MARTIN-LÖF RANDOMNESS IN SPACES OF CLOSED SETS

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Abstract. Algorithmic randomness was originally defined for Cantor space with the fair-coin measure. Recent work has examined algorithmic randomness in new contexts, in particular closed subsets of 2^ω ([2] and [8]). In this paper we use the probability theory of closed set-valued random variables (RACS) to extend the definition of Martin-Löf randomness to spaces of closed subsets of locally compact, Hausdorff, second countable topological spaces. This allows for the study of Martin-Löf randomness in many new spaces, but also gives a new perspective on Martin-Löf randomness for 2^ω and on the algorithmically random closed sets of [2] and [8]. The first half of this paper is devoted to developing the machinery of Martin-Löf randomness for general spaces of closed sets. We then prove some general results and move on to show how the algorithmically random closed sets of [2] and [8] fit into this new framework.

1. Introduction

It is easy to generate a random binary sequence: simply flip a fair coin and record the outcome of each flip as 0 for heads and 1 for tails. The laws of probability tell us that in the long run we should expect to get the same number of 0s and 1s, that we can expect to eventually see any finite sequence of 0s and 1s, et cetera. What we certainly do not expect to get is the sequence $10101010\ldots$, or any other sequence with a pattern. However, this sequence is as likely to be the outcome of our sequence of coin flips as any other particular binary sequence. Algorithmic randomness (also called effective randomness) formalizes the intuition that some sequences are "more random" than others by combining computability theory with probability.

One of the most important definitions in algorithmic randomness is Martin-Löf randomness, which we introduce in section 2.1. Algorithmically random sequences have been the subject of a great deal of attention recently and much of the work has been collected in [9] and [18].

Increasing interest in algorithmically random sequences has led researchers to ask whether similarly interesting behavior is possible in other settings. One alternative setting is $C[0,1]$, the set of continuous functions from the closed unit interval to $\mathbb{R}$. Algorithmic randomness in this setting was first studied in [11] and [10], and later in [14] which showed that results about algorithmically random functions could be obtained by producing a measure algebra homomorphism between 2^ω and $C[0,1]$. As we will see, this basic idea can be used in other settings, in particular in the space of closed subsets of certain topological spaces.

Algorithmically random closed subsets of Cantor space were first studied in [2] which defines algorithmic randomness in this setting by coding each infinite binary tree without dead ends as a ternary real. Every closed subset of Cantor space can be uniquely represented as the set of paths through such a tree. In [2] a closed set is defined to be algorithmically random if the code for the corresponding tree is Martin-Löf random.

Probability theorists and statisticians have defined a random closed set to be something quite different. A random closed set as defined in the literature (see [15], [16], or [17]) is simply a closed set-valued random variable. That is, a random closed set is a measurable map from a probability space to $\mathcal{F}(\mathbb{E})$, the space of closed subsets of the topological space $\mathbb{E}$. This is formalized using the hit-or-miss topology (also known as the Fell topology) on the space $\mathcal{F}(\mathbb{E})$. We introduce this probabilistic theory of random closed sets in section 2.2. This is obviously a very different idea than that developed in [2], but there are connections. We prove that the coding of closed sets of Cantor space used in [2] is actually an example of a particularly nice random closed set (in the probability theory sense). This is lemma 4.5 in section 4.2.

As in the case of real-valued random variables, random closed sets induce a probability measure on the target space. The connection with algorithmic randomness comes when we use such a measure to produce Martin-Löf tests in the space of closed sets. In section 3 we give some general results about how to do this. One key result here is lemma 3.5, which establishes that Martin-Löf randomness can be defined in the space of closed subsets of a locally compact, Hausdorff, second countable space using the hit-or-miss topology. This allows us to study algorithmic randomness purely from the perspective of the space of closed sets with the hit-or-miss topology. We note that different random closed sets (in the probability sense) give rise to different measures and hence different classes of Martin-Löf random closed sets. Other important results in this section are the technical lemmas 3.6 and 3.7, which are used extensively in the later sections.

Much of this paper is an exploration of examples of specific random closed sets (in the probability sense) and the Martin-Löf random closed sets they give rise to. We begin by looking at the example of Martin-Löf random closed subsets of $\mathbb{N}$ in section 4.1. We prove that the Martin-Löf random closed subsets of $\mathbb{N}$ are exactly those subsets with a Martin-Löf random characteristic function (in 2^ω). This is an easy result, but it is interesting in that it shows that our approach is a generalization of "classic" algorithmic randomness.

Our first major example of a random closed set is the coding defined in [2]. Having established in lemma 4.5 that this coding is a measurable map from $3^\omega \to \mathcal{F}(2^\omega)$ we are then able to prove that a closed set is Martin-Löf random in $\mathcal{F}(2^\omega)$ if and only if it is the image of a Martin-Löf random element of 3^ω (corollary 4.6). This means that our definition of Martin-Löf random closed sets agrees exactly with the definition of algorithmically random closed sets given in [2]. Our approach, however, allows for the use of theorems from probability theory of random closed sets. In proposition 4.9, for instance, we use one of these tools (Robbins' theorem) to prove that for these Martin-Löf random closed sets have measure 0. This is a result originally proved in [2] (although in less generality) by a different technique. The main obstacle in that proof is nicely resolved here by our application of Robbins' theorem.

Our next example, the Galton-Watson random closed sets of [8], is another map defined by coding closed subsets of 2^ω in a Cantor space. This example is explored in section 4.3. In [8] Diamondstone and Kjos-Hanssen prove that a closed set is $\frac{2}{3}$-Galton-Watson random if and only if it is either $\emptyset$ or an algorithmically random closed set in the sense of [2]. Their proof relies, in part, on lemma 4.21. We also prove in lemma 4.19 that the coding used in [8] is a random closed set and that it maps Martin-Löf random reals to Martin-Löf random closed sets.

A sequel to this paper will deal with examples of algorithmically random closed sets in $\mathbb{R}$. In particular it will focus on algorithmic randomness arising from an important class of random closed sets from probability theory called generalized Poisson processes.

2. Background

This section covers the requisite background in the theory of algorithmic randomess and the theory of random closed sets.

2.1. Martin-Löf Randomness.

This section covers the basics of Martin-Löf randomness for the Cantor space 2^ω. Much greater detail can be found in [18] or [9]. The Cantor space 2^ω is the space of infinite binary sequences with the topology generated from the basis of cylinders, here denoted $[\sigma]$ where σ is a finite binary strings. This space is homeomorphic to the Cantor middle thirds set, hence the name Cantor space. We note that Cantor space is compact, Hausdorff, and second countable.

We fix a computable enumeration $\sigma_0, \sigma_1, \sigma_2, \ldots$ of the set of finite binary strings $2^{<\omega}$. This, of course, gives a computable enumeration of the basis for the topology on 2^ω. We will use this to determine the algorithmic complexity of subsets of 2^ω.

Definition 2.1. Let $h \in 2^\omega$. $U \subseteq 2^\omega$ is $\Sigma_1^{0,h}$ if there is an h-computably enumerable $f \in 2^\omega$ such that $U = \bigcup_{f(n)=1} [\sigma_n]$.

Cantor space is usually endowed with a Borel probability measure, m, such that for each $\sigma \in 2^{<\omega}$ $m([\sigma]) = 2^{-|\sigma|}$. This measure is often called the fair coin measure because the probability that a given string σ will be the result of flips of a fair coin (taking heads as 0 and tails as 1, say) is exactly $2^{-|\sigma|}$. There are, however, many other measures on Cantor space and we will eventually wish to consider some of these measures. All measures under consideration will be Borel, even when not explicitly stated.

The idea behind Martin-Löf randomness is that the "laws" of probability theory describe random behavior. For example, the strong law of large numbers says that $f \in 2^\omega$ will m-almost surely satisfy the following equation

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} f(i)}{n} = \frac{1}{2}.$$

A truly random sequence should obey all such laws. In particular, since each law describes a set of measure 1, we would like to say that the intersection of all these measure 1 sets is exactly the set of random sequences. Unfortunately, given any $g \in 2^\omega$ there is a law saying that $f \in 2^\omega$ is almost surely not g. Therefore the intersection of all these measure 1 sets is exactly the empty set. Algorithmic randomness solves this problem by restricting the collection of laws that must be obeyed to just those that are sufficiently effective. It is also traditional to think in terms of the complements of these measure 1 sets: laws of probability determine measure 0 sets of non-random sequences and random sequences must avoid all such sets.

Before giving the definition we must briefly address the use of measures as oracles. When we use a measure μ as an oracle we mean that we have access to an oracle that allows us compute the measure $\mu([\sigma_i])$ uniformly for each $i \in \omega$. We call such an oracle a representation of the measure μ. In [19] it is shown that the particular choice of representation is important since arbitrary information can be encoded in such representations. Moreover, [7] establish the existence of measures for which the representations have no minimum Turing degree. The end result is that attention must be paid to the particular representation of the measure.

Definition 2.2. Let μ be a Borel measure on 2^ω and let $r \in 2^\omega$ be a representation of μ.

(1) An *r-Martin-Löf test (r-ML test)* is a uniformly $\Sigma^{0,r}_1$ sequence of subsets of 2^ω, $\{U_i\}_{i\in\omega}$, such that $\mu(U_i) \leq 2^{-i}$.
(2) $f \in 2^\omega$ is *r-Martin-Löf random (or r-ML random)* if there is no r-Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $f \in \bigcap_{i\in\omega} U_i$.

Definition 2.3. A sequence $f \in 2^\omega$ is *μ-Martin-Löf random (or μ-ML random)* if there is some representation r of μ such that f is r-ML random.

A fundamental result for Martin-Löf randomness is the existence of a universal r-Martin-Löf test:

Lemma 2.4. *For each $r \in 2^\omega$ there is an r-ML test $\{U_i\}_{i\in\omega}$ such that $f \in 2^\omega$ is r-ML random if and only if $f \notin \bigcap_{i\in\omega} U_i$.*

It follows immediately that for any measure μ and any representation r, μ-almost every element of Cantor space is r-ML random.

2.2. **Spaces of closed sets.** In this section we cover the basics of random closed sets in probability theory. Greater detail can be found in either [15] or [16] (we follow the notational conventions of the latter).

We begin with an arbitrary topological space $\mathbb{E}$.

Definition 2.5.

(1) $\mathcal{F}(\mathbb{E})$ is the collection of closed subsets of $\mathbb{E}$.
(2) $\mathcal{K}(\mathbb{E})$ is the collection of compact subsets of $\mathbb{E}$.
(3) $\mathcal{G}(\mathbb{E})$ is the collection of open subsets of $\mathbb{E}$.

Where the underlying space is clear from context we will omit $\mathbb{E}$ and write only $\mathcal{F}$, $\mathcal{K}$, and $\mathcal{G}$.

$\mathcal{F}$ can be given a variety of topologies but we will focus on the Fell topology. The Fell topology is also known as the hit-or-miss topology because it is generated from so-called hitting and missing sets:

Definition 2.6. Let $A \subseteq \mathbb{E}$.

(1) $\mathcal{F}_A = \{F \in \mathcal{F} : F \cap A \neq \emptyset\}$ (the hitting set of A).
(2) $\mathcal{F}^A = \{F \in \mathcal{F} : F \cap A = \emptyset\}$ (the missing set of A).

Note that $\mathcal{F}^A$ is the complement of $\mathcal{F}_A$ in the space $\mathcal{F}$. The following proposition clarifies how the sets $\mathcal{F}_A$ and $\mathcal{F}^A$ combine under unions and intersections.

Proposition 2.7. *If $\{A_i : i \in I\}$ is a collection of subsets of $\mathbb{E}$ then:*

(1) $\bigcup_{i\in I} \mathcal{F}_{A_i} = \mathcal{F}_{\bigcup_{i\in I} A_i}$;

(2) $\bigcap_{i\in I} \mathcal{F}^{A_i} = \mathcal{F}^{\bigcup_{i\in I} A_i}$;
(3) $\bigcap_{i\in I} \mathcal{F}_{A_i} \supseteq \mathcal{F}_{\bigcap_{i\in I} A_i}$;
(4) $\bigcup_{i\in I} \mathcal{F}^{A_i} \subseteq \mathcal{F}^{\bigcap_{i\in I} A_i}$.

It is important to realize that in general $\mathcal{F}_{A_1} \cap \mathcal{F}_{A_2} \neq \mathcal{F}_{A_1\cap A_2}$. For example, in the space $\mathcal{F}(\mathbb{R})$, $\{0,1\} \in \mathcal{F}_{[0,\frac{1}{2}]} \cap \mathcal{F}_{[\frac{1}{2},1]}$ but $\{0,1\} \notin \mathcal{F}_{\{\frac{1}{2}\}}$. It is thus convenient to introduce the following notation.

Definition 2.8. Let $A, B_1, \ldots, B_n \subseteq \mathbb{E}$. The set $\mathcal{F}^A \cap \mathcal{F}_{B_1} \cap \cdots \cap \mathcal{F}_{B_n}$ will be denoted $\mathcal{F}^A_{B_1,\ldots,B_n}$.

We are now ready to define a topology for $\mathcal{F}$.

Definition 2.9. The *Fell topology* on $\mathcal{F}$ is generated by the sub-base of sets of the form $\mathcal{F}^K$ and $\mathcal{F}_G$ where K is compact and G is open. Hence sets of the form $\mathcal{F}^K_{G_1,\ldots,G_n}$ with K compact and $G_1, \ldots, G_n$ open form a basis for the Fell topology.

We now state some of the basic results from the theory of RACS that will help us to work with the Fell topology. A proof of lemma 2.10 can be found on page 13 of [16] where it is the first step of the first proof of the Choquet capacity theorem. A complete proof of lemma 2.11 can be found on page 3 of [15].

Lemma 2.10. *Let $\mathcal{V}$ be a any family of subsets of $\mathbb{E}$. Let $\mathfrak{B}$ be the family containing $\mathcal{F}_V$ and $\mathcal{F}^V$ for $V \in \mathcal{V}$ and closed under finite intersections. Then each $\mathcal{Y} \in \mathfrak{B}$ can be represented as*
$$\mathcal{Y} = \mathcal{F}^{V_{n+1}\cup\cdots\cup V_k}_{V_1,\ldots,V_n}$$
for some $k \geq n \geq 0$ and $V_1, \ldots, V_k \in \mathcal{V}$ with $V_i \not\subseteq V_j \cup (V_{n+1} \cup \cdots \cup V_k)$ for $i, j \leq n$ with $i \neq j$ (such a representation is called reduced*). Moreover, if $\mathcal{Y} = \mathcal{F}^{V'_{m+1}\cup\cdots\cup V'_l}_{V'_1,\ldots,V'_m}$ is another reduced representation, then $V_{n+1} \cup \cdots \cup V_k = V'_{m+1} \cup \cdots \cup V'_l$, $n = m$, and for each $i \in \{1, \ldots, n\}$ there is $j_i \in \{1, \ldots, m\}$ such that $V_i \cup V_{n+1} \cup \cdots \cup V_k = V'_{j_i} \cup V_{n+1} \cup \cdots \cup V_k$.*

It follows from lemma 2.10 that finding a reduced representation is computable using only knowledge about intersections and unions of the sets in $\mathcal{V}$: If $\mathcal{F}^V_{V_1,\ldots,V_n}$ is not reduced, then there are $i, j \leq n$ such that $V_i \subseteq V \cup V_j$. Thus removing V_j does not change the set, i.e.
$$\mathcal{F}^V_{V_1,\ldots,V_n} = \mathcal{F}^V_{V_1,\ldots,V_{j-1},V_{j+1},\ldots,V_n}.$$
Dropping V_j and iterating the procedure will eventually produce a reduced representation of any element of $\mathfrak{B}$. Ultimately we will use this to show that it is possible to computably determine if two basic open sets of the Fell topology are equal in certain situations (to be made more precise later). This will be important in our use of the Fell topology for computable analysis.

Random closed sets in probability are typically limited to closed subsets of locally compact, Hausdorff, and second countable spaces. One reason for this is the following lemma.

Lemma 2.11. *Let be $\mathbb{E}$ a locally compact, Hausdorff, second countable (LCHS) space and let $\mathfrak{B}$ be a countable basis for $\mathbb{E}$ such that the closure $\overline{B}$ is compact for each $B \in \mathfrak{B}$. Then $\mathcal{F}(\mathbb{E})$ is compact, Hausdorff, and second countable with a sub-basis consisting of the sets $\mathcal{F}_B$ and $\mathcal{F}^{\overline{B}}$, $B \in \mathfrak{B}$.*

We now give the fundamental definition for the probability theory of such spaces.

Definition 2.12. A *random closed set (or RACS)* is a measurable map from a probability space to $\mathcal{F}(\mathbb{E})$ (where $\mathcal{F}(\mathbb{E})$ is equipped with the Borel σ-algebra of the Fell topology).

Because [2] used "random closed set" to refer to something very different, we will usually use the abbreviation "RACS" for such a measurable map. This abbreviation is standard in the literature (see [15] and [16]) and should help to avoid confusion. To further distinguish between these definitions we will use "BBCDW-random closed set" to describe the objects studied in [2].

In general, any measurable map from a probability space to $\mathbb{R}$ is called a random variable. In the case of a RACS we have something like a random variable except that the measurable map takes values in $\mathcal{F}$ instead of $\mathbb{R}$. Hence the name "random closed set". A RACS can, like a random variable, be thought of as an assignment of probability to each event, where an event is a measurable subset of $\mathcal{F}$.

The fundamental result in the study of RACS is the Choquet capacity theorem. This theorem completely characterizes all possible measures on $\mathcal{F}(\mathbb{E})$ as functionals from the collection of compact sets of $\mathbb{E}$ to the unit interval $[0, 1]$. From the probability perspective the interesting part of a RACS is the measure it gives rise to and so the Choquet theorem is thought of as a characterization of all RACS. A much more detailed exposition of the Choquet capacity theorem can be found in Matheron [15], Molchanov [16], or Choquet's original paper [5].

For the remainder of this section we will consider only locally compact, Hausdorff, and second countable (LCHS) topological spaces. In these spaces we find that the Borel σ-algebra of $\mathcal{F}$ is generated by the hitting sets of compact sets ($\mathcal{F}_K$ for compact K). This suggests that any measure on $\mathcal{F}$ could be fully characterized by its behavior on these sets and this is part of what the Choquet capacity theorem proves. The Choquet capacity theorem also proves that any functional on $\mathcal{K}$ with certain properties will give rise to a RACS. Before we can state the theorem, however, we need to develop some machinery.

The word "capacity" comes from the capacity functional associated with a RACS.

Definition 2.13. Let (Ω, P) be a probability space, let $\mathbb{E}$ be an LCHS space, and let $X : \Omega \to \mathcal{F}(\mathbb{E})$ be a RACS. Then X induces a functional $T_X : \mathcal{K}(\mathbb{E}) \to [0, 1]$ given by $T_X(K) = P\left(X^{-1}(\mathcal{F}_K)\right)$, called the *capacity functional* of X. Where the RACS X is clear from context we use only T instead of T_X.

Note that the capacity functional T_X does not actually depend on the map X but instead on the measure $P \circ X^{-1}$. This measure on $\mathcal{F}$ will show up repeatedly and we will denote it $\mathbf{P}_X$.

The Choquet capacity theorem gives necessary and sufficient conditions for a functional $T : \mathcal{K} \to \mathbb{R}$ to be the capacity functional of a RACS. Some necessary conditions are obvious: $T(\emptyset) = 0$; if $K_1, K_2 \in \mathcal{K}$ and $K_1 \subseteq K_2$ then $T(K_1) \leq T_X(K_2)$, etc. Other properties that turn out to be important are not so obvious. The completely alternating property in particular takes some work to even state.

Definition 2.14. Let $T : \mathcal{K} \to \mathbb{R}$ be any functional and let $K, K_1, K_2, \ldots, K_n \in \mathcal{K}$. Define:

(1) $\Delta_{K_1} T(K) = T(K) - T(K \cup K_1)$;
(2) $\Delta_{K_n} \ldots \Delta_{K_1} T(K) = \Delta_{K_{n-1}} \ldots \Delta_{K_1} T(K) - \Delta_{K_{n-1}} \ldots \Delta_{K_1} T(K \cup K_n)$ for $n \geq 2$.

The functional T is *completely alternating* if for every $K, K_1, \dots, K_n \in \mathcal{K}$,

$$\Delta_{K_n} \dots \Delta_{K_1} T(K) \leq 0.$$

Complete alternation is complicated enough to warrant some explanation. Let X be a RACS and let $\mathbf{P}_X$ be the probability measure induced on $\mathcal{F}$ by X (so $\mathbf{P}_X(\mathcal{S}) = P\left(X^{-1}(\mathcal{S})\right)$ for a measurable $\mathcal{S} \subseteq \mathcal{F}$). If T_X is the capacity of X, then for every $K_1, \dots, K_n, K \in \mathcal{K}$

$$\Delta_{K_n} \dots \Delta_{K_1} T(K) = -\mathbf{P}_X\left(\mathcal{F}^K_{K_1,\dots,K_n}\right).$$

The capacity of any RACS is hence completely alternating simply by virtue of the fact that any probability measure cannot take negative values.

We are now ready to state the Choquet capacity theorem. This theorem completely characterizes those functionals $T : \mathcal{K} \to [0,1]$ that are the capacities of RACS. Proofs can be found in [15], [16], or [5]. An effective version of the theorem has been proved in [4].

Theorem 2.15 (Choquet capacity theorem). *Let $\mathbb{E}$ be an LCHS space and let $T : \mathcal{K} \to [0,1]$. Then T gives rise to a Borel probability measure $\mathbf{P}$ on $\mathcal{F}(\mathbb{E})$ such that $\mathbf{P}(\mathcal{F}_K) = T(K)$ for $K \in \mathcal{K}$ if and only if T satisfies the following conditions:*

1. *$T(\emptyset) = 0$;*
2. *T is upper semi-continuous on $\mathcal{K}$: if $K_0 \supseteq K_1 \supseteq \dots \in \mathcal{K}$ such that $\bigcap_{i\in\omega} K_i = K \in \mathcal{K}$ then $\lim_{i\to\infty} T(K_i) = T(K)$;*
3. *T is completely alternating on $\mathcal{K}$ (definition 2.14).*

Moreover, the probability measure $\mathbf{P}$ is unique.

The proof that these are necessary conditions is straightforward. We outline briefly the proof of sufficiency. Suppose that T is a functional satisfying conditions 1–3. It can be shown that T may be extended to the class of open subsets of $\mathbb{E}$ by setting

$$T(G) = \sup\{T(K) : K \in \mathcal{K} \;\&\; K \subseteq G\}$$

for $G \in \mathcal{G}$. A Borel probability measure $\mathbf{P}$ on $\mathcal{F}$ is then defined by assigning $\mathbf{P}(\mathcal{F}^K_{G_1,\dots,G_n}) = -\Delta_{G_n} \dots \Delta_{G_1} T(K)$ for $G_1, \dots, G_n \in \mathcal{G}$ and $K \in \mathcal{K}$. This produces a RACS on $\mathcal{F}$ in the sense that $\mathcal{F}$ becomes a probability space with the measure $\mathbf{P}$ (and consequently the identity map on $\mathcal{F}$ is a RACS).

Definition 2.16. If $T : \mathcal{K} \to [0,1]$ satisfies the conditions of the Choquet Capacity Theorem then we call T a *Choquet capacity.*

The following theorem, known as Robbins' theorem, is a basic result in the theory of RACS which we will find useful. It applies when the underlying topological space $\mathbb{E}$ is equipped with a measure ν. In this case the composition of a RACS $X : \Omega \to \mathcal{F}(\mathbb{E})$ with the measure ν gives rise to a random variable $\nu \circ X$. The expected value of the resultant random variable is by definition $E(\nu \circ X) = \int_\Omega (\nu \circ X)(\omega) d\omega$. Robbins' theorem shows how to calculate $E(\nu \circ X)$ as an integral in the space $\mathbb{E}$.

Theorem 2.17 (Robbins' Theorem). *Let (Ω, P) be a probability space. If ν is a σ-finite Borel measure on Polish space $\mathbb{E}$ and $X : \Omega \to \mathcal{F}(\mathbb{E})$ is a RACS, then the composite function $\nu \circ X : \Omega \to \mathbb{R}$ is a random variable and*

$$E(\nu \circ X) = \int_{\mathbb{E}} P(x \in X) d\nu(x).$$

Note: it is possible that $E(\nu \circ X) = \infty$ and in that case $\int_{\mathbb{E}} P(x \in X) d\nu(x) = \infty$. The notation $P(x \in X)$ as used in the theorem is the standard abbreviation for $P(\{\omega \in \Omega : x \in X(\omega)\})$.

3. A framework for Martin-Löf randomness in spaces of closed sets

In this section we use the theory of random closed sets to build a framework for Martin-Löf randomness in the spaces of closed sets of a fairly general class of underlying spaces. The following sections will deal specifically with the underlying spaces $\mathbb{N}$ and 2^ω and a sequel to this paper will address the underlying space $\mathbb{R}$.

Our first step is to produce a framework for Martin-Löf randomness which is general enough to apply to a space of closed sets equipped with the Fell topology. Our approach is compatible with that of [12] but slightly more tailored to our work with spaces of closed sets.

Definition 3.1. Let $\mathbb{X}$ be a topological space with Borel measure μ. We say that $\mathbb{X}$ (together with μ) is a *Martin-Löf space* if it has a countable basis $B_0, B_1, B_2, \ldots$ meeting the following conditions:

(1) $\mu(B_i) < \infty$ for each $i \in \omega$;
(2) The set $\{B_0, B_1, B_2, \ldots\}$ is closed under finite intersections of its members;
(3) For any representation r of μ there is an r-computable intersection function $\mathfrak{g} : \omega^2 \to \omega$ such that
$$B_i \cap B_j = B_{\mathfrak{g}(i,j)}.$$

If in addition $\mu(\mathbb{X}) = 1$, then we call $\mathbb{X}$ a *Martin-Löf probability space.*

In this definition the representation of a measure is relative to the particular enumeration of the basis $B_0, B_1, B_2, \ldots$. That is, $r \in 2^\omega$ is a representation of μ if $\mu(B_i)$ is uniformly r-computable. As in the case of measures on Cantor space, there may not be a canonical representation of a given measure or even a representation of lowest Turing degree.

We can now define Martin-Löf randomness for general Martin-Löf spaces.

Definition 3.2. Let $\mathbb{X}$ be a Martin-Löf space with measure μ and basis $B_0, B_1, B_2, \ldots$ and let $r \in 2^\omega$ be a representation of μ (for this basis).

(1) An *r-Martin-Löf test (r-ML test)* is a uniformly $\Sigma_1^{0,r}$ sequence of subsets of $\mathbb{X}$, $\{U_i\}_{i\in\omega}$, such that $\mu(U_i) \le 2^{-i}$.
(2) $x \in \mathbb{X}$ is *r-Martin-Löf random (or r-ML random)* if there is no r-Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $x \in \bigcap_{i\in\omega} U_i$.

Definition 3.3. Let $\mathbb{X}$ be a Martin-Löf space with measure μ. An element $x \in \mathbb{X}$ is *μ-Martin-Löf random (or μ-ML random)* if there is some representation r of μ such that x is r-ML random.

If the oracles are omitted in each of the preceding 3 definitions (3.1, 3.2, and 3.3) then we arrive at the definition of *μ-Hippocrates randomness* (Hippocrates famously ignored the oracles). For computable measures μ, an element is μ-Martin-Löf random if and only if it is μ-Hippocrates random. However, for non-computable measures these two definitions of algorithmic randomness may be different. In general the set of Martin-Löf random elements of a space is contained in the set of Hippocrates random elements of the space (though both have measure 1).

A key result for traditional Martin-Löf randomness in Cantor space is the existence of a universal Martin-Löf test. Our general framework for Martin-Löf randomness was ensures that this is still the case, as we show in lemma 3.4 below. One important distinction between Martin-Löf randomness and Hippocrates randomness is the non-existence of a universal Hippocrates test as proved in [13]. Hippocrates randomness is useful here in part because it allows for an easy statement of theorems. Adding appropriate oracles to theorems stated in terms of Hippocrates randomness often gives theorems involving Martin-Löf randomness.

Lemma 3.4. *Let $\mathbb{X}$ be a Martin-Löf space and let r be a representation of its measure μ. There is a single r-ML test $\{U_i\}_{i\in\omega}$ such that $x \in \mathbb{X}$ is r-ML random if and only if $x \notin \bigcap_{i\in\omega} U_i$ (we call such a test a universal r-ML test).*

Proof. Definitions 3.1 and 3.2 ensure that the standard proof works in this situation. Start with an r-computable enumeration of all the uniformly $\Sigma_1^{0,r}$ sequences. Modify each $\Sigma_1^{0,r}$ sequence into an r-ML test to achieve an enumeration of (essentially) all r-ML tests. Then diagonalize as usual. We provide the details of the proof because of the new, more general context.

Let $\{V_i\}_{i\in\omega}$ be a $\Sigma_1^{0,r}$ sequence. By definition there is $f \in 2^\omega$ such that

$$V_i = \bigcup_{f(\langle i,j\rangle)=1} B_j.$$

Recursively define $\hat{V}_i$ as follows. Begin with $\hat{V}_{i,0} = \emptyset$ and at stage $s+1$ let k be the least natural number such that all of the following hold:

(1) $k \leq s$,
(2) $f(\langle i,k\rangle) = 1$,
(3) B_k has not already been added to $\hat{V}_{i,s}$, and
(4) $\mu\left(\hat{V}_{i,s} \cup B_k\right) \restriction s < 2^{-i} - 2^{-s}$.

Let $\hat{V}_{i,s+1} = \hat{V}_{i,s} \cup B_k$ when such a k exists, otherwise $\hat{V}_{i,s+1} = \hat{V}_{i,s}$. We define $\hat{V}_i = \bigcup_{s\in\omega} \hat{V}_{i,s}$.

We note that although the oracle r allows us to compute $\mu\left(\hat{V}_{i,s} \cup B_k\right)$ with arbitrary precision (using inclusion-exclusion and the r-computable intersection function of definition 3.1), this does not mean that the inequality $\mu\left(\hat{V}_{i,s} \cup B_k\right) < 2^{-i}$ is r-computable. We avoid this problem by restricting ourselves to just the first s bits of $\mu\left(\hat{V}_{i,s} \cup B_k\right)$ and asking whether this rational number is less than $2^{-i} - 2^{-s}$, a question that we can computably answer. When we answer in the affirmative we can conclude that $\mu\left(\hat{V}_{i,s} \cup B_k\right) < 2^{-i}$. Consequently $\mu(\hat{V}_i) \leq 2^{-i}$ and the sequence $\{\hat{V}_i\}_{i\in\omega}$ is $\Sigma_1^{0,r}$.

We have thus converted an enumeration of all $\Sigma_1^{0,r}$ sequences $\{V_i\}_{i\in\omega}$ into an enumeration of some r-ML tests $\{\hat{V}_i\}_{i\in\omega}$. If $\{U_i\}_{i\in\omega}$ is a r-ML test, then it appears in our enumeration of r-ML tests as long as $\mu(U_i) < 2^{-i}$ for all i. If this fails to be the case, then the equivalent test $\{U_{i+1}\}_{i\in\omega}$ appears in our enumeration (the test is equivalent because its intersection is the same). We have therefore produced a r-computable enumeration of r-ML tests $\{\hat{V}_i^0\}_{i\in\omega}$, $\{\hat{V}_i^1\}_{i\in\omega} \ldots$ that will capture all non-r-ML random elements of $\mathbb{X}$.

Diagonalize by defining

$$U_i = \bigcup_{j\in\omega} \hat{V}^j_{j+i+1}.$$

Then $\{U_i\}_{i\in\omega}$ is clearly uniformly $\Sigma^{0,r}_1$ and

$$\begin{aligned}\mu(U_i) &\leq \sum_{j\in\omega} \mu\left(\hat{V}^j_{j+i+1}\right)\\ &\leq \sum_{j\in\omega} 2^{-(j+i+1)}\\ &= 2^{-i}.\end{aligned}$$

Therefore $\{U_i\}_{i\in\omega}$ is an r-Martin-Löf test.

Now suppose that $x \in \mathbb{X}$ is not r-Martin-Löf random. There is a r-Martin-Löf test $\{\hat{V}^j_i\}_{i\in\omega}$ such that $x \in \bigcap_{i\in\omega} \hat{V}^j_i$. Hence $x \in \hat{V}^j_{i+j+1} \subseteq U_i$ for all $i \in \omega$. In other words $x \in \bigcap_{i\in\omega} U_i$. Therefore $\{U_i\}_{i\in\omega}$ is universal. □

We now show that $\mathcal{F}(\mathbb{E})$ with the Fell topology is a Martin-Löf space for locally compact, Hausdorff, second countable underlying spaces $\mathbb{E}$.

Lemma 3.5. *If $\mathbb{E}$ is an LCHS space, then $\mathcal{F}(\mathbb{E})$ is a Martin-Löf probability space under any Borel probability measure.*

Proof. The space $\mathbb{E}$ is LCHS and so it has a countable basis $\{B_0, B_1, \dots\}$ such that $\overline{B_i}$ is compact for all $i \in \omega$. By lemma 2.11 $\mathcal{F}(\mathbb{E})$ has a basis consisting of the sets

$$\mathcal{F}^{\overline{B}_{j_1}\cup\cdots\cup\overline{B}_{j_k}}_{B_{i_1},\dots,B_{i_n}}.$$

This basis has an obvious enumeration $\mathcal{B}_0, \mathcal{B}_1, \dots$ where $\mathcal{B}_i$ corresponds to

$$\mathcal{F}^{\overline{B}_{j_1}\cup\cdots\cup\overline{B}_{j_k}}_{B_{i_1},\dots,B_{i_n}}$$

exactly when $i = \langle\langle i_1, \dots, i_n\rangle, \langle j_1, \dots, j_k\rangle\rangle$ (with $\langle\ ,\ \rangle$ a computable pairing function). This collection of sets is closed under finite intersections and a computable intersection function is then given by

$$\begin{aligned}&\mathfrak{g}\left(\langle\langle i_1, \dots, i_n\rangle, \langle j_1, \dots, j_k\rangle\rangle, \langle\langle i'_1, \dots, i'_m\rangle, \langle j'_1, \dots, j'_l\rangle\rangle\right) =\\ &\qquad\qquad\langle\langle i_1, \dots, i_n, i'_1, \dots, i'_n\rangle, \langle j_1, \dots, j_k, j'_1, \dots, j'_l\rangle\rangle.\end{aligned}$$

Clearly any Borel probability measure $\mathbf{P}$ satisfies $\mathbf{P}(\mathcal{B}_i) \leq 1$. Therefore $\mathcal{F}(\mathbb{E})$ is a Martin-Löf space. □

The preceding lemma and definition 3.3 allow for the study of Martin-Löf random closed subsets of LCHS spaces. Given an enumeration of a basis for $\mathbb{E}$, the lemma produces an enumeration of a basis for $\mathcal{F}(\mathbb{E})$ which we will call the *standard enumeration* of the *standard basis*. In fact, all of the following work assumes the use of the standard enumeration of the standard basis, so we rarely specify either.

There are a great variety of LCHS spaces (of particular interest are 2^ω, $\mathbb{R}^n$, and $[0, 1]^n$) as well as a great variety of RACS on the corresponding spaces of closed sets, so really general results are difficult to state. However, when a RACS is given as a map $X : \Omega \to \mathcal{F}(\mathbb{E})$ where Ω is a Martin-Löf space, then we can connect Martin-Löf randomness in Ω to Martin-Löf randomness in $\mathcal{F}(\mathbb{E})$. Our first lemma in this vein looks at when such a RACS maps algorithmically random elements of Ω to algorithmically random elements of $\mathcal{F}(\mathbb{E})$. The second lemma looks at when algorithmically random elements of $\mathcal{F}(\mathbb{E})$ are the image of algorithmically random elements of Ω. The general statement of each of these lemmas is awkward due in part to the (possibly) different computational power of the measures in Ω and $\mathcal{F}(\mathbb{E})$. In many situations these two measures have the same computational power (or one computes the other) and so adding the measure as an oracle in the lemma gives a useful statement about Martin-Löf randomness.

Lemma 3.6. *Let Ω be a Martin-Löf probability space with measure P. Let $\mathbb{E}$ be an LCHS space with basis $B_0, B_1, \dots$ such that $\overline{B}_i$ is compact for every i. Let $X : \Omega \to \mathcal{F}(\mathbb{E})$ be a map such that $X^{-1}(\mathcal{F}_{B_i})$ and $X^{-1}\left(\mathcal{F}^{\overline{B_i}}\right)$ are both uniformly Σ^0_1. Then for any P-Hippocrates random $\rho \in \Omega$, $X(\rho)$ is $\mathbf{P}_X$-Hippocrates random (where $\mathbf{P}_X$ is the probability measure on $\mathcal{F}(\mathbb{E})$ induced by X).*

Proof. By lemma 3.5, $\mathcal{F}(\mathbb{E})$ with $\mathbf{P}_X$ is a Martin-Löf space. We prove the equivalent statement that if $F \in \mathcal{F}(\mathbb{E})$ is not $\mathbf{P}_X$-Hippocrates random, then either $X^{-1}(\{F\})$ consists entirely of non-P-Hippocrates random elements of Ω or is empty. We accomplish this by showing that if $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}_X$-Hippocrates test, then $\left\{X^{-1}(\mathcal{V}_i)\right\}_{i\in\omega}$ is a P-Hippocrates test.

Consider a basic open set of $\mathcal{F}(\mathbb{E})$: $\mathcal{F}^{\overline{B}_{j_1}\cup\dots\cup\overline{B}_{j_k}}_{B_{i_1},\dots B_{i_n}}$. By definition,

$$X^{-1}\left(\mathcal{F}^{\overline{B}_{j_1}\cup\dots\cup\overline{B}_{j_k}}_{B_{i_1},\dots B_{i_n}}\right) = X^{-1}\left(\mathcal{F}_{B_{i_1}}\right)\cap\dots\cap X^{-1}\left(\mathcal{F}_{B_{i_n}}\right)\cap X^{-1}\left(\mathcal{F}^{\overline{B}_{j_1}}\right)\cap\dots\cap X^{-1}\left(\mathcal{F}^{\overline{B}_{j_k}}\right).$$

The right hand side of this equation is a finite intersection of Σ^0_1 sets. Such a finite intersection is again Σ^0_1 because Ω has a computable intersection function. Therefore if $\mathcal{B}$ is a basic open of $\mathcal{F}(\mathbb{E})$, then $X^{-1}(\mathcal{B})$ is a Σ^0_1 subset of Ω. Moreover, this correspondence is uniform over the standard enumeration of the standard basis for $\mathcal{F}(\mathbb{E})$.

Let $\mathcal{B}_0, \mathcal{B}_1, \dots$ be the standard enumeration of the standard basis for $\mathcal{F}(\mathbb{E})$. Let $\{\mathcal{V}_i\}_{i\in\omega}$ be a $\mathbf{P}_X$-Hippocrates test. By the preceding, $\left\{X^{-1}(\mathcal{B}_j)\right\}$ is a uniformly Σ^0_1 sequence of subsets of $\mathbb{E}$. Furthermore, by definition

$$P\left(X^{-1}(\mathcal{V}_i)\right) = \mathbf{P}_X(\mathcal{V}_i) \le 2^{-i}.$$

Therefore $\left\{X^{-1}(\mathcal{V}_i)\right\}_{i\in\omega}$ is a P-Hippocrates test.

Suppose $F \in \mathcal{F}(\mathbb{E})$ is not $\mathbf{P}_X$-Hippocrates random. Then there is an $\mathbf{P}_X$-Hippocrates test $\{\mathcal{V}_i\}_{i\in\omega}$ such that $F \in \bigcap_{i\in\omega}\mathcal{V}_i$. Hence

$$X^{-1}(\{F\}) \subseteq X^{-1}\left(\bigcap_{i\in\omega}\mathcal{V}_i\right) = \bigcap_{i\in\omega} X^{-1}(\mathcal{V}_i).$$

But $\{X^{-1}(\mathcal{V}_i)\}_{i\in\omega}$ is a P-Hippocrates test and therefore $X^{-1}(\{F\})$ consists entirely of non-P-Hippocrates random elements (or is $\emptyset$). □

Lemma 3.7. *Let Ω be a Martin-Löf probability space with basis $B_0, B_1, \dots$ and measure P, let $\mathbb{E}$ be an LCHS space, and let $X : \Omega \to \mathcal{F}(\mathbb{E})$ be a measurable map such that for each $i \in \omega$ there is a uniformly computable Σ^0_1 set $\mathcal{H}_i \subseteq \mathcal{F}(\mathbb{E})$ with $X^{-1}(\mathcal{H}_i) = B_i$. Under these conditions, if $F \in \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_X$-Hippocrates random, then $X^{-1}(\{F\})$ consists entirely of P-Hippocrates random elements of Ω (or is empty).*

Proof. For each $i \in \omega$ let $\mathcal{H}_i \subseteq \mathcal{F}(\mathbb{E})$ be the Σ^0_1 set such that $X^{-1}(\mathcal{H}_i) = B_i$. Let $\{U_i\}_{i\in\omega}$ be a P-Hippocrates test. By definition there is a c.e. $f \in 2^\omega$ such that $U_i = \bigcup_{f(\langle i,j\rangle)=1} B_j$. Define for each $i \in \omega$

$$\mathcal{V}_i = \bigcup_{f(\langle i,j\rangle)=1} \mathcal{H}_j.$$

Our hypotheses ensure that $\{\mathcal{V}_i\}_{i\in\omega}$ is uniformly Σ^0_1. Furthermore $X^{-1}(\mathcal{V}_i) = U_i$ and thus,

$$\mathbf{P}_X\left(\mathcal{V}_i\right) = P\left(X^{-1}\left(\mathcal{V}_i\right)\right) = P\left(U_i\right) \leq 2^{-i}.$$

Therefore $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}_X$-Hippocrates test.

Now suppose that $F \in \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_X$-Hippocrates random. Then $F \notin \bigcap_{i\in\omega} \mathcal{V}_i$. Equivalently, $\{F\} \cap \left(\bigcap_{i\in\omega} \mathcal{V}_i\right) = \emptyset$. By applying X^{-1} to this equation we find that $X^{-1}\left(\{F\}\right) \cap \left(\bigcap_{i\in\omega} U_i\right) = \emptyset$. This is true for every P-Hippocrates test $\{U_i\}_{i\in\omega}$ and therefore $X^{-1}(\{F\})$ consists entirely of P-Hippocrates random elements of Ω (or is empty). □

4. Martin-Löf random closed sets

In this section we apply the framework developed in the previous section to some specific spaces and RACS. Of particular interest are RACS in Cantor space, in part because of previous work in the study of algorithmically random closed subsets of Cantor space. We will examine some of these previous approaches. In sections 4.2 and 4.3 we explore the approaches of [2] and of [8] and show that they are compatible with the framework developed here. It is also worth noting that the random fractal constructions of [6] are also compatible, though we will not address this in any detail here. Greater detail on Martin-Löf randomness arising from all of these RACS can be found in [1]. We are thus able to unite all of these definitions for algorithmically closed subsets of 2^ω under a unified framework as Martin-Löf randomness arising from different measures on the space of closed sets $\mathcal{F}(2^\omega)$.

Working in this framework we are able to take advantage of the theorems of probability theory, in particular of the theory of RACS. This allows for new (simple) proofs of some existing results as well as some new results. A sequel to this paper deals with Martin-Löf random closed sets for generalized Poisson processes, an important measure from the study of RACS. This measure has not been used in algorithmic randomness before and its importance as an example in probability theory makes it worthy of consideration in such a context.

4.1. **Martin-Löf random closed subsets of $\mathbb{N}$.** We begin with what might be considered a test-case for the framework we have developed: the closed subsets of $\mathbb{N}$ with the discrete topology. In this case every subset is closed and we can thus identify $\mathcal{F}(\mathbb{N})$ with 2^ω. We already have (many) notions of algorithmic randomness for the space 2^ω and our new framework potentially gives rise to a new algorithmic randomness for 2^ω. As it turns out, however, the Fell topology for $\mathcal{F}(\mathbb{N})$ is exactly the standard topology for 2^ω, even considering computational issues. Consequently, the Martin-Löf random elements of 2^ω are identical whether we proceed by the usual route (as described in section 2.1) or by the novel route developed in section 3.

Proposition 4.1. *The topology generated by the cylinders $[\sigma]$ for $\sigma \in 2^{<\omega}$ (the standard topology) is the same as the Fell topology for $\mathcal{F}(\mathbb{N}) = 2^\omega$.*

Proof. We begin by proving that the standard topology contains the Fell topology. By theorem 2.11 the Fell topology for $\mathcal{F}(\mathbb{N}) = 2^\omega$ has a sub-basis consisting of sets of the form $\mathcal{F}^{\{a\}}$ and $\mathcal{F}_{\{b\}}$ where $a, b \in \mathbb{N}$. The basic open sets of the Fell topology are finite intersections of such sets, so it will suffice to prove that $\mathcal{F}_{\{b\}}$ is clopen in the standard topology. This is sufficient because $\mathcal{F}^{\{a\}} = \left(\mathcal{F}_{\{a\}}\right)^{\complement}$ and the complement of a clopen set is itself clopen. The following calculation shows that $\mathcal{F}_{\{b\}}$ is a finite union of cylinders and hence clopen:

$$\mathcal{F}_{\{b\}} = \bigcup \{[\sigma] : |\sigma| = b + 1 \ \& \ \sigma(b) = 1\} .$$

To show that the standard topology is contained in the Fell topology we show that for any $\sigma \in 2^{<\omega}$ the cylinder $[\sigma]$ is open in the Fell topology.

$$[\sigma] = \left(\bigcap \left\{\mathcal{F}^{\{n\}} : \sigma(n) = 0\right\}\right) \cap \left(\bigcap \left\{\mathcal{F}_{\{n\}} : \sigma(n) = 1\right\}\right) .$$

This set is (basic) open in the Fell topology.

Therefore the two topologies coincide. □

A closer look at the proof reveals that we have shown that $[\sigma]$ is a basic open set in the Fell topology and that any basic open in the Fell topology is clopen in the standard topology. Moreover the correspondence is computable (by following the calculations above). Hence any subset of 2^ω is Σ^0_1 in the standard topology if and only if it is Σ^0_1 in the Fell topology. This means that Martin-Löf tests in the Fell topology are exactly the same as Martin-Löf tests in the standard topology.

4.2. **BBCDW-random closed sets.** This section deals with the approach to algorithmically random closed subsets of Cantor space taken in [2] which relies on a coding the closed subsets of 2^ω as ternary reals. An algorithmically random closed set is then defined to be any closed set with a Martin-Löf random code (a more detailed review of [2] follows shortly). Because we wish to discuss other approaches to algorithmically random closed sets, we will instead call the closed sets of [2] BBCDW-random closed sets.

We prove (in lemma 4.5) that the coding of closed sets as ternary reals used in [2] is compatible with the Fell topology and in fact gives rise to a RACS which we call the "canonical decoding" and denote by Z. We note that this result was also proved (in a different way) by [3]. A direct result of lemma 4.5 is that a closed set $F \subseteq 2^\omega$ is BBCDW-random if and only if it is $\mathbf{P}_Z$-Martin-Löf random, where $\mathbf{P}_Z$ is the measure induced by the canonical decoding, Z, on $\mathcal{F}(2^\omega)$. This allows us to explore the BBCDW-random closed sets using the theory of random closed sets.

The development of BBCDW-random closed sets depends on the characterization of the closed sets of Cantor space as the sets of paths through trees. In particular we use the one-to-one correspondence of extensible trees and closed subsets of Cantor space.

Definition 4.2 (Barmpalias et al. [2])**.** Let $F \subseteq 2^\omega$ be nonempty and closed and let $T_F \subseteq 2^{<\omega}$ be the unique extensible tree such that $F = [T_F]$ (the set of paths through the tree). We code T_F as a ternary real, h_F, as follows. Enumerate the nodes of T_F in order (by length and then lexicographically), starting with the empty string, $\lambda = \tau_0, \tau_1, \tau_2, \ldots$.

$$h_F(n) = \begin{cases} 2 & \text{if } \tau_n{}^\frown 0 \in T_F \text{ and } \tau_n{}^\frown 1 \in T_F \\ 1 & \text{if } \tau_n{}^\frown 0 \notin T_F \text{ and } \tau_n{}^\frown 1 \in T_F \\ 0 & \text{if } \tau_n{}^\frown 0 \in T_F \text{ and } \tau_n{}^\frown 1 \notin T_F \end{cases}$$

This coding is called the *canonical coding* of F as a ternary real.

Proposition 4.3 (Barmpalias et al. [2])**.** *The canonical coding is a bijection between the collection of nonempty closed subsets of* 2^ω *and* 3^ω*.*

The central definition of [2] is the following.

Definition 4.4 (Barmpalias et al. [2])**.** A nonempty closed set $F \subseteq 2^\omega$ is *BBCDW-random* if its canonical code $h_F \in 3^\omega$ is Martin-Löf random (with respect to the fair 3-sided coin measure on 3^ω).

To bring the canonical coding of a closed set into the probability framework let

$$Z : 3^\omega \to \mathcal{F}(2^\omega)$$

be the inverse map of the canonical coding (the *canonical decoding*). We wish to prove that Z is a RACS. In fact we can do better: we prove that Z is a homeomorphism between 3^ω and $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ and moreover, Z and Z^{-1} both preserve the complexity of sets.

By lemma 3.5 $\mathcal{F}(2^\omega)$ is a Martin-Löf space with the standard basis consisting of the sets

$$\mathcal{F}^{[\sigma_{m+1}]\cup\cdots\cup[\sigma_n]}_{[\sigma_1],\ldots,[\sigma_m]}$$

for $\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \ldots, \sigma_n \in 2^{<\omega}$. Let $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ be the standard enumeration of this basis.

Lemma 4.5. *$Z : 3^\omega \to \mathcal{F}(2^\omega) \setminus \{\emptyset\}$ is a homeomorphism that preserves the complexity of sets, i.e. if $A \subseteq 3^\omega$ is Σ^0_1, then $Z(A) \subseteq \mathcal{F}(2^\omega)$ is Σ^0_1 and if $\mathcal{A} \subseteq \mathcal{F}$ is Σ^0_1, then $Z^{-1}(\mathcal{A})$ is Σ^0_1.*

Proof. We already know that Z is a bijection. It remains to be shown that Z and Z^{-1} are both continuous and that both preserve complexity.

We first prove that for any $\sigma \in 2^{<\omega}$ $Z^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ is clopen. The map Z is given by decoding a real, f, into an extensible tree, T_f, and then taking the paths through that tree. That is, $f \mapsto [T_f]$. The tree T_f is extensible (has no dead ends) and hence,

$$Z^{-1}(\mathcal{F}_{[\sigma]}) = \{f \in 3^\omega : \sigma \in T_f\}.$$

But $\sigma \in T_f$ if and only if $\forall n \leq |\sigma|$ if k is the coding location for $\sigma \upharpoonright n$, then $f(k) = \sigma(n)$ or $f(k) = 2$. The coding locations for $\sigma \upharpoonright n$ with $n \leq |\sigma|$ all occur in the first $2^{|\sigma|+1}$ bits of f.

It follows that if $g \in Z^{-1}(\mathcal{F}_{[\sigma]})$, then

$$[g \upharpoonright 2^{|\sigma|+1}] \subseteq Z^{-1}\left(\mathcal{F}_{[\sigma]}\right).$$

Consequently there are $\tau_1, \ldots, \tau_m \in 2^{|\sigma|+1}$ such that

$$Z^{-1}\left(\mathcal{F}_{[\sigma]}\right) = [\tau_1] \cup \cdots \cup [\tau_m].$$

Therefore $Z^{-1}(\mathcal{F}_{[\sigma]})$ is clopen. Moreover, finding $\tau_1, \ldots, \tau_m \in 2^{|\sigma|+1}$ is uniformly computable (over $\sigma \in 2^{<\omega}$): we simply decode each $\tau \in 2^{|\sigma|+1}$ into a (finite) tree and check to see if σ is in that tree.

Now $Z^{-1}\left(\mathcal{F}^{[\sigma]}\right) = \left[Z^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right]^{\complement}$ and hence is also clopen and uniformly computable from σ. Therefore if $\mathcal{B} = \mathcal{F}^{[\sigma_{n+1}]\cup\cdots\cup[\sigma_k]}_{[\sigma_1],\ldots,[\sigma_n]}$ is a basic open set of $\mathcal{F}$, then

$$Z^{-1}(\mathcal{B}) = Z^{-1}\left(\mathcal{F}_{[\sigma_1]}\right) \cap \cdots \cap Z^{-1}\left(\mathcal{F}_{[\sigma_n]}\right) \cap Z^{-1}\left(\mathcal{F}^{[\sigma_{n+1}]}\right) \cap \cdots \cap Z^{-1}\left(\mathcal{F}^{[\sigma_k]}\right),$$

a finite intersection of uniformly computable clopen sets in 3^ω. Such an intersection is again uniformly computably clopen. Therefore for any basic open set $\mathcal{B}$, $Z^{-1}(\mathcal{B})$ is clopen and uniformly computable from the canonical index for $\mathcal{B}$. This means that Z is continuous and that if $\mathcal{A} \subseteq \mathcal{F}$ is Σ^0_1, then $Z^{-1}(\mathcal{A})$ is Σ^0_1.

We now consider $Z([\tau])$ for $\tau \in 3^{<\omega}$. We prove by induction on τ that $Z([\tau])$ is a basic open set of $\mathcal{F}$.

Base case: $\tau = \lambda$ (the empty string).

$$Z([\lambda]) = \mathcal{F} \setminus \{\emptyset\} = \mathcal{F}_{2^\omega}.$$

Suppose now that

$$Z([\tau]) = \mathcal{F}^{[\sigma_{m+1}]\cup\cdots\cup[\sigma_n]}_{[\sigma_1],\ldots,[\sigma_m]}$$

where $\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \ldots, \sigma_n \in 2^{<\omega}$. Then $\tau^\frown i$ codes for the branching at some node $\sigma \in 2^{<\omega}$ which we can compute uniformly from τ. We the have the following:

$$\begin{aligned} Z([\tau^\frown 0]) &= \mathcal{F}^{[\sigma_{m+1}]\cup\cdots\cup[\sigma_n]\cup[\sigma^\frown 1]}_{[\sigma_1],\ldots,[\sigma_m],[\sigma^\frown 0]} \\ Z([\tau^\frown 1]) &= \mathcal{F}^{[\sigma_{m+1}]\cup\cdots\cup[\sigma_n]\cup[\sigma^\frown 0]}_{[\sigma_1],\ldots,[\sigma_m],[\sigma^\frown 1]} \\ Z([\tau^\frown 2]) &= \mathcal{F}^{[\sigma_{m+1}]\cup\cdots\cup[\sigma_n]}_{[\sigma_1],\ldots,[\sigma_m],[\sigma^\frown 0],[\sigma^\frown 1]}. \end{aligned}$$

Hence $Z([\tau])$ is a basic open set for any $\tau \in 3^\omega$ and moreover, the canonical index for this basic open set is uniformly computable from τ. Therefore Z^{-1} is continuous and so we have shown that Z is a homeomorphism. In addition, it follows that if $A \subseteq 3^\omega$ is Σ^0_1, then $Z(A)$ is uniformly Σ^0_1. □

We first note that a direct consequence of lemma 4.5 is that the measure $\mathbf{P}_Z$ is Turing equivalent to the measure on 3^ω (in the sense that any representation of one of the measures must be a representation of the other). This simplifies many computability concerns, in particular those in the statements of lemmas 3.6 and 3.7 (which we apply shortly). In this section we are mostly thinking of the fair-coin measure on 3^ω since that is what was considered in [2], though we note that these results hold for other measures. With the fair coin measure on 3^ω the following corollary asserts that a closed set is BBCDW-random if and only if it is $\mathbf{P}_Z$-Martin-Löf random.

Corollary 4.6. *A element $f \in 3^\omega$ is Martin-Löf random if and only if $Z(f)$ is $\mathbf{P}_Z$-Martin-Löf random.*

Proof. ($\Rightarrow$) By lemma 4.5 $Z^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ and $Z^{-1}\left(\mathcal{F}^{[\sigma]}\right)$ are uniformly Σ^0_1 for $\sigma \in 2^{<\omega}$. Therefore by lemma 3.6, if $f \in 3^\omega$ is Martin-Löf random, then $Z(f)$ is $\mathbf{P}_Z$-Martin-Löf random.

($\Leftarrow$) Suppose that $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-Martin-Löf random. First note that $\mathcal{F}^{2^\omega} = \{\emptyset\}$ is a basic open set in $\mathcal{F}(2^\omega)$. Furthermore $Z^{-1}\left(\mathcal{F}^{2^\omega}\right) = \emptyset$ and so $\mathbf{P}_Z\left(\mathcal{F}^{2^\omega}\right) = 0$. Therefore $\emptyset$ is not $\mathbf{P}_Z$-random and hence $F \neq \emptyset$.

By proposition 4.3 Z is a bijection between 3^ω and $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ and so we know that $Z^{-1}(F)$ exists. We now wish to show that Z satisfies the hypotheses of lemma

3.7. Suppose that $\tau \in 3^{<\omega}$. Then τ codes for a finite tree with terminal nodes $\sigma_1, \sigma_2, \ldots, \sigma_n \in 2^{<\omega}$ (and these nodes are uniformly computable from τ). By the definition of the decoding map

$$Z([\tau]) = \mathcal{F}_{[\sigma_1],[\sigma_2],\ldots,[\sigma_n]}^{([\sigma_1]\cup[\sigma_2]\cup\cdots\cup[\sigma_n])^{\complement}}.$$

But Z is a bijection and thus

$$[\tau] = Z^{-1}\left(\mathcal{F}_{[\sigma_1],[\sigma_2],\ldots,[\sigma_n]}^{([\sigma_1]\cup[\sigma_2]\cup\cdots\cup[\sigma_n])^{\complement}}\right).$$

In other words, for each $\tau \in 3^{<\omega}$ there is a basic open set $\mathcal{B}$ such that $Z^{-1}(\mathcal{B}) = [\tau]$ and this basic open set is uniformly computable from τ. Therefore by lemma 3.7 $Z^{-1}(F)$ is Martin-Löf random. ☐

By lemma 4.5 and corollary 4.6 any topological property of the Martin-Löf random elements of 3^ω must be shared by the $\mathbf{P}_Z$-Martin-Löf random closed sets. For example:

Corollary 4.7. *The class of $\mathbf{P}_Z$-Martin-Löf random closed sets is dense in $\mathcal{F}(2^\omega)$.*

This means that for all $\sigma_1, \ldots \sigma_n, \tau_1, \ldots, \tau_k \in 2^{<\omega}$ such that $\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}^{[\tau_1]\cup\cdots\cup[\tau_k]} \neq \emptyset$, there is a $\mathbf{P}_Z$-ML random closed set $F \in \mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}^{[\tau_1]\cup\cdots\cup[\tau_k]}$. In other words, we can specify a finite number of cylinders and know that there is a $\mathbf{P}_Z$-Martin-Löf random closed set contained in the union of those cylinders. For example, there must be a $\mathbf{P}_Z$-Martin-Löf random closed set whose members all have initial bit 0, another whose members all have initial bits 00, and so on.

Lemma 4.5 also means that any notion of randomness based on Martin-Löf tests is the same for the space 3^ω and the space $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ with measure $\mathbf{P}_Z$. In particular, relativizing corollary 4.6 will show that $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-n-Martin-Löf random relative to oracle g if and only if $Z^{-1}(F)$ is n-Martin-Löf random relative to g. If we wish to consider a different measure P on 3^ω, then we can take the oracle to be a representation of P (and $n = 1$) to find the following corollary.

Corollary 4.8. *Let P be any probability measure on 3^ω and let $\mathbf{P}_Z$ be defined by $\mathbf{P}_Z(\mathcal{A}) = P\left(Z^{-1}(\mathcal{A})\right)$ for measurable $\mathcal{A} \subseteq \mathcal{F}(2^\omega)$. Then $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-Martin-Löf random if and only if $Z^{-1}(F)$ is P-Martin-Löf random.*

Proof. Simply add a representation of the measure P everywhere in the proof of corollary 4.6. As already noted (as a consequence of the proof of lemma 4.5) any representation of P is also a representation of $\mathbf{P}_Z$ (and vice versa), so the result follows. ☐

The following theorem about the measure of $\mathbf{P}_Z$-ML random closed sets was proved for the fair coin measures on 2^ω and 3^ω in [2]. We give a different proof (relying on Robbins' theorem) in order to illustrate the utility of our approach to algorithmically random closed sets. The following proof also works naturally for non-computable measures and so we state the theorem in some generality.

Proposition 4.9. *Let μ be any σ-finite Borel measure on 2^ω and let P be any computable, Borel, probability measure on 3^ω such that for μ-almost every $f \in 2^\omega$ $P(\{g \in 3^\omega : f \in Z(g)\}) = 0$. If $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-Martin-Löf random relative to a representation of μ, then $\mu(F) = 0$.*

Proof. By Robbins' theorem (2.17):

$$\begin{aligned}E(\mu \circ Z) &= \int_{2^\omega} P(\{g \in 3^\omega : f \in Z(g)\})\, df \\ &= \int_{2^\omega} 0 df \\ &= 0.\end{aligned}$$

Consequently $\mathbf{P}_Z$-almost every closed set has measure 0. Then by lemma 4.5, P-almost every $g \in 3^\omega$ maps to a closed set with measure 0. We will use this fact to build P-Martin-Löf tests that catch each $g \in 3^\omega$ such that $Z(g)$ has positive measure. It will then follow from corollary 4.8 if F has positive measure, then F is not $\mathbf{P}_Z$-Martin-Löf random.

As in the proof of lemma 4.5, let T_g be the extensible tree coded by $g \in 3^\omega$. For $n \in \omega$ define the level n approximation of $[T_g] = Z(g)$ as follows:

$$A_{g,n} = \bigcup_{\sigma \in T_g \cap 2^n} [\sigma].$$

Then $A_{g,n} \subseteq 2^\omega$ is clopen for each $n \in \omega$ and $\bigcap_{n\in\omega} A_{g,n} = [T_g]$. Moreover, $\mu(A_{g,n})$ is computable from any representation of μ and

$$\mu([T_g]) = \lim_{n\to\infty} \mu(A_{g,n}).$$

Fix $k \in 2^\omega$. Define the set

$$U_n = \left\{g \in 3^\omega : \mu(A_{g,n}) > 2^{-k}\right\}.$$

The set $A_{g,n}$ depends on at most the first 2^{n+1} bits of g and thus if $g \in U_n$, then $\left[g \restriction 2^{n+1}\right] \subseteq U_n$. It follows that U_n is clopen and its P-measure is uniformly computable from any representation of both μ and P. In addition, $U_1 \supseteq U_2 \supseteq \dots$ and $\bigcap_{n\in\omega} U_n = \{g \in 3^\omega : \mu(Z(g)) \geq 2^{-k}\}$. Thus $\lim_{n\to\infty} \mu(U_n) = \mu\left(\{g \in 3^\omega : \mu(Z(g)) \geq 2^{-k}\}\right) = 0$.

Let $V_i = U_n$ where n is the least number such that $P(U_n) \leq 2^{-i}$. Then $\{V_i\}_{i\in\omega}$ is a P-Martin-Löf test relative to any representation of μ and

$$\bigcap_{i\in\omega} V_i = \bigcap_{n\in\omega} U_n = \left\{g \in 3^\omega : \mu(Z(g)) \geq 2^{-k}\right\}.$$

Suppose that $F \in \mathcal{F}$ has $\mu(F) \geq 2^{-k}$. Then $Z^{-1}(F) \in \bigcap_i V_i$. By definition $Z^{-1}(F)$ is non P-Martin-Löf random relative to any representation of μ. Therefore by corollary 4.8 F is not $\mathbf{P}_Z$-Martin-Löf random relative to any representation of μ.

This holds for any $k \in \omega$ and therefore if F is $\mathbf{P}_Z$-Martin-Löf random relative to a representation of μ, then $\mu(F) = 0$. □

Note that the condition that $P(\{g \in 3^\omega : f \in Z(g)\}) = 0$ is simply a way of writing $\mathbf{P}_Z\left(\mathcal{F}_{\{f\}}\right) = 0$. This is not a strong condition. For example, any Bernoulli measure on 3^ω satisfies this condition.

4.3. **Galton-Watson random closed sets.** This section explores the approach to algorithmically random closed sets taken by Kjos-Hanssen and Diamondstone [8]. Our goal once again is to show that this approach is compatible with the framework developed in section 3. As in the previous section, the correspondence between binary-branching trees and closed sets of 2^ω will be used to define a RACS. This time the trees are coded in 2^ω (rather than 3^ω) and Kjos-Hanssen and Diamondstone allow for non-extensible trees. The idea is to construct a tree by extending each node according to a pair of coin flips.

Let $f \in 2^\omega$. At stage s we determine which strings of length s are members of the finite tree $T(f)[s]$. The potentially infinite tree $T(f)$ is then defined as $\bigcup_{s\in\omega} T(f)[s]$. This gives rise to a map $X : 2^\omega \to \mathcal{F}(2^\omega)$ defined by $X(f) = [T(f)]$.

We begin by setting $T(f)[0] = \{\lambda\}$. At stage 1 we determine which of the strings 0 and 1 are in $T(f)[1]$: $0 \in T(f)[1]$ if and only if $f(0) = 1$; $1 \in T(f)[1]$ if and only if $f(1) = 1$.

For later stages we simply continue this process. Let $T(f)[s] \subseteq 2^{<\omega}$ be the tree after stage s of the construction. At this point $T(f)[s]$ contains some number of strings, $n(s)$, added according to the first $n(s) - 1$ bits of f (the empty string is always in $T(f)[s]$). The tree $T(f)[s]$ also contains some number of strings of length s: $\sigma_0, \sigma_1, \dots, \sigma_k$ (ordered lexicographically). The next $2(k + 1)$ bits of f are used to determine which extensions of $\sigma_0, \sigma_1, \dots, \sigma_k$ are in the tree $T(f)[s + 1]$.

Definition 4.10. The map $X : 2^\omega \to \mathcal{F}(2^\omega)$ is given by $X(f) = [T(f)]$.

As already mentioned, this map is very similar to the canonical decoding of section 4.2. In this case, however, not every tree is extensible. Classically these two maps are almost identical since in this construction the three possible ways of extending a string given that it does have an extension are equiprobable. The possibility of non-extension does add considerable complication from the computability perspective, however. In [8] these complications are resolved and it is shown that if $f \in 2^\omega$ is Martin-Löf random, then $X(f)$ is either $\emptyset$ or is BBCDW-random (a version of their result is stated as theorem 4.15 below). We will prove that the map X is a RACS and use this to translate the results of [8] into the context of Martin-Löf random closed sets. We also prove a result, lemma 4.21, that used in [8] to prove theorem 4.15.

Throughout the following we wish to consider a more general measure on 2^ω than the fair coin measure. We will generalize to Bernoulli measures, i.e. measures generated by flipping a biased coin.

Definition 4.11. Let $p \in (0, 1)$. The *Bernoulli measure with parameter* p is the Borel measure μ_p on 2^ω such that:

(1) $\forall \sigma \in 2^{<\omega}\ \mu_p\left([\sigma^\frown 1]\right) = p\mu_p([\sigma])$ and
(2) $\mu_p(2^\omega) = 1$.

Random trees produced from 2^ω with the measure μ_p in the method described above are called (binary branching) Galton-Watson trees with survival probability p. Such trees were originally studied in the 19^{th} century (by Sir Francis Galton) in the context of the extinction of noble surnames. A basic result in the theory of these trees is the following lemma about the probability of the existence of a path through a Galton-Watson tree (the existence of such a path means that a surname does not become extinct). The result that we will actually use is the easy corollary 4.13, also due to Galton and Watson.

Lemma 4.12 (Galton and Watson)**.** *The probability that a (binary branching) Galton-Watson tree with survival probability $p \in (0, 1)$ has no (infinite) paths is the least positive solution to the equation*

$$x = p^2x^2 + 2p(1 - p)x + (1 - p)^2.$$

Corollary 4.13. *If T is a (binary branching) Galton-Watson tree with survival probability $p \in (0,1)$, then the probability that T has no infinite paths is 1 if and only if $p \leq \frac{1}{2}$. Otherwise the probability that T has no infinite paths is* $\left(\frac{1-p}{p}\right)^2$.

Having dispensed with these preliminaries, we are now ready to define the Galton-Watson random closed sets of [8] and to state the result.

Definition 4.14 (Diamondstone and Kjos-Hanssen [8])**.** Let $p \in (0,1)$. A closed set $F \subseteq 2^\omega$ is *p-Galton-Watson random* (p-GW random) if there is a μ_p-ML random $f \in 2^\omega$ such that $X(f) = F$.

Theorem 4.15 (Diamondstone and Kjos-Hanssen [8])**.** *A closed set $F \subseteq 2^\omega$ is BBCDW-random if and only if F is $\frac{2}{3}$-GW random and $F \neq \emptyset$.*

We wish to bring this result into the context of Martin-Löf random closed sets. The first step is to apply corollary 4.5. This gives the following.

Corollary 4.16. *Let $Z : 3^\omega \to \mathcal{F}(2^\omega)$ be the canonical decoding map of section 4.2. $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-ML random if and only if F is $\frac{2}{3}$-GW random and $F \neq \emptyset$.*

Proof. This is a direct consequence of theorems 4.15 and corollary 4.6 which showed that a closed set $F \subseteq 2^\omega$ is BBCDW-random if and only if F is $\mathbf{P}_Z$-ML random. □

The next step is to prove that the map X is measurable.

Proposition 4.17. *The map X is a RACS.*

Proof. Consider $X^{-1}\left(\mathcal{F}^{[\sigma]}\right)$. By definition $X^{-1}\left(\mathcal{F}^{[\sigma]}\right) = \{f \in 2^\omega : [T(f)] \cap [\sigma] = \emptyset\}$. The cylinder $[\sigma]$ is compact and so

$$[T(f)] \cap [\sigma] = \emptyset \iff (\exists n \in \omega)\,(\forall \tau \in 2^n)\,(\tau \succeq \sigma \implies \tau \notin T(f)).$$

Thus

$$X^{-1}\left(\mathcal{F}^{[\sigma]}\right) = \{f \in 2^\omega : (\exists n \in \omega)\,(\forall \tau \in 2^n)\,(\tau \succeq \sigma \implies \tau \notin T(f))\}. \tag{1}$$

There are $2^{n+1} - 1$ strings of length at most n. Thus, if $f \in 2^\omega$ and $n \in \omega$ are such that $(\forall \tau \in 2^n)\,(\tau \succeq \sigma \implies \tau \notin T(f))$, then

$$\left[f \upharpoonright 2^{n+1}\right] \subseteq X^{-1}\left(\mathcal{F}^{[\sigma]}\right).$$

Hence the set $X^{-1}\left(\mathcal{F}^{[\sigma]}\right)$ is open (actually Σ^0_1) and thus measurable. Because sets of the form $\mathcal{F}^{[\sigma]}$ generate the Borel σ-algebra on $\mathcal{F}$ this suffices to prove the proposition. □

Let $\mathbf{P}_X$ be the measure induced on $\mathcal{F}(2^\omega)$ by X. Ideally the p-GW random closed sets would be exactly the $\mathbf{P}_X$-Martin-Löf random closed sets. Unfortunately we have only been able to show containment in one direction: every p-GW random closed set is $\mathbf{P}_X$-ML random. Before we prove this we must determine the computational power of $\mathbf{P}_X$.

Lemma 4.18. *Let μ_p the Bernoulli measure on 2^ω with parameter $p \in (\frac{1}{2}, 1)$ and let $\mathbf{P}_X$ be the corresponding measure on $\mathcal{F}(2^\omega)$. Then p is a representation of minimal degree of μ_p and of $\mathbf{P}_X$.*

Proof. It is clear from the definition of Bernoulli measure that if r is a representation of μ_p, then $r \geq_T p$ and, conversely, that p is a representation of μ_p.

Now suppose that r is a representation of the measure $\mathbf{P}_X$ (with respect to the standard basis). The quantity $c = \mathbf{P}_X\left(\mathcal{F}^{2^\omega}\right)$ is the measure of a basic open set of $\mathcal{F}(2^\omega)$ and hence computable from r. But $\mathcal{F}^{2^\omega} = \{\emptyset\}$ and so by corollary 4.13 $c = \left(\frac{1-p}{p}\right)^2$. Consequently $1 - 2p + (1-c)p^2 = 0$ and p can be computed from c using the quadratic equation. Therefore $p \leq_T r$.

Now we wish to show that p is a representation of $\mathbf{P}_X$. We claim that it suffices to show that $p \geq_T \mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],[\sigma_2],\ldots,[\sigma_n]}\right)$ for any $\sigma_1, \sigma_2, \ldots, \sigma_n \in 2^{<\omega}$. Recall the definition of the capacity T_X: $T_X([\sigma]) = \mathbf{P}_X\left(\mathcal{F}_{[\sigma]}\right)$. Complete alternation (definition 2.14) gives a recursive algorithm for computing $\mathbf{P}_X$ for any basic open set of $\mathcal{F}(2^\omega)$ from $T_X([\sigma_1] \cup [\sigma_2] \cup \cdots \cup [\sigma_n])$ for any $\sigma_1, \sigma_2, \ldots, \sigma_n \in 2^{<\omega}$. By applying inclusion-exclusion to the definition of $T_X([\sigma_1] \cup [\sigma_2] \cup \cdots \cup [\sigma_k])$ we find

$$\begin{aligned}
T_X([\sigma_1] \cup [\sigma_2] \cup \cdots \cup [\sigma_k]) &= \mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]\cup[\sigma_2]\cup\cdots\cup[\sigma_k]}\right) \\
&= \mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]} \cup \mathcal{F}_{[\sigma_2]} \cup \cdots \cup \mathcal{F}_{[\sigma_k]}\right) \\
&= \sum_{1\leq i\leq k} \mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]}\right) \\
&\quad - \sum_{1\leq i<j\leq k} \mathbf{P}_X\left(\mathcal{F}_{[\sigma_i],[\sigma_j]}\right) + \ldots \\
&\quad + (-1)^{k-1}\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],\ldots,[\sigma_k]}\right).
\end{aligned}$$

Therefore the capacity $T_X([\sigma_1] \cup [\sigma_2] \cup \cdots \cup [\sigma_k])$ can be computed if we know $\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],[\sigma_2],\ldots,[\sigma_n]}\right)$ for every $\sigma_1, \sigma_2, \ldots, \sigma_n \in 2^{<\omega}$. This proves the claim that we need only to show that p allows us to compute the measures of such sets.

We now find a formula for calculating $\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}\right)$ from p. If there are $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $[\sigma_i] \cap [\sigma_j] \neq \emptyset$, then we know that $[\sigma_i] \cap [\sigma_j] = [\sigma_i]$ (by swapping indices if necessary). Consequently $\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]} = \mathcal{F}_{[\sigma_1],\ldots,[\sigma_{j-1}],[\sigma_{j+1}],\ldots,[\sigma_n]}$ and so we can drop $[\sigma_j]$ from the expression. We can thus assume, without loss of generality, that $[\sigma_i] \cap [\sigma_j] = \emptyset$ for every $i \neq j \in \{1, \ldots, n\}$.

Let $S = \{\tau \in 2^{<\omega} : (\exists i \leq n)\tau \preceq \sigma_i\}$, i.e. the finite tree of all predecessors of $\sigma_1, \ldots, \sigma_n$. Let $N = |S| - 1$ (we subtract 1 because by construction all trees include the empty string λ). In particular, if $S \subseteq T(f)$, then the N bits of f coding for the non-empty strings of S must be 1. Hence $\mu_p(\{f \in 2^\omega : S \subseteq T(f)\}) = p^N$.

Our construction is self-similar in the sense that if we know $\sigma \in 2^{<\omega}$ is in our tree, then the probability that the sub-tree of extensions of σ has an infinite path is exactly $1 - \left(\frac{1-p}{p}\right)^2$. In addition, the sub-trees of extensions of $\sigma_1, \ldots, \sigma_n$ are (statistically) independent.

Hence

$$\begin{aligned}
\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}\right) &= \mu_p\left(X^{-1}\left(\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}\right)\right) \\
&= \mu_p\left(\{f \in 2^\omega : S \subseteq T(f) \;\&\; (\forall i \leq n)[T(f)] \cap [\sigma_i] \neq \emptyset\}\right) \\
&= p^N\left(1 - \left(\frac{1-p}{p}\right)^2\right)^n.
\end{aligned} \tag{2}$$

We have thus shown how to compute the measure of the set $\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}$ from p. Therefore p is a representation of $\mathbf{P}_X$. $\square$

A direct consequence of this lemma is that computational concerns regarding representations of the measures μ_p and $\mathbf{P}_X$ are greatly simplified. In this case representations of these measures have the same minimal Turing degree, and hence the only representation we need to consider is p itself.

We are now ready to prove that the $\mathbf{P}_X$-ML random closed sets include the p-GW random closed sets. Recall that the p-GW random closed sets are exactly the image under X of μ_p-ML random elements of 2^ω. Consequently, we actually need to show if $f \in 2^\omega$ is μ_p-ML random, then $X(f)$ is $\mathbf{P}_X$-ML random. This is very much in the vein of lemma 3.6, however, the hypotheses of the lemma are not satisfied by the RACS X.

Lemma 4.19. *Let $p \in (0,1)$. If $f \in 2^\omega$ is μ_p-ML random, then $X(f)$ is $\mathbf{P}_X$-ML.*

Proof. We have two cases to consider: $p \leq \frac{1}{2}$ and $p > \frac{1}{2}$.

Case 1: $p \leq \frac{1}{2}$. In this case $\mathbf{P}_X(\{\emptyset\}) = 1$ by corollary 4.13. This means that $\mathbf{P}_X\left(\mathcal{F}_{2^\omega}\right) = 0$. Every non-empty closed subset of 2^ω is in $\mathcal{F}_{2^\omega}$ and therefore $\emptyset$ is the only $\mathbf{P}_X$-ML random closed set. Thus we need to prove that if $f \in 2^\omega$ is μ_p-ML random, then $X(f) = \emptyset$.

Applying equation 1 of proposition 4.17 to $\{\emptyset\} = \mathcal{F}^{2^\omega}$ gives

$$X^{-1}\left(\{\emptyset\}\right) = \{f \in 2^\omega : (\exists n \in \omega)(\forall \tau \in 2^n)\tau \notin T(f)\}$$

and shows that this set is Σ^0_1. We also know that $\mu_p\left(X^{-1}\left(\{\emptyset\}\right)\right) = 1$. A Σ^0_1 set of measure 1 must contain every μ_p-ML random $f \in 2^\omega$. Therefore if $f \in 2^\omega$ is μ_p-ML random, then $X(f) = \emptyset$. This finishes case 1.

Case 2: $p > \frac{1}{2}$. In this case $0 < \mathbf{P}_X(\{\emptyset\}) = \left(\frac{1-p}{p}\right)^2 < 1$ by corollary 4.13. We prove the contrapositive: if $F \in \mathcal{F}(2^\omega)$ is not $\mathbf{P}_X$-ML random and $X(f) = F$, then f is not p-ML random. We would like to apply lemma 3.6 but this is not possible: $X^{-1}(\mathcal{F}_{[\sigma]})$ is Π^0_1 and not Σ^0_1 as required in the hypotheses of the lemma. The technique of this proof, however, is similar to the technique of the proof of lemma 3.6. In particular, the goal is to take a $\mathbf{P}_X$-ML test in $\mathcal{F}(2^\omega)$ and pull it back via X to a μ_p-ML test in 2^ω. As a result of lemma 4.18 it is sufficient to work with the single oracle p, rather than different representations of the measures μ_p and $\mathbf{P}_X$.

In order to proceed we approximate the Π^0_1 sets $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ by clopen sets:

$$A_{\sigma,s} = \left\{f \in 2^\omega : \left(\exists \tau \in 2^{|\sigma|+s}\right) \tau \succeq \sigma \;\&\; \tau \in T(f)\right\}.$$

The construction of $T(f)$ ensures that for each $\sigma \in 2^{<\omega}$ and each $s \in \omega$, $A_{\sigma,s}$ is clopen and $A_{\sigma,0} \supseteq A_{\sigma,1} \supseteq \cdots \supseteq X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$. Furthermore, if $f \in \bigcap_{s\in\omega} A_{\sigma,s}$, then $T(f)$ contains an extension of σ of every length. Thus $[T(f)] \cap [\sigma] \neq \emptyset$. By definition $X(f) = [T(f)]$ and so it follows that

$$\bigcap_{s\in\omega} A_{\sigma,s} = X^{-1}\left(\mathcal{F}_{[\sigma]}\right).$$

Now we wish to calculate $\mu_p\left(X^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right)$. We can apply equation 2 from the proof of lemma 4.18

$$\mu_p\left(X^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right) = p^{|\sigma|}\left(1 - \left(\frac{1-p}{p}\right)^2\right),$$

which is computable using oracle p. Because $A_{\sigma,s}$ is clopen for every $s \in \omega$ we can compute (using oracle p) the measure $\mu_p\left(A_{\sigma,s}\right)$ and the exact error of the approximation

$$\epsilon(\sigma, s) = \mu_p\left(A_{\sigma,s}\right) - p^{|\sigma|}\left(1 - \left(\frac{1-p}{p}\right)^2\right).$$

Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a $\mathbf{P}_X$-ML test. We produce a μ_p-ML test $\{V_i\}_{i\in\omega}$. To define V_k we consider $\mathcal{U}_{k+1}$. If $\mathcal{F}^C_{[\sigma_1],\ldots,[\sigma_n]}$ is the s^{th} basic open set enumerated into $\mathcal{U}_{k+1}$ then we find $t \in \omega$ large enough so that

$$\sum_{j=1}^{n} \epsilon(\sigma_j, t) \leq 2^{-(k+1+s)}. \tag{3}$$

This ensures that the total error of approximating $X^{-1}\left(\mathcal{F}^C_{[\sigma_1],\ldots,[\sigma_n]}\right)$ by the Σ^0_1 set $X^{-1}\left(\mathcal{F}^C\right) \cap A_{\sigma_1,t} \cap \cdots \cap A_{\sigma_n,t}$ is sufficiently small.

We then add the Σ^0_1 set $X^{-1}\left(\mathcal{F}^C\right) \cap A_{\sigma_1,t} \cap \cdots \cap A_{\sigma_n,t}$ to V_k. By equation 3

$$\mu_p(V_k) \leq \mathbf{P}_X(\mathcal{U}_{k+1}) + \sum_{s=1}^{\infty} 2^{-(k+1+s)} \leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

Furthermore, $\{V_i\}_{i\in\omega}$ is uniformly $\Sigma^{0,p}_1$ and is thus a p-ML test.

Suppose that $F \in \bigcap_{i\in\omega} \mathcal{U}_i$. Then $X^{-1}(\{F\}) \subseteq \bigcap_{i\in\omega} V_i$ by our definition of V_i. Moreover, this holds for every $\mathbf{P}_X$-ML test $\{\mathcal{U}_i\}_{i\in\omega}$. Therefore if $F \in \mathcal{F}(2^\omega)$ is not $\mathbf{P}_X$-ML random and $X(f) = F$, then f is not μ_p-ML random. This completes case 2 and the proof as a whole. $\square$

Because $\emptyset$ is an atom of the measure $\mathbf{P}_X$ it must be $\mathbf{P}_X$-ML random. Note that $X^{-1}(\{\emptyset\})$ is Σ^0_1 and has positive measure and so must contain both random and non-random elements of 2^ω. This is worth emphasizing: in this case non-μ_p-ML random elements of 2^ω map to a $\mathbf{P}_X$-ML random element of $\mathcal{F}(2^\omega)$. Of course, this $\mathbf{P}_X$-ML random closed set is $\emptyset$, which is unusual in that it is isolated in $\mathcal{F}(2^\omega)$ and and atom of the measure $\mathbf{P}_X$. The question of whether other $\mathbf{P}_X$-ML random closed sets are also the image of non-μ_p-ML random elements of 2^ω remains open.

The following lemma (4.21) in the proof of theorem 4.15 in [8]. This lemma connects the map X and the canonical decoding $Z : 3^\omega \to \mathcal{F}(2^\omega)$ of section 4.2. As in section 4.2, let $\mathbf{P}_Z$ be the measure induced on $\mathcal{F}(2^\omega)$ by Z (and the "fair coin" measure on 3^ω). We establish this connection by considering the space $(\mathcal{F}(2^\omega), \mathbf{P}_Z)$ and a measure, ν_Z, on 2^ω such that $\mathbf{P}_Z(\mathcal{H}) = \nu_Z\left(X^{-1}(\mathcal{H})\right)$ for measurable $\mathcal{H} \subseteq \mathcal{F}(2^\omega)$.

Definition 4.20 (Diamondstone and Kjos-Hanssen [8])**.** Define a Borel probability measure ν_Z on 2^ω as follows:

1. $\nu_Z\left(2^\omega\right) = 1$
2. If σ has even length, then

$$\nu_Z\left([\sigma^\frown 01]\right) = \nu_Z\left([\sigma^\frown 10]\right) = \nu_Z\left([\sigma^\frown 11]\right) = \frac{1}{3}\nu_Z\left([\sigma]\right)$$

and

$$\nu_Z\left([\sigma^\frown 00]\right) = 0.$$

Note that this defines ν_Z on the entire Borel σ-algebra since the measure of the cylinders determined by strings with odd length is implicitly set. For example, $\nu_Z([0]) = \nu_Z([00]) + \nu_Z([01]) = \frac{1}{3}$ and $\nu_Z([1]) = \nu_Z([10]) + \nu_Z([11]) = \frac{2}{3}$.

Lemma 4.21. *The following are equivalent:*

(1) *A closed set $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-ML random;*
(2) *There is some ν_Z-ML random $f \in 2^\omega$ such that $X(f) = F$;*
(3) *Every $f \in 2^\omega$ such that $X(f) = F$ is ν_Z-ML random.*

Proof. We first note that both $\mathbf{P}_Z$ and ν_Z are computable measures (the computability of $\mathbf{P}_Z$ follows from lemma 4.5 and that of ν_Z is clear from the definition). The bulk of this proof consists of showing that $X(f)$ is $\mathbf{P}_Z$-ML random if and only if f is ν_Z-ML random. Because X is surjective the three statements are then clearly equivalent.

We begin by proving that if $X(f) \in \mathcal{F}(2^\omega)$ is not $\mathbf{P}_Z$-ML random, then f is not ν_Z-ML random. As in the proof of lemma 4.19 we must work around the fact that $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ is Π^0_1. As before we approximate this set by clopen sets. This time, however, the approximation is much easier. Define

$$A_\sigma = \{f \in 2^\omega : \sigma \in T(f)\}.$$

Then $A_\sigma \supseteq X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ and A_σ is clopen. Moreover

$$\nu_Z(A_\sigma) = \nu_Z\left(X^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right). \tag{4}$$

These measures are equal because under the measure ν_Z almost every tree is extensible.

Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a $\mathbf{P}_Z$-ML test in $\mathcal{F}(2^\omega)$. Because $\mathbf{P}_Z$ and ν_Z are computable we do not need to worry about the oracles $\mathbf{P}_Z$ or ν_Z. We construct a ν_Z-ML test $\{V_i\}_{i\in\omega}$ as follows. If $\mathcal{F}^C_{[\sigma_1],\ldots,[\sigma_n]}$ is enumerated into $\mathcal{U}_i$, then we enumerate

$$A_{\sigma_1} \cap \cdots \cap A_{\sigma_n} \cap X^{-1}\left(\mathcal{F}^C\right)$$

into V_i. It is clear that $\{V_i\}_{i\in\omega}$ is a uniformly Σ^0_1 sequence. By equation 4

$$\begin{aligned}\nu_Z(V_i) &= \nu_Z\left(X^{-1}(\mathcal{U}_i)\right)\\ &= \mathbf{P}_Z(\mathcal{U}_i)\\ &\leq 2^{-i}.\end{aligned}$$

Therefore $\{V_i\}_{i\in\omega}$ is a ν_Z-ML test.

By construction, if $X(f) \in \bigcap_{i\in\omega}\mathcal{U}_i$, then $f \in \bigcap_{i\in\omega} V_i$. This holds for every $\mathbf{P}_Z$-ML test $\{\mathcal{U}_i\}_{i\in\omega}$. Therefore, if $X(f)$ is not $\mathbf{P}_Z$-ML random, then f is not ν_Z-ML random.

Now we prove the converse: if $f \in 2^\omega$ is not ν_Z-ML random, then $X(f)$ is not $\mathbf{P}_Z$-ML random. Each $\sigma \in 2^{<\omega}$ determines a finite binary tree $T(\sigma)$ as described in definition 4.10. This tree has a finite number of nodes, $\tau_1, \tau_2, \ldots, \tau_n$, whose extensions have not yet been completely determined. Define $C_\sigma = ([\tau_1] \cup \cdots \cup [\tau_n])^{\complement}$. The set C_σ is the largest set such that for any $f \succ \sigma$, $X(f) \cap C_\sigma = \emptyset$. This means that $X([\sigma]) \subseteq \mathcal{F}^{C_\sigma}$. C_σ is also clopen and so $\mathcal{F}^{C_\sigma}$ is a basic open set of $\mathcal{F}(2^\omega)$.

Let $f \in 2^\omega$ and let $\{U_i\}_{i\in\omega}$ be a ν_Z-ML test such that $f \in \bigcap_{i\in\omega} U_i$. Construct a $\mathbf{P}_Z$-ML test $\{\mathcal{V}_i\}_{i\in\omega}$ as follows. If $[\sigma]$ is enumerated into U_i, then $\mathcal{F}^{C_\sigma}$ is enumerated into $\mathcal{V}_i$. Clearly $\{\mathcal{V}_i\}_{i\in\omega}$ is a uniformly Σ^0_1 sequence. Additionally, $X^{-1}(\mathcal{V}_i) = U_i \cup E$ where $E \subseteq \{f \in 2^\omega : T(f)$ is not extensible$\}$. Non-extensible trees occur with probability 0 under ν_Z and hence $\nu_Z(E) = 0$. Consequently $\mathbf{P}_Z(\mathcal{V}_i) = \nu_Z(U_i \cup E) = \nu_Z(U_i) \leq 2^{-i}$. Therefore $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}_Z$-ML test.

Now because $f \in \bigcap_{i\in\omega} U_i$ if follows that $X(f) \in \bigcap_{i\in\omega} \mathcal{V}_i$. Therefore if $f \in 2^\omega$ is not ν_Z-ML random, then $X(f)$ is not a $\mathbf{P}_Z$-ML random closed set.

We have now shown that $X(f) = F$ is $\mathbf{P}_Z$-ML random if and only if f is ν_Z-ML random. Finally, X is a surjection and hence we conclude that statements 1-3 are equivalent.

□

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