frac{x}{5} + 1$, $a = -5$, and $L = 1$.

a) Who wins?

b) Can either player guarantee that he/she wins?

c) If so, what's the winning strategy?

Another round of the game

Have the students play with a couple of functions, including non-limits.

Example

Play the ϵ-δ game with $f(x) = \frac{x}{5} + 1$, $a = -5$, and $L = 1$.

a) Who wins?

b) Can either player guarantee that he/she wins?

c) If so, what's the winning strategy?

Player 1 can win by selecting $\epsilon = 1$ and any allowed number less than -5.

Connection with limits

After playing a few rounds, students should be able to apply intuitive understanding of limits to conclude:

Observation

If Player 2 has a winning strategy for the ϵ-δ game with f, a, and L, then L is the limit of $f(x)$ as x approaches a:

$$\lim_{x \to a} f(x) = L$$

Connection with limits

What if Player 1 has a winning strategy?

Observation

If Player 1 has a winning strategy for the ϵ-δ game with f, a, and L, then L is **not** the limit of $f(x)$ as x approaches a:

$$\lim_{x \to a} f(x) \neq L$$

Connection with limits

What if Player 1 has a winning strategy?

Observation

If Player 1 has a winning strategy for the ϵ-δ game with f, a, and L, then L is **not** the limit of $f(x)$ as x approaches a:

$$\lim_{x \to a} f(x) \neq L$$

One of the players always has a winning strategy. Is it obvious (at a Calc I level) why?

Did it work?

The goal is to provide a bridge from the intuitive idea of a limit to the formal mathematical definition.

- The definition shouldn't surprise students after the ϵ-δ game.
- The ϵ-δ game is no less complicated than the ϵ-δ definition, but it gives students a way to interact with the definition.
- Particularly helpful for students less accustomed to mathematical definitions.
- It's not a very fun game (may still exceed expectations for a calculus class).
- Maybe it needs a better name.

Gambling version

Rules

Play with a function f and numbers a and L (all determined in advance).

To play:

1. Player 1 picks a positive integer: n
2. Player 2 picks a positive integer: k
3. Player 1 selects a number b less than distance $\frac{1}{k}$ from a, but not equal to a (in symbols: $0 < |a - b| < \frac{1}{k}$)

Player 2 wins if the distance from L to $f(b)$ is less than $\frac{1}{n}$ (in symbols: $|L - f(b)| < \frac{1}{n}$). In this case Player 2 wins $\frac{n}{k}$ dollars. Otherwise Player 1 wins $\frac{k}{n}$ dollars.

Encourages making ϵ and δ as large as possible?

The advanced ϵ-δ game

Rules

Play with a function f.
To play:

1. Player 1 picks a number a and a distance ϵ
2. Player 2 picks a distance δ
3. Player 1 selects a number b less than distance δ from a but not equal to a.

Player 2 wins if the distance from $f(a)$ to $f(b)$ is less than ϵ (in symbols: $|f(a) - f(b)| < \epsilon$). Otherwise Player 1 wins.

If Player 2 has a winning strategy, then f is continuous. If Player 1 has a winning strategy, then f is not continuous.

Graphical versions

Graphical versions now exist (thanks to whom?):

- `http://www.desmos.com/calculator/zn6sn1ocjm`
- `http://www.geogebra.org/m/tCnmrWg2`

A Mathematica notebook:
`http://library.wolfram.com/infocenter/Demos/4734/`