Measure and Domination

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Notation

$X \in 2^{\omega}$ is a real.
$\mathcal{A} \subseteq 2^{\omega}$ is a class.
$f \in \omega^{\omega}$ is a function.
$\sigma \in 2^{<\omega}$ is a string.
Measure

Definition

Cantor Space is the set $2^\omega$ with the topology given by the basic open sets $\{[\sigma] : \sigma \in 2^{<\omega}\}$ where $[\sigma] := \{X \in 2^\omega : \sigma \preceq X\}$.

1. Cantor Space is compact.
2. Each basic open set is also closed.
3. A complete Borel measure is defined on Cantor Space by $\mu([\sigma]) = 2^{-|\sigma|}$. 
Domination

Definition
For \( f, g \in \omega^\omega \) we say that \( f \) dominates \( g \) (\( f \succ^* g \)) if \( f(n) > g(n) \) for all but finitely many \( n \in \omega \).

Theorem (Martin)
\( f \) dominates every computable function if and only if \( 0'' \leq_T f' \) (\( f \) is high).

Proof.
\((\Rightarrow)\) Define \( g(s, e) = \{1 \text{ if } \forall x \leq s \phi_{e,f(s)}(x) \downarrow, 0 \text{ otherwise.} \)
\( g \leq_T f \). Define \( u(s) = \mu n(\phi_{e,n} \downarrow \forall x \leq s) \).

□
Measure and Domination

Definition
$Z \in 2^\omega$ is almost everywhere dominating (a. e. dominating) if for almost every $X \in 2^\omega$ if $\Phi^X_e$ is total then there is a total $f \leq_T Z$ such that $\Phi^X_e <^* f$.

Definition
$Z \in 2^\omega$ is almost everywhere uniformly dominating if for almost every $X \in 2^\omega$ there is a total $f \leq_T Z$ such that $\forall e \in \omega$ if $\Phi^X_e$ is total then $\Phi^X_e <^* f$.

Definition
$Z \in 2^\omega$ is uniformly almost everywhere dominating (uniformly a. e. dominating) if there is a total $f \leq_T Z$ such that for almost every $X \in 2^\omega \forall e \in \omega$ if $\Phi^X_e$ is total then $\Phi^X_e <^* f$. 
Proposition

$Z \in 2^\omega$ is uniformly a. e. dominating if and only if $Z$ is a. e. uniformly dominating.

Proof.

That uniformly a. e. dominating implies a. e. uniformly dominating is clear from the definitions. To prove the converse suppose $Z$ is a. e. uniformly dominating. Define

$A_e = \{X \in 2^\omega : \forall n \in \omega \Phi^X_n < * \Phi^Z_e\}$. Then $\mu(\bigcup_e A_e) = 1$. By $\sigma$-additivity there is $e_0$ such that $\mu(A_{e_0}) > 0$. Now if $\sigma, \tau \in 2^{<\omega}$ and $|\sigma| = |\tau|$, then $\forall X \in 2^\omega \sigma X =_T \tau X$. Hence, $\sigma X \in A_{e_0} \Rightarrow \tau X \in A_{e_0}$. In other words $A_{e_0}$ is a tail set. By the Kolmogorov $0 - 1$ law the measure of a tail set is either 0 or 1. Therefore $\mu(A_{e_0}) = 1$ and so $Z$ is uniformly a. e. dominating (as witnessed by $\Phi^Z_{e_0}$).
Theorem (Kurtz)
0′ is uniformly a.e. dominating.

Proposition
Z uniformly a.e. dominating implies that Z is high.

Proof.
Z uniformly a.e. dominating implies that Z computes a function that dominates every computable function. By Martin’s result Z is high.
Conjecture (Dobrinen and Simpson)

The following are equivalent.

1. $Z$ is a. e. dominating.
2. $Z$ is uniformly a. e. dominating.
3. $Z$ is high.

This turns out to be false. We show that 3 does not imply 1 (and hence also does not imply 2). It is not known if 1 and 2 are equivalent.
First we need some theorems connecting domination and regularity of the Lebesgue measure.

**Theorem (Dobrinen and Simpson)**

\(Z \in 2^\omega\) is uniformly a. e. dominating if and only if for any \(\Pi_2\) class \(\mathcal{P} \subseteq 2^\omega\) there is a \(\Sigma_2^Z\) class \(\mathcal{S} \subseteq \mathcal{P}\) such that \(\mu(\mathcal{S}) = \mu(\mathcal{P})\).

**Theorem (Dobrinen and Simpson)**

\(Z \in 2^\omega\) is a. e. dominating if and only if for any \(\Pi_2\) class \(\mathcal{P} \subseteq 2^\omega\) and any \(\epsilon > 0\) there is a \(\Pi_1^Z\) class \(\mathcal{Q} \subseteq \mathcal{P}\) such that \(\mu(\mathcal{Q}) \geq \mu(\mathcal{P}) - \epsilon\).
Binns, Kjos-Hanssen, Lerman, and Solomon were able to the idea of low for randomness along with the above results to prove that there is a high degree below $0'$ that is not a.e. dominating.

**Definition**

For $X, Y \in 2^\omega$ we say $X \leq_{LR} Y$ if $R^Y_1 \subseteq R^X_1$ (that is, every 1-$Y$-random is 1-$X$-random).

**Theorem (Kautz, Kucera)**

Let $Z \in 2^\omega$ and let $Q \subseteq 2^\omega$ be $\Pi^Z_1$ with positive measure. Then $\forall X \in R^Z_1 \exists \sigma \in 2^{<\omega}$ and $\exists Y \in Q$ such that $X = \sigma Y$. 
**Theorem**

If $Z$ is a.e. dominating then $0' \leq_{LR} Z$.

**Proof.**

Let $\{U_n\}_{n \in \omega}$ be a universal $\Sigma^0_1$ Martin-Löf test. Let $P = \omega - U_1$.

1. $P$ is $\Pi^0_2$ (Since $\Pi^0_1 = \Pi^0_2$).
2. $P \subseteq R^0_1$.
3. $\mu(P) > 0$.

Since $Z$ is a.e. dominating there is a $\Pi^Z_1$ class $Q \subseteq P$ with $\mu(Q) > 0$.

Let $X \in R^Z_1$. Then there is a string $\sigma$ and $Y \in Q$ such that $X = \sigma Y$. Then $Y \in R^0_1$ implies $X \in R^0_1$. Therefore $R^Z_1 \subseteq R^0_1$ and $0' \leq_{LR} Z$. 

\[\square\]
Theorem (Nies)

Let $X, Y \in 2^\omega$. Then $X + Y \leq_{LR} Y$ implies $X' \leq_{tt} Y'$.

Corollary

If $Z \leq_T 0'$ is a.e. dominating then $Z' =_{tt} 0''$.

Proof.

$Z \leq_T 0' \Rightarrow Z' \leq_1 0'' \Rightarrow Z' \leq_{tt} 0''$. We also know that $0' + Z \leq_T 0'$. Since $Z$ is a.e. dominating $0' \leq_{LR} Z$ and hence $0' + Z \leq_{LR} Z$. Thus $0'' \leq_{tt} Z'$.
Definition
\[ H_T = \{ e \in \omega : 0'' \leq_T W_e \} \]
\[ H_{tt} = \{ e \in \omega : 0'' \leq_{tt} W_e \} \]

Theorem (Schwarz)
\( H_T \) is \( \Sigma_5 \) complete.

Proposition
\( H_{tt} \) is \( \Sigma_4 \).
Let \( \{ \psi_n \}_{n \in \omega} \) be an enumeration of the w. f. f. s of sentential logic with letters \( a_m, n \in \omega \). For \( X \in 2^\omega \) let \( \nu \) be a truth assignment such that \( \nu(a_n) = T \) if and only if \( n \in X \). Let \( \bar{\nu} \) be the extension of \( \nu \) to all w. f. f. s. Write \( X \models \psi_n \) if \( \bar{\nu}(\psi_n) = T \).

**Definition**

\( Y \leq_{tt} X \) if there is a computable function \( f \) such that \( n \in Y \iff X \models \psi_{f(n)} \).

**Proof.**

\( 0'' \leq_{tt} W'_e \iff (\exists n)(\forall x)[x \in W_n \& (x \in 0'' \iff W'_e \models \psi_{\phi_n(x)})] \).

1. \( x \in W_n \) is \( \Sigma_1 \).
2. \( x \in 0'' \) is \( \Sigma_2 \).
3. \( W'_e \models \psi_{\phi_n(x)} \) is \( \Sigma_2 \).

Therefore \( H_{tt} \) is \( \Sigma_4 \).
Theorem
There is a high c. e. set that is not a. e. dominating.

Proof.
$H_{tt} \subseteq H_T$ and as just shown $H_{tt} \neq H_T$. Let $e \in H_T - H_{tt}$. $W_e$ is high and c. e. but not super high and hence not a. e. dominating. \qed
Other results

1. Binns et. al.
   1.1 $X \in \mathcal{R}_2$ implies $X$ is not a.e. dominating.
   1.2 $X$ 2-generic implies $X$ is not a.e. dominating.

2. Cholak, Greenberg, and Miller: There is an incomplete (c. e. )
   uniformly a.e. dominating set.