Measure and Domination

Logan Axon

February 9, 2006

Notation

$X \in 2^\omega$ is a real.
$\mathcal{A} \subseteq 2^\omega$ is a class.
$f \in \omega^\omega$ is a function.
$\sigma \in 2^{<\omega}$ is a string.

Measure

Definition

Cantor Space is the set 2^ω with the topology given by the basic open sets $\{[\sigma] : \sigma \in 2^{<\omega}\}$ where $[\sigma] := \{X \in 2^\omega : \sigma \preccurlyeq X\}$.

1. Cantor Space is compact.
2. Each basic open set is also closed.
3. A complete Borel measure is defined on Cantor Space by $\mu([\sigma]) = 2^{-|\sigma|}$.

Domination

Definition

For $f, g \in \omega^\omega$ we say that f dominates g ($f >^* g$) if $f(n) > g(n)$ for all but finitely many $n \in \omega$.

Theorem (Martin)

f dominates every computable function if and only if $0'' \leq_T f'$ (f is high).

Proof.

($\Rightarrow$) Define $g(s, e) = \{1$ if $\forall x \leq s\ \phi_{e,f(s)}(x) \downarrow$, 0 otherwise.
$g \leq_T f$. Define $u(s) = \mu n(\phi_{e,n} \downarrow\ \forall x \leq s)$. □

Measure and Domination

Definition

$Z \in 2^\omega$ is almost everywhere dominating (a. e. dominating) if for almost every $X \in 2^\omega$ if Φ_e^X is total then there is a total $f \leq_T Z$ such that $\Phi_e^X <^* f$.

Definition

$Z \in 2^\omega$ is almost everywhere uniformly dominating if for almost every $X \in 2^\omega$ there is a total $f \leq_T Z$ such that $\forall e \in \omega$ if Φ_e^X is total then $\Phi_e^X <^* f$.

Definition

$Z \in 2^\omega$ is uniformly almost everywhere dominating (uniformly a. e. dominating) if there is a total $f \leq_T Z$ such that for almost every $X \in 2^\omega$ $\forall e \in \omega$ if Φ_e^X is total then $\Phi_e^X <^* f$.

Proposition

$Z \in 2^\omega$ is uniformly a. e. dominatingif and only if Z is a. e. uniformly dominating.

Proof.

That uniformly a. e. dominating implies a. e. uniformly dominating is clear from the definitions. To prove the converse suppose Z is a. e. uniformly dominating. Define $\mathcal{A}_e = \{X \in 2^\omega : \forall n \in \omega \Phi_n^X <^* \Phi_e^Z\}$. Then $\mu(\bigcup_e \mathcal{A}_e) = 1$. By σ-additivity there is e_0 such that $\mu(\mathcal{A}_{e_0}) > 0$. Now if $\sigma,\ \tau \in 2^{<\omega}$ and $|\sigma| = |\tau|$, then $\forall X \in 2^\omega\ \sigma X =_T \tau X$. Hence, $\sigma X \in \mathcal{A}_{e_0} \Rightarrow \tau X \in \mathcal{A}_{e_0}$. In other words $\mathcal{A}_{e_0}$ is a tail set. By the Kolmogorov $0-1$ law the measure of a tail set is either 0 or 1. Therefore $\mu(\mathcal{A}_{e_0}) = 1$ and so Z is uniformly a. e. dominating (as witnessed by $\Phi_{e_0}^Z$). □

Theorem (Kurtz)

$0'$ *is uniformly a. e. dominating.*

Proposition

Z uniformly a. e. dominating implies that Z is high.

Proof.

Z uniformly a. e. dominating implies that Z computes a funtion that dominates every computable function. By Martin's result Z is high. □

Conjecture (Dobrinen and Simpson)

The following are equivalent.

1. *Z is a. e. dominating.*
2. *Z is uniformly a. e. dominating.*
3. *Z is high.*

This turns out to be false. We show that 3 does not imply 1 (and hence also does not imply 2). It is not known if 1 and 2 are equivalent.

First we need some theorems connecting domination and regularity of the Legesgue measure.

Theorem (Dobrinen and Simpson)

$Z \in 2^\omega$ is uniformly a. e. dominating if and only if for any Π_2 class $\mathcal{P} \subseteq 2^\omega$ there is a Σ_2^Z class $\mathcal{S} \subseteq \mathcal{P}$ such that $\mu(\mathcal{S}) = \mu(\mathcal{P})$.

Theorem (Dobrinen and Simpson)

$Zin2^\omega$ is a. e. dominating if and only if for any Π_2 class $\mathcal{P} \subseteq 2^\omega$ and any $\epsilon > 0$ there is a Π_1^Z class $\mathcal{Q} \subseteq \mathcal{P}$ such that $\mu(\mathcal{Q}) \geq \mu(\mathcal{P}) - \epsilon$.

Binns, Kjos-Hanssen, Lerman, and Solomon were able to the idea of low for randomness along with the above results to prove that there is a high degree below $0'$ that is not a. e. dominating.

Definition

For $X, Y \in 2^\omega$ we say $X \leq_{LR} Y$ if $\mathcal{R}_1^Y \subseteq \mathcal{R}_1^X$ (that is, every 1-Y-random is 1-X-random).

Theorem (Kautz, Kucera)

Let $Z \in 2^\omega$ and let $\mathcal{Q} \subseteq 2^\omega$ be Π_1^Z with positive measure. Then $\forall X \in \mathcal{R}_1^Z$ $\exists \sigma \in 2^{<\omega}$ and $\exists Y \in \mathcal{Q}$ such that $X = \sigma Y$.

Theorem

If Z is a. e. dominating then $0' \leq_{LR} Z$.

Proof.

Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a universal $\Sigma_1^{0'}$ Martin-Löf test. Let $\mathcal{P} = \omega - \mathcal{U}_1$.

1. $\mathcal{P}$ is Π_2 (Since $\Pi_1^{0'} = \Pi_2$).
2. $\mathcal{P} \subseteq \mathcal{R}_1^{0'}$.
3. $\mu(\mathcal{P}) > 0$.

Since Z is a. e. dominating there is a Π_1^Z class $\mathcal{Q} \subseteq \mathcal{P}$ with $\mu(\mathcal{Q}) > 0$.
Let $X \in \mathcal{R}_1^Z$. Then there is a string σ and $Y \in \mathcal{Q}$ such that $X = \sigma Y$. Then $Y \in \mathcal{R}_1^{0'}$ implies $X \in \mathcal{R}_1^{0'}$. Therefore $\mathcal{R}_1^Z \subseteq \mathcal{R}_1^{0'}$ and $0' \leq_{LR} Z$. □

Theorem (Nies)

Let $X, Y \in 2^\omega$. Then $X + Y \leq_{LR} Y$ implies $X' \leq_{tt} Y'$.

Corollary

If $Z \leq_T 0'$ is a. e. dominating then $Z' =_{tt} 0''$.

Proof.

$Z \leq_T 0' \Rightarrow Z' \leq_1 0'' \Rightarrow Z' \leq_{tt} 0''$. We also know that $0' + Z \leq_T 0'$. Since Z is a. e. dominating $0' \leq_{LR} Z$ and hence $0' + Z \leq_{LR} Z$. Thus $0'' \leq_{tt} Z'$. □

Definition

$H_T = \{e \in \omega : 0'' \leq_T W_e'\}$. $H_{tt} = \{e \in \omega : 0'' \leq_{tt} W_e\}$

Theorem (Schwarz)

H_T is Σ_5 complete.

Proposition

H_{tt} is Σ_4.

Let $\{\psi_n\}_{n\in\omega}$ be an enumeration of the w. f. f. s of sentential logic with letters a_m, $n \in \omega$. For $X \in 2^\omega$ let ν be a truth assigment such that $\nu(a_n) = T$ if and only if $n \in X$. Let $\overline{\nu}$ be the extension of ν to all w. f. f. s. Write $X \models \psi_n$ if $\overline{\nu}(\psi_n) = T$.

Definition

$Y \leq_{tt} X$ if there is a computable function f such that $n \in Y \iff X \models \psi_{f(n)}$.

Proof.

$0'' \leq_{tt} W'_e \iff (\exists n)(\forall x)[x \in W_n \& (x \in 0'' \iff W'_e \models \psi_{\phi_n(x)})]$.

1. $x \in W_n$ is Σ_1.
2. $x \in 0''$ is Σ_2.
3. $W'_e \models \psi_{\phi_n(x)}$ is Σ_2.

Therefore H_{tt} is Σ_4. □

Theorem

There is a high c. e. set that is not a. e. dominating.

Proof.

$H_{tt} \subseteq H_T$ and as just shown $H_{tt} \neq H_T$. Let $e \in H_T - H_{tt}$. W_e is high and c. e. but not super high and hence not a. e. dominating. □

Other results

1. Binns et. al.
   1.1 $X \in \mathcal{R}_2$ implies X is not a. e. dominating.
   1.2 X 2-generic implies X is not a. e. dominating.
2. Cholak, Greenberg, and Miller: There is an incomplete (c. e.) uniformly a. e. dominating set.