Random fractals

Logan Axon

Notre Dame

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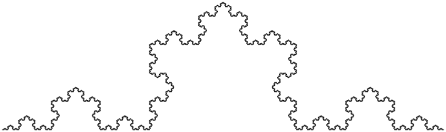
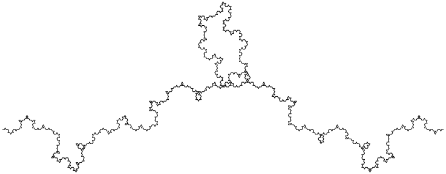




Introduction

What is a fractal?

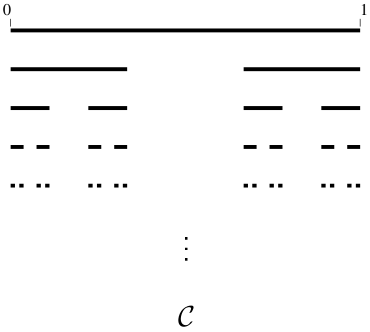
There is no universally agreed upon definition of “fractal”. Mandelbrot took the from the Latin “fractus” meaning broken. He used it to describe sets that were too badly behaved for traditional geometry.

Outline

1. Bad behavior by example
2. Fractal dimension
3. Random fractals

The Cantor middle thirds set

Start with the unit interval $[0,1]$. Remove the middle third to get $\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$. Continue removing middle thirds. The Cantor set $\mathcal{C}$ is what is left.



The Cantor middle thirds set

In symbols . . .

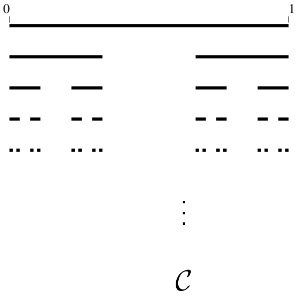
$$E_0 = [0,1]$$

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

$$\vdots$$

$$\mathcal{C} = \bigcap_{i=0}^{\infty} E_i$$



Exercise

What is the total length of all the intervals removed in the construction of $\mathcal{C}$?

Properties of the Cantor set $\mathcal{C}$

1. It is self-similar: The part of $\mathcal{C}$ in $\left[0, \frac{1}{3}\right]$ is a scaled copy of all of $\mathcal{C}$.
2. It has detail at all scales.
3. It is simple to define but hard to describe geometrically.
4. Its "local geometry" is strange: every open interval containing one point of $\mathcal{C}$ must contain other points of $\mathcal{C}$ as well as points not in $\mathcal{C}$.

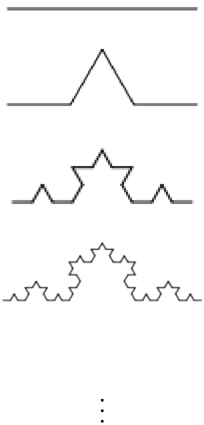
All fractals should have these properties.

The von Koch curve

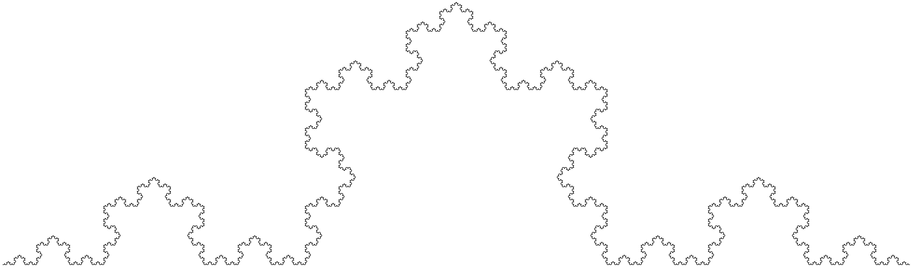
0) Begin with a unit line.
1) Replace the middle third with the top of an equilateral triangle.
2) Replace the middle third of each of the 4 line segments.

⋮

The result is a continuous, nowhere differentiable curve with infinite length called the von Koch curve.



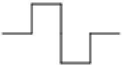
The von Koch curve

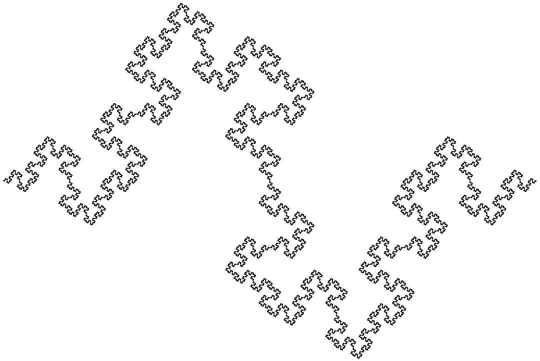


Generalizations of the von Koch curve

The construction of the von Koch curve can be modified in a number of ways.

The quadratic von Koch curve or Minkowski sausage:





The dragon curve

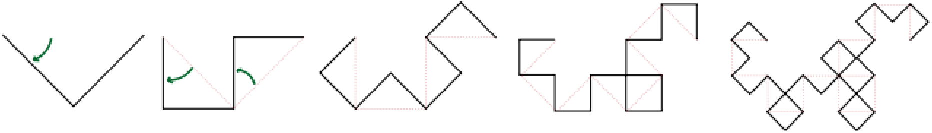
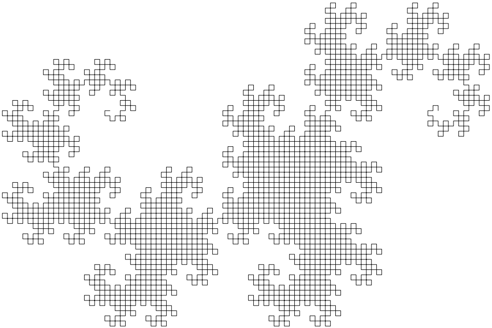


Image source: Wikipedia



The Mandelbrot set

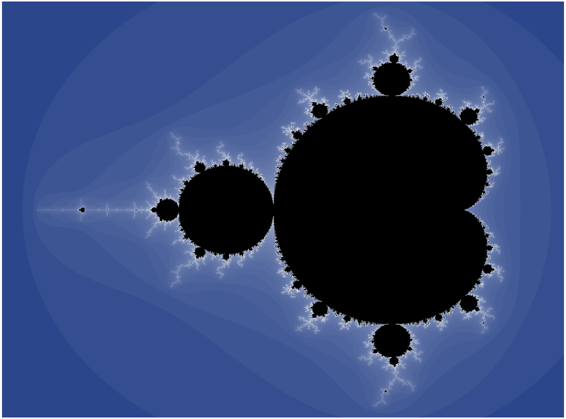


Image source: http://www.math.utah.edu/~pa/math/mandelbrot/mandelbrot.html

Constructed by looking at iterations of a function on $\mathbb{C}$. We won't deal with this kind of fractal.

Dimension of a fractal

One of the main tools in fractal geometry is *fractal dimension*.

Warning: There are a lot of different ways to define fractal dimension (Hausdorff dimension is the most widely used). Our definition is almost Hausdorff dimension but only works for certain kinds of fractals.

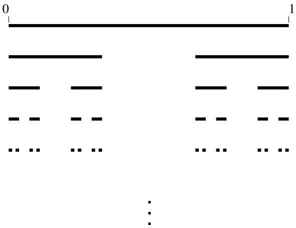
Idea

Look at covering the fractal with smaller and smaller sets. As the sets get smaller we need more of them to cover the fractal. Measure how fast the number of sets increases as the size of the sets decreases.

Covering the Cantor set

The construction of the Cantor set shows how to cover it with smaller and smaller sets.

We can cover $\mathcal{C}$ with:



0) 1 set of size 1;

1) 2 sets of size $\frac{1}{3}$;

2) 4 sets of size $\frac{1}{9}$;

$\vdots$

n) 2^n sets of size ...

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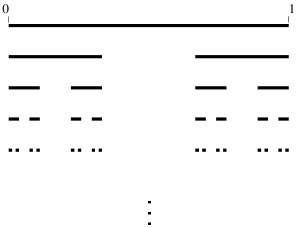

0) 1 set of size 1;
1) 2 sets of size $\frac{1}{3}$;
2) 4 sets of size $\frac{1}{9}$;

$\vdots$

n) 2^n sets of size ... 3^{-n}.

Fractal dimension of the Cantor set

We want to measure how fast 2^n grows relative to how fast 3^{-n} shrinks. Do this by looking at

$$\lim_{n\to\infty} 3^{-nx}2^n$$

for different values of x.

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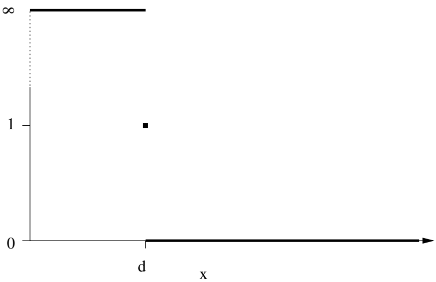
for different values of x.

$$\lim_{n\to\infty} 3^{-nx}\left(2^n\right) = \lim_{n\to\infty}\left(2\cdot 3^{-x}\right)^n = \begin{cases} \infty & \text{if } 2\cdot 3^{-x} > 1 \\ 1 & \text{if } 2\cdot 3^{-x} = 1 \\ 0 & \text{if } 2\cdot 3^{-x} < 1 \end{cases}$$

Fractal dimension of the Cantor set



$$y = \lim_{n\to\infty} 3^{-nx} 2^n$$

The fractal dimension of the Cantor set is the value d where $\lim_{n\to\infty} 3^{-nx}(2^n)$ switches from ∞ to 0. This happens when $2 \cdot 3^{-x} = 1$.

Fractal dimension of the Cantor set

$$\begin{aligned} 2 \cdot 3^{-x} = 1 &\iff 2 = 3^x \\ &\iff \log 2 = x \log 3 \\ &\iff x = \frac{\log 2}{\log 3} \end{aligned}$$

Therefore the fractal dimension of the Cantor set is $\frac{\log 2}{\log 3} \approx 0.6309$.

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We wish to do this more generally. We won't always be covering with straight line segments so we need a good definition for the size of a more general set.

Diameter

Definition

The *diameter* of a set is the largest distance between two points in the set.

To be technically correct, the diameter of a set is the supremum of the distances between two points in the set.

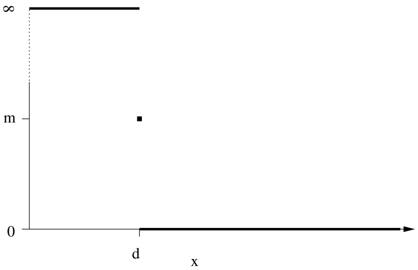
For example:

- The diameter of the line segment $[0, 1]$ is 1;
- The diameter of a triangle is the length of its longest side;
- The diameter of a rectangle is the length of its diagonal;
- The diameter of a disk is twice the radius.

Fractal dimension

Suppose N_δ is the minimum number of sets of diameter δ needed to cover the set F. We compare how fast N_δ grows relative to the how fast δ shrinks by looking at

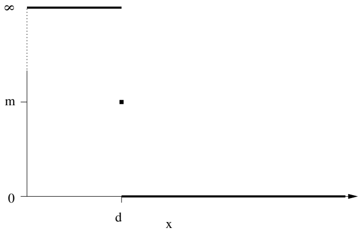
$$\lim_{\delta \to 0+} \delta^x N_\delta.$$



$$y = \lim_{\delta \to 0+} \delta^x N_\delta$$

Fractal dimension

The *fractal dimension* of F is the value d where $\lim_{\delta \to 0+} \delta^x N_\delta$ switches from ∞ to 0.



In symbols:

$$d = \inf \left\{ x : \lim_{\delta \to 0+} \delta^x N_\delta = 0 \right\}$$

The actual value $\lim_{\delta \to 0+} \delta^d N_\delta$ may be 0, ∞, or anything in between. This value is called the d-dimensional measure of the set.

Why "dimension"?

Why is this called fractal dimension?

It agrees with our ideas of what dimension should be:

- The fractal dimension of a point is 0;
- The fractal dimension of a line is 1;

$\vdots$

And the fractal dimension of the Cantor set is ≈ 0.6309. The Cantor set had properties "between" those of a point and a line. It makes sense that the fractal dimension of the Cantor set is between 0 and 1.

Fractal dimension of the von Koch curve

Generalize the technique from the calculation for $\mathcal{C}$. Again the construction tells us how to get covers of the curve.

We can cover the curve with:

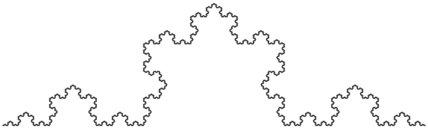
(0) 1 full-sized copy of the curve;

(1) 4 copies of the curve scaled by $\frac{1}{3}$;

(2) 16 copies of the curve scaled by $\frac{1}{9}$;

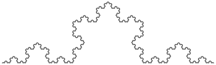
$\vdots$

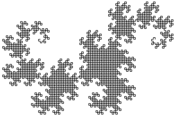
(n) 4^n copies of the curve scaled by 3^{-n}.



The diameter of the curve itself is 1. So ...

Bestiary





- The dimension of a point is 0.
- The dimension of the Cantor middle thirds set is $\frac{\log 2}{\log 3} \approx 0.6309$.
- The dimension of a line is 1.
- The dimension of the von Koch curve is $\frac{\log 4}{\log 3} \approx 1.262$.
- The dimension of the Minkowski sausage is 1.5.
- The dimension of the dragon curve is 2.
- Many more on Wikipedia.

Romanesco broccoli (image source: Wikipedia)

The coast of Britain

In one of his early papers on fractals Mandelbrot addressed the question, "How long is the coast of Britain?" This is a hard question in part because you get different answers depending on how much detail you include.

Image source: Wikipedia

The coast of Britain

How long is the coast of Britain?

Mandelbrot's answer was that the coast of Britain is a fractal and so asking for the length is the wrong question. Instead a better measurement is the fractal dimension, which Richardson estimated to be 1.25.

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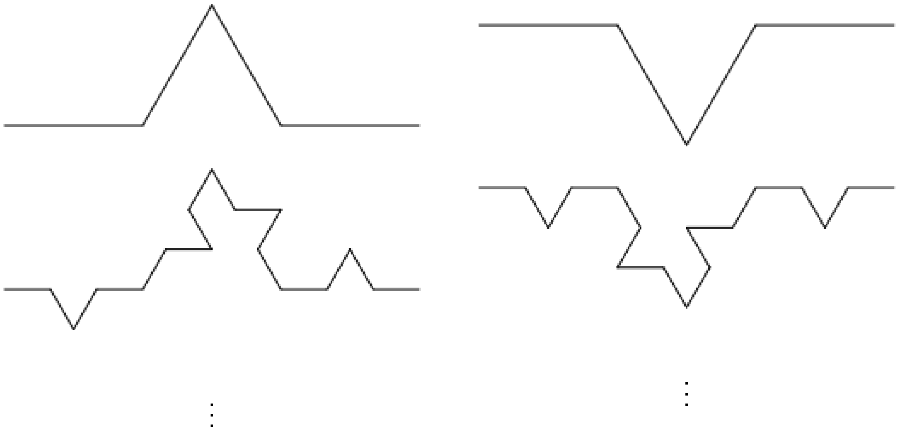
The coast of Britain is not strictly self-similar: it varies randomly. It is "stochastically self-similar".

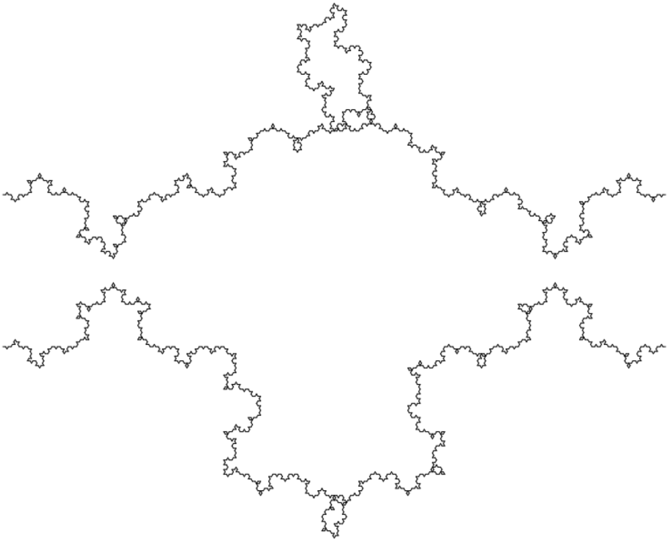
We can get the same kind of stochastic self-similarity by adding an element of randomness to our constructions.

A random von Koch curve

Add randomness to the construction of the von Koch curve by flipping a coin to decide if each triangle we add should point up or down.

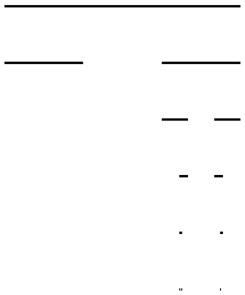


Two random von Koch curves



A random Cantor set

As we construct the Cantor set, flip a biased coin to decide whether or not to include each interval (after starting with $[0, 1]$). Include the interval if the flip gives heads. For example, if the probability of heads is $\frac{2}{3}$:



H, H

T, T, H, H

T, H, H, T

H, T, T, H

H, H, H, T

$$\vdots$$

A random Cantor set

Let $\mathcal{R}$ be the random Cantor set obtained from this construction.

Question

It's possible that $\mathcal{R} = \emptyset$. How likely is this?

We can calculate the exact probability that $\mathcal{R} = \emptyset$, $\mathbf{P}(\mathcal{R} = \emptyset)$, using the self-similarity of the construction. The result depends on the bias of the coin. Let p be the probability that the coin is heads.

Calculating $\mathbf{P}(\mathcal{R} = \emptyset)$

There are 3 ways that $\mathcal{R}$ can be empty:

1. Neither $[0, \frac{1}{3}]$ nor $[\frac{2}{3}, 1]$ is included.
2. Exactly one of $[0, \frac{1}{3}]$ or $[\frac{2}{3}, 1]$ is included and $\mathcal{R}$ is eventually empty below that interval.
3. Both of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ is included and $\mathcal{R}$ is eventually empty below both intervals.

The probability that $\mathcal{R}$ is eventually empty below $[0, \frac{1}{3}]$ given that $[0, \frac{1}{3}]$ was included in the construction is exactly $\mathbf{P}(\mathcal{R} = \emptyset)$ by self-similarity. The same holds for $[\frac{2}{3}, 1]$.

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Let $q = \mathbf{P}(\mathcal{R} = \emptyset)$. Recall that p is the probability of heads. These events happen with probabilities:

1. $(1-p)^2$;
2. $2p(1-p)q$;
3. p^2q^2.

Calculating $\mathbf{P}(\mathcal{R} = \emptyset)$

Therefore $q = \mathbf{P}(\mathcal{R} = \emptyset)$ is a solution to the equation

$$x = p^2x^2 + 2p(1-p)x + (1-p)^2.$$

The solutions to this equation are 1 and $\left(\frac{1-p}{p}\right)^2$. Which solution is $\mathbf{P}(\mathcal{R} = \emptyset)$?

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If $p \leq \frac{1}{2}$, then $\left(\frac{1-p}{p}\right)^2 \geq 1$. Hence

$$\mathbf{P}(\mathcal{R} = \emptyset) = 1.$$

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If $p \leq \frac{1}{2}$, then $\left(\frac{1-p}{p}\right)^2 \geq 1$. Hence

$$\mathbf{P}(\mathcal{R} = \emptyset) = 1.$$

On the other hand, $\mathbf{P}(\mathcal{R} = \emptyset) \to 0$ as $p \to 1$. Hence for $p > \frac{1}{2}$

$$\mathbf{P}(\mathcal{R} = \emptyset) = \left(\frac{1-p}{p}\right)^2 < 1.$$

The dimension of a random Cantor set

If the random Cantor set is $\emptyset$, then there's no point in asking about its dimension. A theorem (next slide) tells us that if the random Cantor set is not $\emptyset$, then with probability 1 its dimension is the solution d of

$$3^{-d}\left(2p(1-p)\right) + 3^{-d}(2)p^2 = 1.$$

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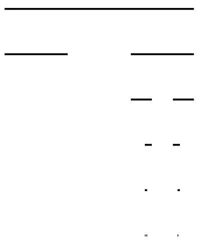
For $p = \frac{2}{3}$, $d \approx 0.2619$.
For $p = 1$, $d = \frac{\log 2}{\log 3} \approx 0.6309$, as we calculated before.

A random Cantor set

For our random Cantor set we think of the construction as replacing each component with 2 stochastically similar components. Each replacement piece was either a $\frac{1}{3}$-scaled copy or a 0-scaled copy (i.e. $\emptyset$).



The theorem

Consider a random Cantor set in which each component of the construction is replaced by 2 copies with random scaling s_1 and s_2. Let N be the (random) number of these copies that is non-empty.

Theorem (Falconer, Graf, Mauldin, Williams)

This random Cantor set has probability q of being empty, where $q = x$ is the smallest non-negative solution to the equation

$$\mathbf{P}(N = 0) + \mathbf{P}(N = 1)x + \mathbf{P}(N = 2)x^2 = x.$$

The theorem continued

N is the (random) number of the copies that is non-empty and s_1, s_2 are the (random) scaling of the copies.

Theorem (Falconer, Graf, Mauldin, Williams)

This random Cantor set has probability q of being empty, where $q = x$ is the smallest non-negative solution to the equation

$$\mathbf{P}(N = 0) + \mathbf{P}(N = 1)x + \mathbf{P}(N = 2)x^2 = x.$$

If it is not $\emptyset$, then with probability 1 the random Cantor set has dimension d, where $d = y$ is the solution of

$$\mathbf{E}\left(s_1^y + s_2^y\right) = 1.$$

References

Kenneth Falconer. *Fractal Geometry*. Wiley, 2003.

Benoit B. Mandelbrot. How long is the coast of Britain? Statistical self-similarity and fractional dimension. *Science:* 156, 1967.

Benoit B. Mandelbrot. `http://www.math.yale.edu/~bbm3/index.html`.