MARTIN-LÖF RANDOM GENERALIZED POISSON PROCESSES

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Abstract. Martin-Löf randomness was originally defined and studied in the context of the Cantor space 2^ω. In [1] probability theoretic random closed sets (RACS) are used as the foundation for the study of Martin-Löf randomness in spaces of closed sets. Here we focus on the space of closed subsets of $\mathbb{R}$ and a particular family of measures on this space, the generalized Poisson processes. This gives a novel class of Martin-Löf random closed subsets of $\mathbb{R}$. We begin studying these Martin-Löf random closed sets, proving in part that the elements of these sets are themselves Martin-Löf random.

1. Introduction

Previous work ([1]) used the probability theory of random closed sets (RACS) to extend the definitions of Martin-Löf randomness to spaces of closed sets. This paper continues to study the resultant Martin-Löf random closed sets. Where [1] focused on developing the general theory for closed subsets of locally compact, Hausdorff, second countable spaces we now address only closed subsets of $\mathbb{R}$. In fact, we will deal with only one specific class of measures on the space of closed subsets of $\mathbb{R}$: the generalized Poisson processes. Generalized Poisson processes are an important example from probability theory and we establish a basic understanding of the Martin-Löf random closed sets to which they give rise.

Effort has been made to make this paper reasonably self-contained. Definitions and theorems that were stated in their full (or nearly so) generality in [1] are here stated only for $\mathbb{R}$. In particular, sections 2.2 and 3 deal with special cases of the more general situation described in [1]. These sections give the background necessary for the following sections without dealing with some of the finer details.

It is worth mentioning related work of [5] and [4]. In the former, Martin-Löf randomness is extended to metric spaces. The spaces we study here can be viewed as metric spaces, however our approach does not rely on metrics in any way, instead using only topology and measures. In the latter, Martin-Löf randomness is extended to effective topological spaces equipped with a measure, again a more general class of spaces than is addressed here or in [1]. Our approach is similar to that taken in this paper, but tailored to spaces of closed sets.

2. Background

We give the definitions of Martin-Löf randomness for $\mathbb{R}$ and then cover the Fell topology of the space of closed subsets of $\mathbb{R}$.

2.1. **Martin-Löf Randomness in $\mathbb{R}$.** We fix a computable enumeration $I_0, I_1, I_2, \ldots$ of the set of open rational intervals (that is, open intervals with rational end-points). This means that we can uniformly computably extract from the index i the end-points of the interval I_i. This gives a computable enumeration of a basis for the canonical topology on $\mathbb{R}$: we will use this to determine the algorithmic complexity of subsets of $\mathbb{R}$.

Definition 2.1. Let $h \in 2^\omega$. A set $U \subseteq \mathbb{R}$ is $\Sigma_1^{0,h}$ if there is an h-c.e. $f \in 2^\omega$ such that $U = \bigcup_{f(i)=1} I_i$.

Martin-Löf randomness requires that $\mathbb{R}$ be equipped with a Borel measure, usually the Lebesgue measure, m, which assigns the measure $b - a$ to the interval (a, b). In this case computational concerns are minimized. Other Borel measures may assign different measures to rational intervals, in which case we must deal with the computational issues involved in computing the measures of these intervals. In general, if an oracle allows us compute the measure $\Lambda(I_i)$ uniformly for each $i \in \omega$, then we call the oracle a *representation of the measure* Λ. In [10] it is shown that arbitrary information can be encoded in representations of Borel measures on 2^ω and thus the choice of representation may matter. Moreover, results in [3] establish the existence of Borel measures on 2^ω for which the representations have no minimum Turing degree.

Definition 2.2. Let Λ be a Borel measure on $\mathbb{R}$.

(1) Let $r \in 2^\omega$ be a representation of Λ. An *r-Martin-Löf test (r-ML test)* is a uniformly $\Sigma_1^{0,r}$ sequence of subsets of $\mathbb{R}$, $\{U_i\}_{i\in\omega}$, such that $\Lambda(U_i) \leq 2^{-i}$.
(2) A number $x \in \mathbb{R}$ is *Λ-Martin-Löf random (or Λ-ML random)* if for some representation r of Λ there is no r-Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $x \in \bigcap_{i\in\omega} U_i$.

Note that leaving off the oracle in the preceding definitions gives *Hippocrates randomness* (after Hippocrates who famously ignored oracles) instead of Martin-Löf randomness. More precisely, a *Hippocrates test* (under the measure Λ) is a Σ_1^0 sequence $\{U_i\}_{i\in\omega}$ such that $\Lambda(U_i) \leq 2^{-i}$.

A fundamental result for Martin-Löf randomness is the existence of a universal r-ML test for any representation of the measure Λ.

Lemma 2.3. *Let Λ be a Borel measure on $\mathbb{R}$ and let $r \in 2^\omega$ be a representation of Λ. There is an r-ML test $\{U_i\}_{i\in\omega}$ such that for any r-ML test $\{V_i\}_{i\in\omega}$, if $x \in \bigcap_{i\in\omega} V_i$ then $x \in \bigcap_{i\in\omega} U_i$. (In this case $\{U_i\}_{i\in\omega}$ is called a universal r-ML test).*

2.2. **Random Closed Sets.** In [1] Martin-Löf randomness was extended to spaces of closed subsets of locally compact, Hausdorff, second countable topological spaces. Here we will deal only with the space of closed subsets of $\mathbb{R}$ and so we will provide only the material necessary for work in this particular setting rather than the full generality provided in [1].

Martin-Löf randomness requires a topology and a Borel measure. We take advantage of the standard probability theoretic constructions for random closed sets (usually called RACS in the literature). Good introductions to the theory of RACS can be found in [7], [8], or [9]. In general we follow standard notational conventions.

Definition 2.4. The family of all closed subsets of $\mathbb{R}$ is denoted $\mathcal{F}(\mathbb{R})$ or, for brevity, just $\mathcal{F}$. The family of all compact subsets of $\mathbb{R}$ is $\mathcal{K}(\mathbb{R})$ or just $\mathcal{K}$. Let $A \subseteq \mathbb{R}$.

(1) The hitting set for A is $\mathcal{F}_A = \{F \in \mathcal{F} : F \cap A \neq \emptyset\}$.
(2) The missing set for A is $\mathcal{F}^A = \{F \in \mathcal{F} : F \cap A = \emptyset\}$.

Note that the complement of the hitting set $\mathcal{F}_A$ in $\mathcal{F}$ is the missing set $\mathcal{F}^A$.

Definition 2.5. The *Fell topology* (also called the *hit-or-miss topology* is generated by the sub-basis of sets of the form $\mathcal{F}^K$ where $K \subseteq \mathbb{R}$ is compact and $\mathcal{F}_G$ where $G \subseteq \mathbb{R}$ is open.

This means that basic open sets in the Fell topology are of the form

$$\mathcal{F}^{K_1} \cap \mathcal{F}^{K_2} \cap \cdots \cap \mathcal{F}^{K_m} \cap \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} \cap \cdots \cap \mathcal{F}_{G_n}$$

where the sets $K_1, K_2, \ldots, K_m \subseteq \mathbb{R}$ are compact and the sets $G_1, G_2, \ldots, G_n \subseteq \mathbb{R}$ are open. Conveniently

$$\mathcal{F}^{K_1} \cap \mathcal{F}^{K_2} \cap \cdots \cap \mathcal{F}^{K_m} = \mathcal{F}^{K_1 \cup K_2 \cup \cdots \cup K_m}$$

and the finite union of compact sets is again compact, so a general basic open set has the form

$$\mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} \cap \cdots \cap \mathcal{F}_{G_n}.$$

where $K \subseteq \mathbb{R}$ is compact and the sets $G_1, G_2, \ldots, G_n \subseteq \mathbb{R}$ are open. We now introduce a commonly used notation for such sets.

Definition 2.6. Let $K, G_1, \ldots, G_n \subseteq \mathbb{R}$. The set $\mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \cdots \cap \mathcal{F}_{G_n}$ is denoted $\mathcal{F}^K_{G_1,\ldots,G_n}$.

The Fell topology and the topology on $\mathbb{R}$ are obviously related since the basis for the Fell topology depends on the topology of $\mathbb{R}$. One fundamental result in the study of RACS establishes connections between certain bases for the topology on the underlying space and bases for the Fell topology. For the underlying space $\mathbb{R}$ this gives the following lemma:

Lemma 2.7. *A sub-basis for the Fell topology consists of the sets of the following forms:*

(1) $\mathcal{F}_{(a,b)}$ *where $a < b \in \mathbb{Q}$ (hitting sets of open rational intervals) and*
(2) $\mathcal{F}^{[a,b]}$ *where $a < b \in \mathbb{Q}$ (missing sets of closed rational intervals).*

Consequently a basic open set for the Fell topology has the form

$$\mathcal{F}^{\overline{J}_1 \cup \overline{J}_2 \cup \cdots \cup \overline{J}_m}_{J_{m+1}, J_{m+2}, \ldots, J_n}$$

where $J_1, J_2, \ldots, J_n$ are open intervals with rational endpoints (and $\overline{J_i}$ is the closure of interval J_i). We note that the space $\mathcal{F}$ equipped with this topology is compact, Hausdorff, and second countable.

Any Borel probability measure P on the space $\mathcal{F}$ gives rise to a *capacity functional* on the compact subsets of $\mathbb{R}$, $T : \mathcal{K} \to \mathbb{R}$ defined by $T(K) = P(\mathcal{F}_K)$ for any $K \in \mathcal{K}$. The Choquet Capacity Theorem characterizes the Borel probability measures on $\mathcal{F}$ in terms of their capacity functionals. One of the conditions, complete alternation, requires significant additional explanation which we omit here for the sake of brevity. The Choquet Capacity Theorem is fundamental to the probability theory of random closed sets and much more detail, including a discussion of complete alternation, can be found in [7], [8], [9], or [1].

Theorem 2.8 (Choquet capacity theorem for $\mathbb{R}$)**.** *Let $T : \mathcal{K} \to [0, 1]$ be a functional on the compact subsets of $\mathbb{R}$. Then T gives rise to a Borel probability measure $\mathbf{P}$ on $\mathcal{F}$ such that $\mathbf{P}(\mathcal{F}_K) = T(K)$ for every $K \in \mathcal{K}$ if and only if T satisfies the following conditions:*

(1) $T(\emptyset) = 0$;
(2) *T is upper semi-continuous on $\mathcal{K}$: if $K_0, K_1, K_2, \dots \in \mathcal{K}$ such that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$ and $\bigcap_{i\in\omega} K_i = K$ then $\lim_{i\to\infty} T(K_i) = T(K)$;*
(3) *T is completely alternating on $\mathcal{K}$ (details omitted for brevity).*

Moreover, the probability measure $\mathbf{P}$ is unique.

A functional $T : \mathcal{K} \to [0, 1]$ satisfying the hypotheses of this theorem is called a *Choquet capacity*. One important example of a Choquet capacity is the capacity defined by $T(K) = 1 - e^{\Lambda(K)}$ where Λ is a locally finite Borel measure on $\mathbb{R}$. Such a capacity (and its associated measure) is called a *generalized Poisson processes*. The algorithmically random closed sets for generalized Poisson processes are the main interest of this paper. Some of the probability theory for generalized Poisson processes can be found in [7] or [8].

3. Martin-Löf randomness in the space $\mathcal{F}$

Here we extend the definitions of section 2.1 to the space $\mathcal{F}$ equipped with the Fell topology. Our approach is more or less the same as that taken in [4] but tailored to this specific situation. We begin with our computable enumeration of all open rational intervals in $\mathbb{R}$: $I_0, I_1, I_2, \dots$. By Lemma 2.7 this computable enumeration gives rise to a computable enumeration of a basis for $\mathcal{F}$: the basic open set

$$\mathcal{F}^{\overline{I}_{i_1}\cup\overline{I}_{i_2}\cup\cdots\cup\overline{I}_{i_m}}_{I_{j_1}, I_{j_2}, \dots, I_{j_n}}$$

can be represented as $\langle\langle i_1.i_2, \dots, i_m\rangle, \langle j_1, j_2, \dots, j_n\rangle\rangle$ where $\langle\ ,\ \rangle$ is the usual computable pairing function (and its extension to arbitrary tuples). We will refer to this enumeration of this basis as the *canonical enumeration* of basic open sets for the Fell topology on and denote it $\mathcal{B}_0, \mathcal{B}_1, \dots$.

We note that this basis is closed under finite intersections and, given indices i and j, we can uniformly compute an index k such that $\mathcal{B}_i \cap \mathcal{B}_j = \mathcal{B}_k$. For this reason we say that the canonical enumeration has the *computable intersection property*. This property is key for proving the existence of a universal Martin-Löf test.

We further note that representations of a measure P on $\mathcal{F}$ are representations relative to this canonical enumeration: $r \in 2^\omega$ is a representation of P if $P(\mathcal{B}_i)$ can be uniformly r-computed for each $i \in \omega$. We are now ready to define Martin-Löf randomness for closed subsets of $\mathbb{R}$.

Definition 3.1. Let P be a Borel measure on $\mathcal{F}$ and let r be a representation of P.

(1) A sequence of subsets of $\mathcal{F}$, $\{\mathcal{U}_i\}_{i\in\omega}$, is *uniformly* $\Sigma^{0,r}_1$ if there is an r-c.e. $f \in 2^\omega$ such that
$$\mathcal{U}_i = \bigcup_{f(\langle i,n\rangle)=1} \mathcal{B}_n.$$
(2) An *r-Martin-Löf test (r-ML test)* is a uniformly $\Sigma^{0,r}_1$ sequence of subsets of $\mathcal{F}$, $\{\mathcal{U}_i\}_{i\in\omega}$, such that $P(\mathcal{U}_i) \le 2^{-i}$.
(3) A closed set $F \in \mathcal{F}$ is *P-Martin-Löf random (or P-ML random)* if for some representation r of P there is no r-ML test $\{\mathcal{U}_i\}_{i\in\omega}$ such that $F \in \bigcap_{i\in\omega} \mathcal{U}_i$.

Lemma 3.2. *Let r be any representation of Borel measure P. There is a universal r-ML test.*

Proof. The usual proof works here and we now sketch it only very briefly (greater detail can be found in [1]). Enumerate all $\Sigma_1^{0,r}$ sequences. Convert this into an enumeration (possibly with repetition) of the r-ML tests (this is where the computable intersection property is used), then take diagonal unions. □

4. Martin-Löf randomness for generalized Poisson processes

Here we explore the Martin-Löf random generalized Poisson processes. We will begin by working with the capacity $T(K) = 1 - 2^{-m(K)}$ where m is the Lebesgue measure on $\mathbb{R}$. This simplest generalized Poisson process allows us to focus on topological concerns without worrying about the computational power of the measure.

We then move to the more general case of $T(K) = 1 - e^{-\Lambda(K)}$ where Λ is a locally finite, non-atomic, regular, Borel measure on $\mathbb{R}$. Local finiteness guarantees that we are dealing with a well-defined capacity functional. Both of the properties of non-atomicity and regularity are important at some point, though not always at the same time. In general we care only about Borel measures. Lacking a reason to make finer distinctions, all work proceeds under the assumption that the measure Λ has all of these properties.

We first prove a lemma concerning the computational strength of representations of measures associated with the capacity $T(K) = 1 - e^{-\Lambda(K)}$.

Lemma 4.1. *Let Λ be a locally finite, non-atomic, regular, Borel measure on $\mathbb{R}$ and let $\mathbf{P}$ be the probability measure on $\mathcal{F}$ arising from the capacity $T(K) = 1 - e^{-\Lambda(K)}$. Then $r \in 2^\omega$ is a representation of Λ if and only if r is a representation of $\mathbf{P}$.*

Proof. We first prove that for any $i \in \omega$, the measure $\Lambda(I_i)$ can be uniformly computed from any representation of the measure $\mathbf{P}$. By definition $\mathcal{F}^{\overline{I}_i}$ appears in the canonical enumeration of our basis for the Fell topology and so we can compute its measure using any representation of $\mathbf{P}$. Moreover,

$$\mathbf{P}\left(\mathcal{F}^{\overline{I}_i}\right) = 1 - \mathbf{P}(\mathcal{F}_{\overline{I}_i}) = e^{-\Lambda\left(\overline{I}_i\right)}.$$

Since Λ is non-atomic it follows that $\Lambda\left(\overline{I}_i\right) = \Lambda(I_i)$. Thus it is possible to calculate $\Lambda(I_i)$ from any representation of $\mathbf{P}$.

Now we prove that for any $k \in \omega$, the measure $\mathbf{P}(\mathcal{B}_k)$ can be uniformly computed from any representation of the measure Λ. We can compute $\mathbf{P}(\mathcal{B}_k)$ by first interpreting $\mathcal{B}_k$ as a finite intersection of hitting and missing sets of rational intervals:

$$\mathcal{B}_k = \mathcal{F}^{\overline{I}_{i_1} \cup \overline{I}_{i_2} \cup \cdots \cup \overline{I}_{i_m}}_{I_{j_1}, I_{j_2}, \ldots, I_{j_n}}$$

for some $i_1, i_2, \ldots, i_m, j_i, j_2, \ldots, j_n \in \omega$ (and $I_0, I_1, I_2, \ldots$ the canonical enumeration of open rational intervals). The complete alternation property suggests a recursive formula for calculating $\mathbf{P}(\mathcal{B}_k)$ from the measure of simpler sets: in this case

$$\mathbf{P}(\mathcal{B}_k) = \mathbf{P}\left(\mathcal{F}^{\overline{I}_{i_1} \cup \overline{I}_{i_2} \cup \cdots \cup \overline{I}_{i_m}}_{I_{j_1}, I_{j_2}, \ldots, I_{j_{n-1}}}\right) - \mathbf{P}\left(\mathcal{F}^{\overline{I}_{i_1} \cup \overline{I}_{i_2} \cup \cdots \cup \overline{I}_{i_m} \cup I_{j_n}}_{I_{j_1}, I_{j_2}, \ldots, I_{j_{n-1}}}\right).$$

Iteration moves all of the intervals into the superscript. This means that the measure of the basic open set $\mathcal{B}_k$ can be calculated provided we can compute the measure of any set of the form $\mathcal{F}^{\overline{I}_{i_1} \cup \overline{I}_{i_2} \cup \cdots \cup \overline{I}_{i_m} \cup I_{j_1} \cup I_{j_2} \cup \cdots \cup I_{j_n}}$ where we are re-using the indices to mean any $i_1, i_2, \ldots, i_m, j_i, j_2, \ldots, j_n \in \omega$. Clearly it suffices to compute the measure of the complement of this set. Its complement is

$\mathcal{F}_{\overline{I}_{i_1}\cup\overline{I}_{i_2}\cup\cdots\cup\overline{I}_{i_m}\cup I_{j_1}\cup I_{j_2}\cup\cdots\cup I_{j_n}}$ and because Λ is non-atomic it follows that

$$\mathbf{P}\left(\mathcal{F}_{\overline{I}_{i_1}\cup\overline{I}_{i_2}\cup\cdots\cup\overline{I}_{i_m}\cup I_{j_1}\cup I_{j_2}\cup\cdots\cup I_{j_n}}\right) = 1 - e^{\Lambda\left(I_{i_1}\cup I_{i_2}\cup\cdots\cup I_{i_m}\cup I_{j_1}\cup I_{j_2}\cup\cdots\cup I_{j_n}\right)}.$$

This quantity is computable from any representation of Λ. □

4.1. **The generalized Poisson process $\mathbf{T(K) = 1 - 2^{-m(K)}}$.** We now proceed to our first examination of Martin-Löf randomness for a generalized Poisson process. Let m be Lebesgue measure on $\mathbb{R}$. Define the capacity $T : \mathcal{K} \to [0,1]$ by $T(K) = 1 - 2^{-m(K)}$. Let $\mathbf{P}$ be the Borel measure on $\mathcal{F}$ induced by T (so $\mathbf{P}(\mathcal{F}_K) = T(K)$ for any $K \in \mathcal{K}$). It follows from Lemma 4.1 that $\mathbf{P}$ has a computable representation. Consequently we need only to consider ML randomness without oracles.

We know that $\mathbf{P}$ will inherit some of the properties of the Lebesgue measure. One example is translation invariance, which is called *stability* in the probability theory of random closed sets.

Definition 4.2. Let $c \in \mathbb{R}$, $F \in \mathcal{F}$, and $\mathcal{A} \subseteq \mathcal{F}$. We define the following expressions:

(1) $F + c = \{x + c : x \in F\}$;
(2) $\mathcal{A} + c = \{E + c : E \in \mathcal{A}\}$.

Definition 4.3. A Borel probability measure P on $\mathcal{F}$ is *stable* if for every $c \in \mathbb{R}$ and every Borel $\mathcal{A} \subseteq \mathcal{F}$ it follows that $P(\mathcal{A}) = P(\mathcal{A} + c)$.

Our generalized Poisson processes is a canonical example of a stable measure on $\mathcal{F}$. We now prove an effective version of stability for this measure.

Lemma 4.4. *If $F \in \mathcal{F}$ is $\mathbf{P}$-ML random and $c \in \mathbb{Q}$ then $F + c$ is also $\mathbf{P}$-ML random.*

Proof. We prove the contrapositive. Suppose that $F+c$ is not $\mathbf{P}$-ML random. Then there is a $\mathbf{P}$-ML test $\{\mathcal{U}_i\}_{i\in\omega}$ such that $(F + c) \in \bigcap_{i\in\omega} \mathcal{U}_i$. Define $\mathcal{V}_i = \mathcal{U}_i - c$ for each $i \in \omega$. Observe that for $r_1, \ldots, r_k, q_1, \ldots, q_n \in \mathbb{Q}$,

$$\mathcal{F}^{[r_1,r_2]\cup\cdots\cup[r_{k-1},r_k]}_{(q_1,q_2),\ldots,(q_{n-1},q_n)} \subseteq \mathcal{U}_k \iff \mathcal{F}^{[r_1-c,r_2-c]\cup\cdots\cup[r_{k-1}-c,r_k-c]}_{(q_1-c,q_2-c),\ldots,(q_{n-1}-c,q_n-c)} \subseteq \mathcal{V}_k.$$

Hence $\{\mathcal{V}_i\}_{i\in\omega}$ is a uniformly Σ^0_1 sequence and from the stability of $\mathbf{P}$ it follows that $\mathbf{P}(\mathcal{V}_i) = \mathbf{P}(\mathcal{U}_i) \leq 2^{-i}$. Thus $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}$-ML test and $F \in \bigcap_{i\in\omega} \mathcal{V}_i$. Therefore F is not $\mathbf{P}$-ML random. □

The following is now immediate.

Corollary 4.5. *A closed set $F \in \mathcal{F}$ is $\mathbf{P}$-ML random if and only if $F + c$ is $\mathbf{P}$-ML random for every $c \in \mathbb{Q}$.*

We are also interested in the elements of the $\mathbf{P}$-ML random closed sets. It turns out that they are all m-ML random reals; in fact, inclusion in some $\mathbf{P}$-ML random closed set is a new characterization of the m-ML random reals. This is very different from the random closed subsets of 2^ω studied in [2], which always have non-random elements.

Theorem 4.6. *A real number x is m-ML random if and only if there is some $\mathbf{P}$-ML random closed set F such that $x \in F$.*

Proof. First we prove that if F contains a non m-ML random real x, then F is not $\mathbf{P}$-ML random. Let $x \in F$ and suppose that $\{U_i\}_{i\in\omega}$ is an m-ML test such that $x \in \bigcap_{i\in\omega} U_i$. Let $I_0, I_1, \ldots$ be our standard enumeration of all open rational intervals and let $f \in 2^\omega$ be c.e. such that $U_i = \bigcup_{f(\langle i,j\rangle)=1} I_j$. Consider the sequence of hitting sets $\{\mathcal{F}_{U_i}\}_{i\in\omega}$. This is a uniformly Σ^0_1 sequence: $\mathcal{F}_{U_i} = \bigcup_{f(\langle i,j\rangle)=1} \mathcal{F}_{I_j}$. Because $x \in U_i$ for all $i \in \omega$ and $x \in F$, it follows that $F \in \mathcal{F}_{U_i}$ for all $i \in \omega$. Furthermore, $\mathbf{P}\left(\mathcal{F}_{U_i}\right) = 1 - 2^{-m(U_i)} \leq 1 - 2^{-2^{-i}}$. Consequently, if $n \geq \log_2 \log_2 \left(1 - 2^{-i}\right)$, then $\mathbf{P}\left(\mathcal{F}_{U_n}\right) \leq 2^{-i}$. Define a new sequence of subsets of $\mathcal{F}$:

$$\mathcal{V}_i = \mathcal{F}_{U_{\lceil \log_2 \log_2 \left(1-2^{-i}\right)\rceil}}.$$

Then $\{\mathcal{V}_i\}_{i\in\omega}$ a $\mathbf{P}$-ML test and $F \in \bigcap_{i\in\omega} \mathcal{V}_i$. Therefore F is not $\mathbf{P}$-ML random.

To prove the converse we show that if $x \in \mathbb{R}$ is not an element of any $\mathbf{P}$-ML random closed set, then x is not m-ML random. Suppose that $x \in \mathbb{R}$ is not contained in any $\mathbf{P}$-ML random closed set, that is, every closed set containing x is not $\mathbf{P}$-ML random. Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a universal $\mathbf{P}$-ML test (the existence of which was proved in Lemma 3.2). No closed set containing x is random and thus $\mathcal{F}_{\{x\}} \subseteq \mathcal{U}_i$ for every $i \in \omega$. The first step is to show that open covers of $\mathcal{F}_{\{x\}}$ must in fact cover $\mathcal{F}_I$ for some interval I with $x \in I$. The second step is to use this information to construct an m-ML test the intersection of which contains x.

The set $\{x\}$ is compact in $\mathbb{R}$ and hence $\mathcal{F}_{\{x\}}$ is closed in the Fell topology. Moreover, the space $\mathcal{F}$ is compact, so $\mathcal{F}_{\{x\}}$ is compact. Let $f \in 2^\omega$ be c.e. such that $\mathcal{U}_i = \bigcup_{f(\langle i,j\rangle)=1} \mathcal{B}_j$ (where $\mathcal{B}_0, \mathcal{B}_1, \ldots$ is the standard basis for $\mathcal{F}$). By compactness, there must be some $N \in \omega$ such that $\mathcal{F}_{\{x\}} \subseteq \bigcup_{j\leq N \ \&\ f(\langle i,j\rangle)=1} \mathcal{B}_j$.

By definition $\mathcal{B}_j = \mathcal{F}^{K_j}_{I_{k_{j,1}},\ldots,I_{k_{j,n_j}}}$ where K_j is some finite union of closed rational intervals. We now define

$$I = \bigcap \left\{ I_{k_{j,m}} : j \leq N,\ f\left(\langle i,j\rangle\right) = 1,\ m \leq n_j,\ x \in I_{k_{j,m}} \right\}.$$

From the preceding paragraph we know that $\{x\} \in \mathcal{F}_{\{x\}} \subseteq \bigcup_{j\leq N \ \&\ f(\langle i,j\rangle)=1} \mathcal{B}_j$, hence there is $j_0 \leq N$ such that $f\left(\langle i,j_0\rangle\right) = 1$ and $\{x\} \in \mathcal{B}_{j_0}$. Because $\{x\} \in \mathcal{B}_{j_0}$ we know that $x \in I_{k_{j_0,m}}$ for each $m \leq n_{j_0}$. It follows that I is an intersection over a non-empty set. Furthermore, I is a finite intersection of open rational intervals, each of which contains x. Thus I is itself a nonempty open rational interval containing x.

We now claim that $\mathcal{F}_I \subseteq \mathcal{U}_i$. Suppose that $F \in \mathcal{F}_I$. Note that $F \cup \{x\} \in \mathcal{F}_{\{x\}} \subseteq \bigcup_{j\leq N \ \&\ f(\langle i,j\rangle)=1} \mathcal{B}_j$. Hence there is $j \leq N$ such that $f\left(\langle i,j\rangle\right) = 1$ and $F \cup \{x\} \in \mathcal{B}_j$. Fix this j. By definition $\left(F \cup \{x\}\right) \cap K_j = \emptyset$ and $\left(F \cup \{x\}\right) \cap I_{k_{j,m}} \neq \emptyset$ for each $m \in \{1, \ldots, n_j\}$. Clearly $F \cap K_j = \emptyset$ and hence $F \in \mathcal{F}^{K_j}$. Now, if $x \notin I_{k_{j,m}}$ then it must be the case that $F \cap I_{k_{j,m}} \neq \emptyset$. On the other hand, if $x \in I_{k_{j,m}}$ then $I \subseteq I_{k_{j,m}}$ by the definition of I. But $F \in \mathcal{F}_I$ and so by definition $F \cap I \neq \emptyset$. It then follows that $F \cap I_{k_{j,m}} \neq \emptyset$ in this case as well. Thus $F \cap I_{k_{j,m}} \neq \emptyset$ for each $m \in \{1, \ldots, n_j\}$. Consequently $F \in \mathcal{F}^{K_j}_{I_{k_{j,1}},\ldots,I_{k_{j,n_j}}} = \mathcal{B}_j$ for at least this value of j. Therefore $F \in \bigcup_{j\leq N \ \&\ f(\langle i,j\rangle)=1} \mathcal{B}_j = \mathcal{U}_i$ and we have proved the claim.

At this point we have proved the existence of an open rational interval I such that $\mathcal{F}_I \subseteq \mathcal{U}_i$ and $x \in I$. Taking J to be any closed rational interval such that $J \subseteq I$ and $x \in J$, we find a closed rational interval such that $\mathcal{F}_J \subseteq \mathcal{U}_i$ and $x \in J$. Thus for each $i \in \omega$ there is some closed rational interval J such that $x \in J$ and $\mathcal{F}_J \subseteq \mathcal{U}_i$.

Note that we have simply proved the existence of such a closed rational interval and have not made any claims about being able to find this interval effectively.

Our next step is to construct an m-ML test $\{V_i\}_{i\in\omega}$ such that $x \in \bigcap_{i\in\omega} V_i$. We will use the intervals of the previous step to argue that this construction works as claimed. As above we have $f \in 2^\omega$ be c.e such that $\mathcal{U}_i = \bigcup_{f(\langle i,j\rangle)=1} \mathcal{B}_j$. Let $\mathcal{U}_{i,s}$ be the stage s approximation to $\mathcal{U}_i$, specifically $\mathcal{U}_{i,s} = \bigcup_{f(\langle i,j\rangle)=1\ \&\ j\le_s} \mathcal{B}_j$.

For each $i \in \omega$ the hitting set $\mathcal{F}_{\overline{I}_i}$ is compact in the Fell topology. It thus follows that $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_n$ if and only if there is a stage s such that $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,s}$. The following calculations show how to identify these stages in a computable manner. First, $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,s}$ if and only if $\mathcal{F}_{\overline{I}_i} \setminus \mathcal{U}_{n,s} = \emptyset$. The set $\mathcal{F}_{\overline{I}_i} \setminus \mathcal{U}_{n,s}$ is a closed set of the form

$$\mathcal{F}^{O_1}_{K^1_1,K^1_2,\ldots K^1_{k_1}} \cup \cdots \cup \mathcal{F}^{O_m}_{K^m_1,K^m_2,\ldots K^m_{k_m}} \tag{1}$$

where for each l and j, the set K^j_l is a finite union of closed rational intervals and O_j is a finite union of open rational intervals. Finding such an expression for this set is also uniformly computable from i, n, and s. Clearly $\mathcal{F}_{\overline{I}_i} \setminus \mathcal{U}_{n,s} = \emptyset$ if and only if each set in formula 1 is empty. The set $\mathcal{F}^{O_j}_{K^j_1,K^j_2,\ldots K^j_{k_j}}$ is empty if and only if there is a number $l \le k_j$ such that $K^j_l \subseteq O_j$ (making impossible for any subset of $\mathbb{R}$ to both hit K^j_l and miss O_j). Because each of the sets K^j_l and O_j were obtained uniformly computably we can uniformly computably determine if $K^j_l \subseteq O_j$. Therefore we can uniformly computably determine if $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,s}$.

To construct V_k, the k^{th} set in our m-ML test $\{V_i\}_{i\in\omega}$, we use $\mathcal{U}_n$ where $n = \left\lceil -\log_2\left(1-2^{-2^{-(k+1)}}\right)\right\rceil$. This ensures that $2^{-n} \le 1 - 2^{-2^{-(k+1)}}$. We enumerate the closed rational intervals $\overline{I}_0, \overline{I}_1, \ldots,$ identifying subsets of $\mathcal{U}_{n,s}$ and adding increasingly accurate open covers of these closed sets to V_k. At the same time we construct a set $C_k \subseteq V_k$ that we will use later to calculate the measure of V_k. We build V_k and C_k in stages, starting with $V_{k,0} = C_{k,0} = \emptyset$.

Suppose that we have enumerated j intervals into $V_{k,s}$. Wait for the next stage t such that there is $i \le t$ with $\overline{I}_i \nsubseteq V_{k,s}$ and $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,t}$. Let i be the least index such that I_i satisfies these conditions. We then take an open rational interval I such that $\overline{I}_i \subseteq I$ and $m(I \setminus \overline{I}_i) \le 2^{-(2+k+j)}$ and set $V_{k,t} = V_{k,s} \cup I$. We also set $C_{k,t} = C_{k,s} \cup \overline{I}_i$. Finally, let $V_k = \bigcup_{s\in\omega} V_{k,s}$ and $C_k = \bigcup_{s\in\omega} C_{k,s}$.

From step one, we know that there is a closed rational interval J such that $x \in J$ and $\mathcal{F}_J \subseteq \mathcal{U}_n$. We claim that $J \subseteq V_k$. Because $\mathcal{F}_J \subseteq \mathcal{U}_n$ and $\mathcal{F}_J$ is compact, there is a stage s such that $\mathcal{F}_J \subseteq \mathcal{U}_{n,s}$. There is also some index i_m such that $J = \overline{I}_{i_m}$. If $J \nsubseteq V_{k,s}$ then at some stage $t \ge \max\{s, i_m\}$, the index i_m will be the least i such that $\overline{I}_i \nsubseteq V_{k,t}$ and $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,t}$. At this stage we will set $C_{k,t} = C_{k,s} \cup \overline{I}_{i_m}$. Because $C_{k,t} \subseteq V_{k,t}$ and $J = \overline{I}_{i_m}$ we know that $J \subseteq V_k$. This proves the claim.

A clear consequence of the claim is that $x \in V_k$ for every $k \in \omega$. In other words, $r \in \bigcap_{k\in\omega} V_k$. We must now prove that $\{V_k\}_{k\in\omega}$ is an m-ML test.

By construction V_k is Σ^0_1 and we note that this is uniform over $k \in \omega$. We have also constructed a set $C_k \subseteq V_k$ such that

$$\begin{aligned} m(V_k) &= m(C_k) + m(V_k \setminus C_k) \\ &\leq m(C_k) + \sum_{j=0}^{\infty} 2^{-(2+k+j)} \\ &= m(C_k) + 2^{-(k+1)}. \end{aligned}$$

We also know that $\mathcal{F}_{C_k} \subseteq \mathcal{U}_n$ and hence $\mathbf{P}(\mathcal{F}_{C_k}) \leq 2^{-n}$. By definition $\mathbf{P}(\mathcal{F}_{C_k}) = 1 - 2^{-m(C_k)}$ and so we find that $m(C_k) \leq -\log_2(1 - 2^{-n})$. But n was chosen to be large enough that

$$-\log_2\left(1 - 2^{-n}\right) \leq -\log_2\left(1 - \left(1 - 2^{-2^{-(k+1)}}\right)\right) = 2^{-(k+1)}.$$

Consequently $m(V_k) \leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}$. Thus $\{V_k\}_{k\in\omega}$ is an m-ML test.

We now know that $x \in \bigcap_{k\in\omega} V_k$ and $\{V_k\}_{k\ in \omega}$ is an m-ML test. Therefore x is not m-ML random. □

Thus far our proofs have involved relating $\mathbf{P}$-ML tests to m-ML tests, a reasonable approach since $\mathbb{R}$ is much more familiar than $\mathcal{F}$. However, constructing $\mathbf{P}$-ML tests from scratch gives us a little more insight into the structure of $\mathbf{P}$-ML random closed sets.

Proposition 4.7. *If $F \in \mathcal{F}$ is $\mathbf{P}$-ML random, then F is neither bounded below nor above.*

Proof. We first prove that a $\mathbf{P}$-ML random closed set must not be bounded above. We prove the contrapositive: suppose $F \in \mathcal{F}$ is bounded above. Then there is some $M \in \mathbb{N}$ such that $F \subseteq (-\infty, M)$. By definition the missing set $\mathcal{F}^{[M,i]}$ is part of our canonical basis for the Fell topology for every integer $i > M$. Thus $\left\{\mathcal{F}^{[M,i]}\right\}_{i>M}$ is a Σ^0_1 sequence in $\mathcal{F}$. Moreover, $\mathbf{P}\left(\mathcal{F}^{[M,i]}\right) = 1 - \mathbf{P}\left(\mathcal{F}_{[M,i]}\right) = 2^{-(i-M)}$. Consequently, we can easily extract a $\mathbf{P}$-ML test from this sequence of sets. Because $F \subseteq (-\infty, M)$, we know that $F \in \bigcap_{i>M} \mathcal{F}^{[M,i]}$. Therefore F is not $\mathbf{P}$-ML random.

The proof that a $\mathbf{P}$-ML random closed set must not be bounded below is nearly identical. □

We note that lemma 4.4 allows us to draw conclusions about all elements of $\mathbf{P}$-ML random closed sets from conclusions about elements at specific locations. In particular, if F is a $\mathbf{P}$-ML random closed set and $r \in F$, then we know that there is some $q \in \mathbb{Q}$ such that $r + q \in [0,1]$. Because $F + q$ is a $\mathbf{P}$-ML random closed set and $r + q \in F + q$ conclusions about elements of $\mathbf{P}$-ML random closed sets in the interval $[0,1]$ can be translated into conclusions about r. In particular we have the following theorem.

Theorem 4.8. *If $F \in \mathcal{F}$ is $\mathbf{P}$-Martin-Löf random and $z \in F \cap [0,1]$ then there is an open set $G \subseteq \mathbb{R}$ such that $z \in G$ and $G \cap F = \{z\}$ (i.e. z is not a limit point of F).*

Proof. We will abuse notation somewhat in this proof and identify the finite binary string σ with the rational number in the interval $[0,1]$ with binary expansion given by σ. We will also use $x{\upharpoonright}n$ to refer to the string/rational number given by the first

$n \geq 1$ bits of a binary expansion of a real number $x \in [0,1]$. We note that a m-ML random real number x (or any irrational real number) is an element of closed set F if and only if $F \cap (x \restriction n, (x \restriction n) + 2^{-n}) \neq \emptyset$ for all $n \geq 1$.

Let F be a **P**-ML random closed set and suppose that $z \in F \cap [0,1]$ is a limit point of F. Closed sets contain their limit points and hence $z \in F$. It then follows that z is m-ML random and consequently irrational. Thus for every $i \geq 1$ there is $x \in F \cap [0,1]$ such that $x \restriction i = z \restriction i$ but $x \neq z$. For each such number x there is an integer n for which $x \restriction n = z \restriction n$ but $x \restriction (n+1) \neq z \restriction (n+1)$. For this n we see that $F \cap \left(z \restriction n, (z \restriction n) + 2^{-(n+1)}\right) \neq \emptyset$ and $F \cap \left((z \restriction n) + 2^{-(n+1)}, (z \restriction n) + 2^{-n}\right) \neq \emptyset$. Expressed in terms of the Fell topology, our last conclusion is

$$F \in \mathcal{F}_{\left(z \restriction n, (z \restriction n) + 2^{-(n+1)}\right), \left((z \restriction n) + 2^{-(n+1)}, (z \restriction n) + 2^{-n}\right)}.$$

We can calculate the **P**-measure of this set:

$$\mathbf{P}\left(\mathcal{F}_{\left(z \restriction n, (z \restriction n) + 2^{-(n+1)}\right), \left((z \restriction n) + 2^{-(n+1)}, (z \restriction n) + 2^{-n}\right)}\right) = \left(1 - 2^{2^{-(n+1)}}\right)^2.$$

This motivates the following construction.

For each $i \in \omega$ define

$$\mathcal{U}_i = \bigcup \left\{\mathcal{F}_{(\sigma, \sigma + 2^{-(n+1)}), (\sigma + 2^{-(n+1)}, \sigma + 2^{-n})} : \sigma \in 2^n, \ n \geq i\right\}.$$

We wish to extract a **P**-ML test from the sequence $\{\mathcal{U}_i\}_{i \in \omega}$. It is clear from the definition that $\{\mathcal{U}_i\}_{i \in \omega}$ is a Σ^0_1 sequence. Furthermore,

$$\begin{aligned}
\mathbf{P}(\mathcal{U}_i) &\leq \sum_{n \geq i} \left[\sum_{\sigma \in 2^n} \mathbf{P}\left(\mathcal{F}_{(\sigma, \sigma + 2^{-(n+1)}), (\sigma + 2^{-(n+1)}, \sigma + 2^{-n})}\right)\right] \\
&= \sum_{n \geq i} 2^n \left(1 - 2^{-2^{-(n+1)}}\right)^2.
\end{aligned}$$

By the familiar ratio test of calculus this series converges. In fact, the ratio rest shows that there is $N \in \omega$ and $q \in (0,1) \cap \mathbb{Q}$ such that for $n \geq N$

$$2^n \left(1 - 2^{-2^{-(n+1)}}\right)^2 \leq q^n.$$

Using this bound we can then pick out a sub-sequence $\{\mathcal{U}_{i_j}\}_{j \in \omega}$ such that $\mathbf{P}_T(\mathcal{U}_{i_j}) \leq 2^{-j}$. This gives the desired **P**-ML test.

As already noted, for each $i \in \omega$ there is $n \geq i$ such that

$$F \in \mathcal{F}_{\left(z \restriction n, (z \restriction n) + 2^{-(n+1)}\right), \left((z \restriction n) + 2^{-(n+1)}, (z \restriction n) + 2^{-n}\right)}.$$

Thus $F \in \bigcap_{i \in \omega} \mathcal{U}_i$. This contradicts the assumption that F is **P**-ML random. We therefore conclude that if F is **P**-ML random and $z \in F \cap [0,1]$, then z is not a limit point of F. □

As discussed before, the stability of **P**-ML random closed sets means that this theorem applies to all elements of F, not just those in the interval $[0,1]$.

Corollary 4.9. *If F is a **P**-ML random closed set and $z \in F$, then z is not a limit point of F.*

The following corollaries follow directly from this result. The latter is an effective version of Robbins' Theorem, another basic result of the theory of RACS.

Corollary 4.10. *If F is a **P**-ML random closed set, then F is countable.*

Corollary 4.11. *If F is a $\mathbf{P}$-ML random closed set, then $m(F) = 0$.*

To summarize, every $\mathbf{P}$-ML random closed set is an unbounded, countable collection of points, none of which is a limit point of the set and each of which is m-ML random. More precise characterization might be possible here: for example, classically we would expect the gaps between points of F to have an exponential distribution and it remains to be determined if this holds effectively.

Another result similar to Theorem 4.15 (but not directly related) can be obtained by considering functions called *selections* which are used extensively in the study of RACS.

Definition 4.12. Let P be a probability measure on $\mathcal{F}$. A measurable function $s : \mathcal{F} \to \mathbb{R}$ is a *selection* for P if $s(F) \in F$ for P-almost every $F \in \mathcal{F}$.

As usual, details on the probability theory can be found in in [7], [8], or [9]. For our purposes it is most important to note that a selection induces a measure μ on $\mathbb{R}$ (called the distribution of the selection) defined by $\mu(A) = P(s^{-1}(A))$ for measurable $A \subseteq \mathbb{R}$. From the definition we know that $s^{-1}(A) \subseteq \mathcal{F}_A \cup B$ for some measure 0 set B. Hence, if s is a selection for P, then for any compact $K \subseteq \mathbb{R}$ we find that

$$\begin{aligned} \mu(K) &= P\left(s^{-1}(K)\right) \\ &\leq P\left(\mathcal{F}_K\right) \\ &= T\left(K\right) \end{aligned}$$

where T is the capacity associated with P. This shows that if μ is the distribution of a selection for P, then $\mu(K) \leq T(K)$ for any compact $K \subseteq \mathbb{E}$. A classic theorem of the probability theoretic literature, Artstein's Selectionability Theorem, proves that the converse also holds. We state the theorem in its general form.

Theorem 4.13 (Artstein's Selectionability Theorem)**.** *Let $\mathbb{E}$ be a Polish space, let μ be a probability measure on $\mathbb{E}$, and let T be a Choquet capacity on E. Then there is a RACS X and a selection s for X such that T is the capacity for X and μ is the distribution of s if and only if $T(K) \geq \mu(K)$ for every compact $K \subseteq E$.*

Note that no distinction is made between RACS and the measures they induce in the probability theoretic literature.

We now return to the space of closed subsets of $\mathbb{R}$ with the measure $\mathbf{P}$ induced by the capacity $T(K) = 1 - e^{-m(K)}$. We prove an effective version of Artstein's Selectionability Theorem for this generalized Poisson process and in so doing arrive at another description of elements of $\mathbf{P}$-ML random closed sets.

Theorem 4.14. *Let $x \in \mathbb{R}$. There is a $\mathbf{P}$-ML random closed set F such that $x \in F$ if and only if there is a regular Borel measure μ on $\mathbb{R}$ such that $\mu(K) \leq T(K)$ for every compact $K \subseteq \mathbb{R}$ and x is μ-Hippocrates random.*

Proof. ($\Rightarrow$) Let F be a $\mathbf{P}$-ML random closed set and let $x \in F$. Our goal is to produce a probability measure μ such that x is μ-Hippocrates random and $\mu(K) \leq T(K)$ for each compact set K. We do this by finding a selection function s for $\mathbf{P}$, such that $s(F) = x$. Actually, it is not important for our proof that the function s is a selection for $\mathbf{P}$: it suffices that s be measurable. It seems to be worth noting, in particular in thinking of this result as an effective version of Artstein's Selectionability Theorem, that the function we define is a selection. The desired measure μ is then the distribution of the selection.

Because F is $\mathbf{P}$-ML random we know that x is m-ML random and x is not a limit point of F. Hence here is $q \in \mathbb{Q}$ such that x is the least member of F larger than q. We note that $q \notin F$ because rational numbers are not m-ML random. Define the map $s : \mathcal{F} \to \mathbb{R}$ by

$$s(E) = \begin{cases} \min\left(E \cap [q, \infty)\right) & \text{if } E \cap [q, \infty) \neq \emptyset \\ q & \text{otherwise.} \end{cases}$$

To prove that s is a selection for $\mathbf{P}$ we must show that s is measurable and that $s(E) \in E$ for $\mathbf{P}$-almost every $E \in \mathcal{F}$. To prove that s is measurable is suffices to show that $s^{-1}\left((a, b)\right)$ is measurable for any $a < b \in \mathbb{Q}$. We have three cases:

(1) If $b \leq q$, then $s^{-1}((a, b)) = \emptyset$;
(2) If $a < q < b$, then $s^{-1}((a, b)) = \mathcal{F}_{[q,b)} \cup \mathcal{F}^{[q,\infty)}$;
(3) If $q \leq a$, then $s^{-1}((a, b)) = \mathcal{F}^{[q,a]}_{(a,b)}$.

In the first and third cases $s^{-1}((a, b))$ is a basic open set of the Fell topology by definition. For the second case we may calculate that $\mathcal{F}_{[q,b)} = \bigcap_{i\in\omega} \mathcal{F}_{(q-2^{-i},b)}$ and $\mathcal{F}^{[q,\infty)} = \bigcap_{i\geq 1} \mathcal{F}^{[q,q+i]}$ (we omit proofs of these facts for the sake of brevity). In every case $s^{-1}((a, b))$ is measurable (in fact, $s^{-1}((a, b))$ is always Σ^0_1).

We now prove that $s(E) \in E$ for $\mathbf{P}$-almost every $E \in \mathcal{F}$. As noted above, this is actually not an essential part of the proof and the reader may safely skip to the next paragraph. By the definition of s we know that $s(E) \in E$ if and only if $E \cap [q, \infty) \neq \emptyset$. Hence it suffices to prove that $\mathbf{P}\left(\mathcal{F}^{[q,\infty)}\right) = 0$. Because $\mathcal{F}^{[q,\infty)} = \bigcap_{i\geq 1} \mathcal{F}^{[q,q+i]}$ and $\mathcal{F}^{[q,q+1]} \supseteq \mathcal{F}^{[q,q+2]} \supseteq \mathcal{F}^{[q+3]} \supseteq \ldots$ it follows that $\mathbf{P}\left(\mathcal{F}^{[q,\infty)}\right) = \lim_{i\to\infty} \mathbf{P}\left(\mathcal{F}^{[q,q+i]}\right)$. But $\mathbf{P}\left(\mathcal{F}^{[q,q+i]}\right) = 1 - T([q, q + i]) = 2^{-i}$ and thus $\mathbf{P}\left(\mathcal{F}^{[q,\infty)}\right) = \lim_{i\to\infty} 2^{-i} = 0$. Therefore $s(E) \in E$ for $\mathbf{P}$-almost every $E \in \mathcal{F}$.

We can now define our measure: $\mu(A) = \mathbf{P}\left(s^{-1}(A)\right)$ for any measurable $A \subseteq \mathbb{R}$. The measure μ is regular because $\mathbf{P}$ is regular. We also wish to determine the μ-measure of $K \in \mathcal{K}$ and this entails understanding $s^{-1}(K)$. If $K \subseteq (-\infty, q)$ then $s^{-1}(K) = \emptyset$. On the other hand, if $K \nsubseteq (-\infty, q)$, then the fact that $s(E) \in E$ whenever $E \cap [q, \infty) \neq \emptyset$ ensures that $s^{-1}(K) \subseteq \mathcal{F}_K$. It follows that in either case $s^{-1}(K) \subseteq \mathcal{F}_K$. Hence $\mu(K) = \mathbf{P}\left(s^{-1}(K)\right) \leq \mathbf{P}(\mathcal{F}_K) = T(K)$. We have therefore produced a regular Borel measure μ such that $\mu(K) \leq T(K)$ for every compact $K \subseteq \mathbb{R}$.

It remains to be shown that x is μ-Hippocrates random. We proceed by contradiction. Suppose that $\{U_i\}_{i\in\omega}$ is a μ-Hippocrates test such that $x \in \bigcap_{i\in\omega} U_i$. We follow the enumeration of $\{U_i\}_{i\in\omega}$ and construct a $\mathbf{P}$-ML test $\{\mathcal{V}_i\}_{i\in\omega}$ by adding a basic open set to $\mathcal{V}_i$ for each interval (a, b) enumerated into U_i:

(1) If $b \leq q$, then we add $\emptyset$ to $\mathcal{V}_i$;
(2) If $a < q < b$, then we add $\mathcal{F}_{(q,b)}$ to $\mathcal{V}_i$;
(3) If $q \leq a$, then we add $\mathcal{F}^{[q,a]}_{(a,b)}$ to $\mathcal{V}_i$.

This makes $\{\mathcal{V}_i\}_{i\in\omega}$ a Σ^0_1 sequence and ensures that $\mathcal{V}_i \subseteq s^{-1}(U_i)$ for each $i \in \omega$. It then follows that $\mathbf{P}(\mathcal{V}_i) \leq \mathbf{P}\left(s^{-1}(U_i)\right) = \mu(U_i) \leq 2^{-i}$ for each $i \in \omega$. Therefore $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}$-ML test.

We know that $q < x$, $x \in F$, and $x \in U_i$ for each $i \in \omega$. Hence x is in some interval (a, b) with $q < b$ that is added to U_i. If $a < q$, then $\mathcal{F}_{(q,b)} \subseteq \mathcal{V}_i$ by construction. In this case $F \in \mathcal{V}_i$ because $x \in F$ and $x \in (q, b)$. If $q \leq a$, then

$\mathcal{F}^{[q,a]}_{(a,b)} \subseteq \mathcal{V}_i$ by construction. Because x is the least element of F greater than q we again find that $F \in \mathcal{V}_i$. Hence $F \in \mathcal{V}_i$ in either case and so $F \in \bigcap_{i\in\omega} \mathcal{V}_i$. This contradicts the assumption that F is $\mathbf{P}$-ML random. Therefore x must be μ-Hippocrates random.

($\Leftarrow$) This direction of the proof is implicit in [6]. For completeness we provide a proof (which is substantially different from that in [6] but which is essentially the same as the proof of the corresponding direction of theorem 4.6).

Let μ be a regular Borel measure on $\mathbb{R}$ such that $\mu(K) \leq T(K)$ for all compact $K \subseteq \mathbb{R}$. Suppose that $x \in \mathbb{R}$ is such that no closed set containing x is $\mathbf{P}$-ML random. It follows that $\mathcal{F}_{\{x\}}$ consists entirely of non-$\mathbf{P}$-ML random closed sets. We have exactly the same situation as in the proof of theorem 4.6 and our construction is very much the same. This time, however, we construct a μ-Hippocrates test, not an m-ML test.

Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a universal $\mathbf{P}$-ML test and let $\mathcal{U}_{n,s}$ be the stage s approximation to $\mathcal{U}_n$. We build V_n and C_n in stages. At stage 0 we set $V_{n,0} = C_{n,0} = \emptyset$.

Suppose that we have enumerated j intervals into $V_{n,s}$. We wait for the next stage t such that there is $i \leq t$ with $\overline{I}_i \nsubseteq V_{n,s}$ and $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n+1,t}$. Let i be the least index such that $\overline{I}_i$ satisfies these conditions. We then take an open rational interval I such that $\overline{I}_i \subseteq I$ and

$$T\left(\overline{(I \setminus \overline{I}_i)}\right) \leq 2^{-(2+n+j)}. \tag{2}$$

We then set $V_{n,t} = V_{n,s} \cup I$ and $C_{n,t} = C_{n,s} \cup \overline{I}_i$.

Let $V_n = \bigcup_{s\in\omega} V_{n,s}$ and $C_n = \bigcup_{s\in\omega} C_{n,s}$. Equation 2 ensures that V_n will not be much larger than C_n. The total error, $\mu(V_n \setminus C_n)$, is no more than the sum over $j \in \omega$ of the errors $\mu\left(I \setminus \overline{I}_i\right)$. We may calculate these errors:

$$\mu\left(I \setminus \overline{I}_i\right) \leq \mu\left(\overline{(I \setminus \overline{I}_i)}\right) \leq T\left(\overline{(I \setminus \overline{I}_i)}\right) \leq 2^{-(2+n+j)}.$$

Consequently

$$\mu(V_n \setminus C_n) \leq \sum_{j\in\omega} 2^{-(2+n+j)} = 2^{-(n+1)}.$$

This becomes useful after we have calculated $\mu(C_n)$. By regularity $\mu(C_n) = \lim_{s\to\infty} \mu(C_{n,s})$. Because $C_{n,s}$ is compact for every $s \in \omega$ it follows that $\mu(C_{n,s}) \leq T(C_{n,s})$. Consequently $\mu(C_n) \leq \lim_{s\to\infty} T(C_{n,s})$. But

$$\lim_{s\to\infty} T(C_{n,s}) = \lim_{s\to\infty} \mathbf{P}\left(\mathcal{F}_{C_{n,s}}\right) = \mathbf{P}\left(\bigcup_{s\in\omega} \mathcal{F}_{C_{n,s}}\right) = \mathbf{P}\left(\mathcal{F}_{C_n}\right).$$

By construction $\mathcal{F}_{C_n} \subseteq \mathcal{U}_{n+1}$ and hence $\mathbf{P}(\mathcal{F}_{C_n}) \leq 2^{-(n+1)}$. Therefore $\mu(C_n) \leq 2^{-(n+1)}$.

By construction V_n is Σ^0_1 and we note that this is uniform over $n \in \omega$. Furthermore,

$$\mu(V_n) = \mu(C_n) + \mu(V_n \setminus C_n) \leq 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}.$$

Therefore $\{V_i\}_{i\in\omega}$ is a μ-Hippocrates test.

Exactly as in the proof of theorem 4.6 we know that $x \in V_n$ for each $n \in \omega$. Consequently x in not μ-Hippocrates random. □

4.2. **More generalized Poisson processes.** We now consider the capacity $T : \mathcal{K} \to [0,1]$ given by $T(K) = 1 - e^{-\Lambda(K)}$ where Λ is a locally finite, non-atomic, regular, Borel measure on $\mathbb{R}$. Recall that the measure Λ is locally finite if every real number x has a neighborhood with finite Λ-measure and Λ is non-atomic if for every $x \in \mathbb{R}$ we have $\Lambda(\{x\}) = 0$. The measure Λ is regular if for any compact sets $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$ it follows that $\Lambda\left(\bigcup_{i \ni \omega} K_i\right) = \lim_{i\to\infty} \Lambda(K_i)$ and for any open sets $U_0 \supseteq U_1 \supseteq U_2 \supseteq \dots$ it follows that $\Lambda\left(\bigcap_{i\ni\omega} U_i\right) = \lim_{i\to\infty} \Lambda(U_i)$. As in the preceding section, $\mathbf{P}$ is the measure on $\mathcal{F}$ induced by the capacity T. Some of the results of the preceding section can be shown to hold in this more general case by relativizing to appropriate oracles. In this case we no longer know that the measure Λ is computable, however Lemma 4.1 shows that Λ and $\mathbf{P}$ have the same representations.

We begin with a more general version of Theorem 4.6. The proof is essentially the same so we provide only a brief explanation.

Theorem 4.15. *A real number x is Λ-ML random if and only if there is some $\mathbf{P}$-ML random closed set F such that $x \in F$.*

Proof. First we prove that if F contains a non-Λ-ML random real x, then F is not $\mathbf{P}$-ML random. This means showing that for any representation r of $\mathbf{P}$ there is an r-ML test that catches F.

Suppose that $x \in \mathbb{R}$ is not Λ-ML random and $x \in F$. Let $r \in 2^\omega$ be any representation of $\mathbf{P}$. By Lemma 4.1, r is also a representation of Λ. Because x is not Λ-ML random, there is an r-ML test in $\mathbb{R}$, $\{U_i\}_{i\in\omega}$, such that $x \in \bigcap_{i\in\omega} U_i$. As in the proof of theorem 4.6, the sequence of sets $\{\mathcal{F}_{U_i}\}_{i\in\omega}$ is $\Sigma_1^{0,r}$ and $\mathbf{P}(\mathcal{F}_{U_i}) = 1 - e^{\Lambda(U_i)} \leq 1 - e^{2^{-i}}$. Taking an appropriate sub-sequence of $\{\mathcal{F}_{U_i}\}_{i\in\omega}$ gives an r-ML test in $\mathcal{F}$. As before, $x \in F$ ensures that $F \in \bigcap_{i\in\omega} \mathcal{F}_{U_i}$. Because this works for any representation r of $\mathbf{P}$, F is not $\mathbf{P}$-ML random.

As in the proof of Theorem 4.6 we prove the converse by showing that if every closed set containing x is not $\mathbf{P}$-ML random, then x is not Λ-ML random. Most of the corresponding part of the proof of Theorem 4.6 was purely topological and had nothing to do with the measures in question. Hence we need only to verify that the final construction of an appropriate Martin-Löf test can be completed using a suitable oracle.

Let $x \in \mathbb{R}$ and suppose that no closed set containing x is $\mathbf{P}$-ML random. Let r be a representation of Λ. We wish to construct an r-ML test that catches x. It follows from Lemma 4.1 that r is a representation of $\mathbf{P}$. Thus if F is a closed set containing x, then there is an r-ML test in $\mathcal{F}$, $\{\mathcal{U}_i\}_{i\in\omega}$, such that $F \in \bigcap_{i\in\omega} \mathcal{U}_i$. We construct an r-ML test in $\mathbb{R}$, $\{V_i\}_{i\in\omega}$, exactly as in the proof of Theorem 4.6 except that the construction of V_k now relies on $\mathcal{U}_n$ where $n = \left\lceil -\log_2\left(1 - e^{-2^{-(k+1)}}\right)\right\rceil$. Very similar calculations then show that $\Lambda(V_i) \leq 2^{-i}$ and that $x \in \bigcap_{i\in\omega} V_i$.

It remains only to show that the construction makes $\{V_i\}_{i\in\omega}$ a uniformly $\Sigma_1^{0,r}$ sequence. In this case the approximating sets $\mathcal{U}_{n,s}$ are uniformly r-computable. Note, however, that each approximating set is a finite union of basic open sets and thus working with any one of these is just as effective as it was in the proof of Theorem 4.6. Consequently, determining if $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,t}$ is computable for any given $t \in \omega$ and uniformly computable in r over all $t \in \omega$. Finally, finding an open rational interval I such that $\Lambda(I \setminus \overline{I}_i) \leq 2^{-(2+k+j)}$ is also uniformly computable in r. Hence $\{V_i\}_{i\in\omega}$ is $\Sigma_1^{0,r}$.

We now know that our construction produces an r-ML test that catches x. Since this construction works for any representation r if Λ, it follows that x is not Λ-ML random. □

Theorem 4.14 also extends to the more general case by adding an appropriate oracle everywhere.

Theorem 4.16. *Let $x \in \mathbb{R}$. There is a $\mathbf{P}$-ML random closed set F such that $x \in F$ if and only if there is a regular Borel measure μ on $\mathbb{R}$ such that $\mu(K) \leq T(K)$ for every compact $K \subseteq \mathbb{R}$ and x is μ-Hippocrates random relative to every representation of $\mathbf{P}$.*

We note that Theorems 4.16 and 4.15 are not directly related. In the case of Theorem 4.15 we start with a measure Λ and prove that a real number is Λ-ML random if and only if it is an element of some $\mathbf{P}$-ML random closed set (where $\mathbf{P}$ comes from the capacity $T(K) = 1 - e^{-\Lambda(K)}$). In Theorem 4.16 we prove that a real number is Hippocrates random under *some* measure μ with certain properties if and only it is an element of some $\mathbf{P}$-ML random closed set. One of the properties of the measure μ is the requirement that $\mu(K) \leq T(K)$ for every compact $K \subseteq \mathbb{R}$. This ensures that $\mu(\mathbb{R}) \leq 1$. Thus Theorem 4.16 only applies to finite measures on $\mathbb{R}$. The measure Λ in need not be a probability measure and our most important example, the Lebesgue measure m, is not a probability measure. Combining Theorem 4.6 with Theorem 4.14 gives the following corollary.

Corollary 4.17. *A real number x is m-ML random if and only if there is a regular Borel measure μ on $\mathbb{R}$ such that x is μ-Hippocrates random and $\mu(K) \leq 1-2^{-m(K)}$ for every compact $K \subseteq \mathbb{R}$.*

Not all of the results of the previous section (i.e. Lemma 4.4 and its corollary and Proposition 4.7) hold in this more general situation. These results depend on properties of the measure m which may not be shared by the measure Λ. Lemma 4.1 still applies, however, so when Λ shares these properties these results can be shown to hold. For example, if $\lim_{i\to\infty} \Lambda([M,i]) = \infty$ for any $M \in \omega$, then at least half of Proposition 4.7 will still hold and we can conclude that if F is $\mathbf{P}$-ML random, then F is not bounded above.

Theorem 4.8 and its corollaries do hold in the more general situation. The generalized version of Theorem 4.8 relies on the regularity and non-atomicity of the measure Λ. Corollary 4.9 can also be extended to this situation (though a slightly different method of proof is required if Lemma 4.4 does not extend).

References

[1] Logan Axon. Martin-löf randomness in spaces of closed sets. *Journal of Symbolic Logic*, 80:359–383, 6 2015. ISSN 1943-5886. doi: 10.1017/jsl.2014.76. URL `http://journals.cambridge.org/article_S0022481214000760`.

[2] George Barmpalias, Paul Brodhead, Douglas Cenzer, Seyyed Dashti, and Rebecca Weber. Algorithmic randomness of closed sets. *J. Logic Comput.*, 17(6): 1041–1062, 2007. ISSN 0955-792X.

[3] Adam R. Day and Joseph S. Miller. Randomness for non-computable measures. *Trans. Am. Math. Soc.*, 365:3575–3591, 2013.

[4] Peter Hertling and Klaus Weihrauch. Random elements in effective topological spaces with measure. *Inform. and Comput.*, 181(1):32–56, 2003. ISSN 0890-5401. doi: 10.1016/S0890-5401(02)00034-2. URL `http://dx.doi.org/10.1016/S0890-5401(02)00034-2`.

[5] Mathieu Hoyrup and Cristóbal Rojas. Computability of probability measures and Martin-Löf randomness over metric spaces. *Inform. and Comput.*, 207(7): 830–847, 2009. ISSN 0890-5401. doi: 10.1016/j.ic.2008.12.009. URL `http://dx.doi.org/10.1016/j.ic.2008.12.009`.

[6] Bjørn Kjos-Hanssen. Infinite subsets of random sets of integers. *Math. Res. Lett.*, 16(1):103–110, 2009. ISSN 1073-2780. doi: 10.4310/MRL.2009.v16.n1.a10. URL `http://dx.doi.org/10.4310/MRL.2009.v16.n1.a10`.

[7] G. Matheron. *Random sets and integral geometry.* John Wiley & Sons, New York-London-Sydney, 1975. With a foreword by Geoffrey S. Watson, Wiley Series in Probability and Mathematical Statistics.

[8] Ilya Molchanov. *Theory of random sets.* Probability and its Applications (New York). Springer-Verlag London Ltd., London, 2005. ISBN 978-185223-892-3; 1-85233-892-X.

[9] Hung T. Nguyen. *An introduction to random sets.* Chapman & Hall/CRC, Boca Raton, FL, 2006. ISBN 978-1-58488-519-1; 1-58488-519-X.

[10] Jan Reimann and Theodore A. Slaman. Measures and their random reals. Submitted for publication. Available at `http://www.personal.psu.edu/jsr25/Publications/measures_and_random_reals_preprint.pdf`.

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