ALGORITHMICALLY RANDOM CLOSED SETS AND PROBABILITY

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Abstract

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Algorithmic randomness in the Cantor space, 2^ω, has recently become the subject of intense study. Originally defined in terms of the fair coin measure, algorithmic randomness has since been extended, for example in Reimann and Slaman [22, 23], to more general measures. Others have meanwhile developed definitions of algorithmic randomness for different spaces, for example the space of continuous functions on the unit interval (Fouché [8, 9]), more general topological spaces (Hertling and Weihrauch [12]), and the closed subsets of 2^ω (Barmpalias et al. [1], Kjos-Hanssen and Diamondstone [14]). Our work has also been to develop a definition of algorithmically random closed subsets. We take a very different approach, however, from that taken by Barmpalias et al. [1] and Kjos-Hanssen and Diamondstone [14].

One of the central definitions of algorithmic randomness in Cantor space is Martin-Löf randomness. We use the probability theory of random closed sets (RACS) to prove that Martin-Löf randomness can be defined in the space of closed subsets of any locally compact, Hausdorff, second countable space. We then explore the Martin-Löf random closed subsets of the spaces $\mathbb{N}$, 2^ω, and $\mathbb{R}$ under different measures. In the case of 2^ω we prove that the definitions of Barmpalias et al. [1] and Kjos-Hanssen and Diamondstone [14] are compatible with our approach. In the case of $\mathbb{N}$ we prove that the Martin-Löf random subsets are exactly those with Martin-Löf random characteristic functions. In the case of $\mathbb{R}$ we investigate the Martin-Löf random closed sets under generalized Poisson processes. This leads to a characterization of the Martin-Löf random elements of $\mathbb{R}$ as exactly the reals contained in some Martin-Löf random closed subset of $\mathbb{R}$.

CONTENTS

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CHAPTER 1

INTRODUCTION

It is easy to generate a random binary sequence: simply flip a fair coin and record the outcome of each flip as 0 for heads and 1 for tails. The laws of probability tell us that in the long run we should expect to get the same number of 0s and 1s, that we can expect to see any finite sequence of 0s and 1s, et cetera. What we certainly do not expect to get is a sequence like $101101110111110\ldots$, or any other sequence with a pattern. However, this sequence is as likely to be the outcome of our coin flips as any other sequence (each infinite sequence occurs with probability zero). Of course most sequences look more random than the sequence $101101110111110\ldots$, even if each one is just as likely to occur. Algorithmic randomness (also called effective randomness) formalizes the idea that some sequences are more random than others by combining computability theory with probability.

One of the most important definitions in algorithmic randomness is Martin-Löf randomness which we introduce in section 2.3. Algorithmically random sequences have received a lot of attention recently. Many of the results in the area of algorithmically random sequences have been collected in Downey et al. [5] and Nies [20]. The explosion of interest in algorithmically random sequences has led researchers to ask whether similarly interesting behavior is possible in other settings. One alternative setting is $C[0,1]$, the set of continuous functions from the unit interval to $\mathbb{R}$. This was first studied by Fouché, [9] and [8], and later by Kjos-Hanssen and Nerode [15] who showed that results about algorithmically random functions could be obtained by producing a measure algebra homomorphism between 2^ω and $C[0,1]$. As we will see this basic idea can be used in other settings, in particular in the space of closed subsets of certain topological spaces.

Algorithmically random closed sets were first studied by Barmpalias et al. [1] who defined a notion of randomness for closed subsets of Cantor space by coding each infinite binary tree without dead ends as a ternary real. Every closed subset of Cantor space can be uniquely represented as the set of paths through such a tree. Barmpalias et al. [1] defined a closed set to be random if the code for the corresponding tree was Martin-Löf random. This definition is explored here in section 4.1.

Probability theorists and statisticians, on the other hand, have defined a random closed set to be something quite different. A random closed set as defined in the literature (in particular the books of Matheron [16], Molchanov [18], and Nguyen [19]) is simply a closed set-valued random variable. That is, a random closed set is a measurable map from a probability space to $\mathcal{F}(\mathbb{E})$, the space of closed sets the topological space $\mathbb{E}$. This is formalized using the hit-or-miss topology (also known as the Fell topology) on the space $\mathcal{F}(\mathbb{E})$. We introduce this probabilistic theory of random closed sets in section 2.4. This is obviously a very different idea than that developed by Barmpalias et al. [1] but there are connections. We prove that the coding of closed sets of Cantor space used by Barmpalias et al. [1] is actually an example of a particularly nice random closed set (in the probability theory sense). This is lemma 4.1.4 in section 4.1.

As in the case of real-valued random variables, random closed sets (in the probability sense) induce an probability measure on the target space. The connection with algorithmic randomness comes when we use such a measure to produce Martin-Löf tests in the space of closed sets. This allows for the study Martin-Löf random closed sets purely from the perspective of the space of closed sets and the hit-or-miss topology. In section 3 we develop a general theory of Martin-Löf random closed sets. One key result here is proposition 3.0.9, which establishes that Martin-Löf randomness can be defined in the space of closed subsets of a locally compact, Hausdorff, second countable space using the hit-or-miss topology. This allows us to talk about Martin-Löf random closed subsets of such spaces. We note that different random closed sets give rise to different measures and hence different classes of Martin-Löf random closed sets. Other important results in this section are the technical lemmas 3.0.12 and 3.0.13, which are used extensively in later examples.

The bulk of this paper is an exploration of examples of specific random closed sets (in the probability sense) and the Martin-Löf random closed sets they give rise to. We begin by looking at the example of Martin-Löf random closed subsets of $\mathbb{N}$. In section 3.1 we prove that the Martin-Löf random closed subsets of $\mathbb{N}$ are exactly those subsets with a Martin-Löf random characteristic function (in 2^ω).

Our first major example of a random closed set is the coding defined by Barmpalias et al. [1]. Having established in lemma 4.1.4 that this coding is a measurable map from $3^\omega \to \mathcal{F}(2^\omega)$ we are then able to prove that a closed set is Martin-Löf random in $\mathcal{F}(2^\omega)$ if and only if it is the image of a Martin-Löf random element of 3^ω (corollary 4.1.5). This means that our definition of Martin-Löf random closed sets agrees exactly with the definition of algorithmically random closed sets given by Barmpalias et al. [1]. Our approach, however, allows for the use of theorems from probability theory of random closed sets. In proposition 4.1.9, for instance, we use one of these tools (Robbins' theorem) to prove that for this example Martin-Löf random closed sets have measure 0. This is a result originally proved in Barmpalias et al. [1] (although in less generality) by a different technique. The main obstacle in that proof is nicely resolved by our application of Robbins' theorem.

The coding map of Barmpalias et al. [1] is just one example of a random closed set. Our framework is very flexible and allows for some very different random closed sets. However, our next example, the Galton-Watson random closed sets of Kjos-Hanssen and Diamondstone [14], is another map defined by coding closed subsets of 2^ω in a Cantor space. This example is explored in section 4.2. Kjos-Hanssen and Diamondstone [14] prove that a closed set is Galton-Watson random if and only if it is either $\emptyset$ or an algorithmically random closed set in the sense of Barmpalias et al. [1]. Kjos-Hanssen and Diamondstone cite lemma 4.2.13 in their proof of this result. We also prove that the coding used by Kjos-Hanssen and Diamondstone [14] is a random closed set and that it maps Martin-Löf random reals to Martin-Löf random closed sets (lemma 4.2.11).

Section 4.3 deals with the random fractal constructions of Mauldin and McLinden [4]. We show that this approach fits into our framework and that these constructions preserve Martin-Löf randomness. Sections 4.4 and 4.6 are easy examples of random closed sets that have not been examined from the perspective of algorithmic randomness. In each of these examples we are able to provide a complete characterization of the Martin-Löf random closed sets. Section 4.5 introduces a canonical family of random closed sets, those given by maxitive capacities. These are a generalization of the random closed sets in section 4.4. We have only preliminary results for this example.

The last two sections, 5.1 and 5.2, deal with another canonical family of random closed sets from the probability theory literature. These "generalized Poisson processes" (or "Poisson point processes") give probability measures on the spaces of closed subsets of $\mathbb{R}$, $\mathcal{F}(\mathbb{R})$, under which a closed set is almost surely countable and discrete. We prove that this holds effectively: every Martin-Löf random closed set is countable and discrete. We also prove a new characterization of of the Martin-Löf random reals in theorem 5.1.5 as exactly those points that are members of some Martin-Löf random closed set. In section 5.2 we generalize this result to non-computable measures. We also prove another result, theorem 5.2.6, characterizing the elements of Martin-Löf random closed sets as exactly those points which are Martin-Löf random relative to a probability measure on $\mathbb{R}$ which is "smaller" than our measure on $\mathcal{F}(\mathbb{R})$.

CHAPTER 2

BACKGROUND MATERIAL

2.1 Notational conventions

Because we are working in computability theory we are using some of the slightly non-standard notational conventions of mathematical logic.

1. $\omega = \{0, 1, \dots\}$.

2. 2^ω is the collection of functions $f : \omega \to \{0, 1\}$. This is often denoted $\{0, 1\}^{\mathbb{N}}$ outside of mathematical logic. We think of elements of 2^ω as infinite binary sequences.

3. As in the notation 2^ω, a natural number $n \in \omega$ may be used to refer to the initial segment of ω of length n, i.e. $n = \{0, 1, \dots, n-1\}$.

4. 2^n is sometimes the set of functions $f : n \to 2$, i.e. the set of all binary strings with length n. At other times 2^n is simply the usual cardinal. The distinction should be clear from context.

5. $2^{<\omega} = \bigcup_{n \in \omega} 2^n$, i.e. the set of all finite binary strings.

6. $\langle\ ,\ \rangle : \omega^2 \to \omega$ is a computable pairing function.

7. For $f \in 2^\omega$ and $i \in \omega$ we denote the i^{th} column of f by $f^{[i]}$, i.e. $f^{[i]}(n) = f(\langle i, n \rangle)$ for all $n \in \omega$.

We are also dealing with random closed sets which means that we wish at the same time to talk about a space, elements of the space, subsets of the space, and collections of subsets of the space. Our notational convention here is to use lowercase letters for elements of the space (e.g. $f \in 2^\omega$ or $x \in \mathbb{E}$), plain uppercase for subsets of the space (e.g. $U \subseteq 2^\omega$ or $F \subseteq \mathbb{E}$), and calligraphic uppercase for collections of subsets of the space (e.g. $\mathcal{F}(\mathbb{E}) = \{F \subseteq \mathbb{E} : F \text{ is closed}\}$).

Given a space $\mathbb{E}$ and a subset $A \subseteq \mathbb{E}$ we will denote the closure of A by $\overline{A}$ and the complement of A by A^{C}. Finally, all measures we will deal with are Borel and so even if not explicitly stated it is safe to assume that every σ-algebra mentioned here is Borel.

2.2 A (very) quick review of probability

Some understanding of basic probability theory will be necessary for the following. We provide here some basic definitions and vocabulary of probability theory. This material can be found in any book on probability and statistics. A good general reference is Grimmett and Stirzaker [11].

Definition 2.2.1.

1. Let $(\Omega, \mathcal{A}, P)$ be a measure space with underlying space Ω, σ-algebra $\mathcal{A}$ and measure P. We say that $(\Omega, \mathcal{A}, P)$ is a *probability space* if $P(\Omega) = 1$. Often we will simply say that Ω is a probability space, leaving $\mathcal{A}$ and P implicit.

2. Let $(\Omega, \mathcal{A}, \mu)$ be any measure space (not necessarily a probability space). We say that a property A holds μ-*almost everywhere* if

$$\mu\left(\{\alpha \in \omega : \alpha \text{ does not satisfy property } A\}\right) = 0.$$

When $(\Omega, \mathcal{A}, \mu)$ is a probability space "μ-*almost surely*" may be used in place of "μ-almost everywhere".

3. Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be measure spaces. A map $\varphi : \Omega_1 \to \Omega_2$ is *measurable* if for every $A \in \mathcal{A}_2$ $\varphi^{-1}(A) \in \mathcal{A}_1$.

4. Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be measure spaces and let $\varphi : \Omega_1 \to \Omega_2$ be measurable. Then the *image measure* induced by φ on Ω_2 is defined for all $A \in \mathcal{A}_2$ by

$$\mu_\varphi(A) = \mu_1\left(\varphi^{-1}(A)\right).$$

A particularly important special case of the preceding is the space $\mathbb{R}$. We will always assume that $\mathbb{R}$ is equipped with the Borel σ-algebra and so we will not specify an algebra when dealing with this space.

Definition 2.2.2. Let $(\Omega, \mathcal{A}, P)$ be a probability space.

1. A *random variable* is a measurable map $X : \Omega \to \mathbb{R}$.

2. Every random variable $X : \Omega \to \mathbb{R}$ induces an image measure on $\mathbb{R}$. This measure is called the *distribution* if X and is a probability measure.

Note that every probability measure on $\mathbb{R}$ is the distribution function of a random variable: if $\mathbb{R}$ is equipped with a probability measure, then the identity map is a random variable with this measure as its distribution. We have just seen in the preceding definition that a random variable gives rise to a probability measure on $\mathbb{R}$. Thus random variables and probability measures on $\mathbb{R}$ are equivalent and in fact "random variable" is used in the literature to refer to both.

Definition 2.2.3. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable.

1. For $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ we write $P(X = x)$ instead of $P(\{\alpha \in \Omega : X(\alpha) = x\})$ and $P(X \in A)$ instead of $P(\{\alpha \in \Omega : X(\alpha) \in A\})$.

2. A function $\rho : \mathbb{R} \to \mathbb{R}$ is the *density* function of X if for every Borel $A \subseteq \mathbb{R}$

$$P(X \in A) = \int_A \rho(x)dx.$$

3. If X has density function ρ then the *expected value of* X (or *expectation* of X) is defined by

$$E(X) = \int_{\mathbb{R}} x\rho(x)dx.$$

4. If there are $a_1, a_2, \ldots, a_n$ such that $\sum_{i=1}^{n} P(X = a_i) = 1$, then the *expected value of* X is defined by

$$E(X) = \sum_{n=1}^{n} a_i P(X = a_i).$$

2.3 Algorithmic randomness

This section describes the classical theory of Martin-Löf randomness for 2^ω. Much greater detail can be found in Nies [20] or Downey et al. [5]. The set 2^ω is endowed with the infinite product topology and is homeomorphic to the Cantor middle thirds set. For this reason 2^ω is often called Cantor space.

Definition 2.3.1. $2^{<\omega}$ is the set of finite binary sequences. For $\sigma, \tau \in 2^{<\omega}$ and $h \in 2^\omega$

1. $|\sigma|$ denotes the length of σ;

2. $\sigma \preceq \tau$ if $|\sigma| \leq |\tau|$ and $\forall n < |\sigma| \;\; \sigma(n) = \tau(n)$;

3. $\sigma \prec h$ if $\forall n < |\sigma| \;\; \sigma(n) = h(n)$;

4. $h \upharpoonright n$ is the string of length n such that $h \upharpoonright n \prec h$;

5. $[\sigma] := \{f \in 2^\omega : f \succ \sigma\}$, the cylinder determined by σ.

We prefer to think about the topology on 2^ω using the following basis.

Proposition 2.3.2. *The collection of cylinders $\{[\sigma] : \sigma \in 2^{<\omega}\}$ forms a basis for the topology of 2^ω.*

Proposition 2.3.3 (Some facts about 2^ω)**.**

1. *2^ω is compact.*

2. *2^ω is Hausdorff.*

3. *$[\sigma]$ is closed and open (*clopen*) for each $\sigma \in 2^{<\omega}$.*

We now fix a computable enumeration $\sigma_0, \sigma_1, \sigma_2, \ldots$ of $2^{<\omega}$. This, of course, gives a computable enumeration of our basis for the topology on 2^ω. We will use this to determine the algorithmic complexity of subsets of 2^ω.

Definition 2.3.4.

1. $U \subseteq 2^\omega$ is Σ^0_1 if there is a computably enumerable $f \in 2^\omega$ such that $U = \bigcup_{f(n)=1} [\sigma_n]$.

2. $F \subseteq 2^\omega$ is Π^1_0 if the complement of F is Σ^0_1.

3. $U \subseteq 2^\omega$ is Δ^0_1 if U is both Σ^0_1 and Π^0_1.

Note that the members of a Σ^0_1 class $U \subseteq 2^\omega$ are not necessarily Σ^0_1 themselves. Note also that Σ^0_1 subsets of 2^ω are open (and hence Π^0_1 subsets of 2^ω are closed).

Of course there are only countably many Σ^0_1 subsets while there are uncountably many open sets and so it is not true that open sets must be Σ^0_1. However we can make the following topological characterization of the Δ^0_1 subsets of 2^ω.

Proposition 2.3.5. *$U \subseteq 2^\omega$ is Δ^0_1 if and only if U is clopen.*

Proof. ($\Rightarrow$) Suppose that U is Δ^0_1. Because U is Σ^0_1 there is $f \in 2^\omega$ such that $U = \bigcup_{f(n)=1}[\sigma_n]$. Because U is Π^0_1, U is closed. Cantor space is compact and so U must be compact. Consequently there is $N \in \omega$ such that $U = \bigcup_{f(n)=1,n<N}[\sigma_n]$. Any finite union of clopen sets is clopen and therefore U is clopen.

($\Leftarrow$) Suppose now that U is clopen. U is open and so there is some (possibly non-computable) $f \in 2^\omega$ such that $U = \bigcup_{f(n)=1}[\sigma_n]$. U is closed and hence compact and so we again have $N \in \omega$ such that $U = \bigcup_{f(n)=1,n<N}[\sigma_n]$. Define a new element of 2^ω, $\hat{f} = (f \upharpoonright N)^\frown 0^\omega$. Then $\hat{f}$ is clearly computable and $U = \bigcup_{\hat{f}(n)=1}[\sigma_n]$. Therefore U is Σ^0_1. The same argument applies to $U^\complement$ and so it follows that U is Δ^0_1. $\square$

Definition 2.3.6. A subset $T \subseteq 2^{<\omega}$ is a *tree* if T is downwards closed under $\preceq$:

$$(\forall \sigma \in T)\,(\forall \tau \in 2^{<\omega})\,[\tau \preceq \sigma \implies \tau \in T]\,.$$

Definition 2.3.7.

1. $f \in 2^\omega$ is a *path* through the tree $T \subseteq 2^{<\omega}$ if and only if $\forall n \in \omega\ f \upharpoonright n \in T$.

2. Given a tree $T \subseteq 2^{<\omega}$, let $[T]$ denote the collection of paths through T.

3. A tree $T \subseteq 2^{<\omega}$ is *extensible* if for each $\sigma \in T$ there is $\tau \in T$ such that $\sigma \precneqq \tau$.

Note that square brackets are being used in two ways: $[\sigma]$ for a string σ is the collection of extensions of σ and $[T]$ for a tree T is the collection of paths through T. This distinction is always clear from context.

Proposition 2.3.8. *A subset $F \subseteq 2^\omega$ is closed if and only if there is a tree $T \subseteq 2^{<\omega}$ such that $F = [T]$.*

Proof. ($\Rightarrow$) Given $F \subseteq 2^\omega$ closed, let $T = \{\sigma \in 2^{<\omega} : (\exists f \in F)\ \sigma \prec f\}$. Clearly T is a tree and $[T] \supseteq F$. It remains only to prove that $[T] \subseteq F$. If $F = \emptyset$, then clearly $T = \emptyset$ and therefore $[T] = \emptyset = F$. So suppose that $F \neq \emptyset$ and $g \in [T]$. By definition $g \restriction n \in T$ for each $n \in \omega$. By the definition of T, for each $n \in \omega$ there is $f \in F$ such that $g \restriction n \prec f$. Hence $[g \restriction n] \cap F \neq \emptyset$ for all $n \in \omega$. In other words, g is a limit point of F. Closed sets contain their limit points and therefore $g \in F$.

($\Leftarrow$) Given a tree $T \subseteq 2^{<\omega}$, let $F = [T]$. Then

$$F^{\complement} = \bigcup_{\sigma \notin T} [\sigma].$$

Hence $F^{\complement}$ is open and F is closed. ∎

Cantor space is usually endowed with a Borel probability measure, m, such that for each $\sigma \in 2^{<\omega}$

$$m([\sigma]) = 2^{-|\sigma|}.$$

This measure is sometimes called the fair coin measure because the probability that a given string σ is generated by a sequence of flips of a fair coin (taking heads as 0 and tails as 1, say) is exactly $2^{-|\sigma|}$.

We note that the binary expansion map $q : 2^\omega \to [0,1] \subseteq \mathbb{R}$ given by

$$q(f) = \sum_{n \in \omega} \frac{f(n)}{2^{n+1}}$$

induces a measure algebra isomorphism between the Borel σ-algebra on 2^ω with the coin toss measure and the Borel σ-algebra on $[0,1]$ with Lebesgue measure. This map is the justification behind calling elements of 2^ω "reals". This is fine as long as we keep in mind the fact that the topology on 2^ω is *very* different from the topology on $[0,1]$.

The basic idea behind Martin-Löf randomness is that the "laws" of probability theory describe random behavior. For example, the strong law of large numbers says that an element f of 2^ω will almost surely satisfy the following equation

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} f(i)}{n} = \frac{1}{2}.$$

A truly random sequence should obey all such laws. In particular, if we think of each law as describing a set of measure 1, we would like to say that the intersection of all these measure 1 sets is exactly the set of random sequences. Unfortunately, given any $g \in 2^\omega$ there is a law saying that $f \in 2^\omega$ is almost surely not g. Therefore the intersection of all these measure 1 sets is exactly the empty set.

Algorithmic randomness solves this problem by restricting the collection of laws that must be obeyed to just those that are sufficiently effective. It is also traditional to think in terms of the complements of these measure 1 sets: laws of probability determine measure 0 sets of non-random sequences and random sequences must avoid all such sets. This, then, is the short, informal definition of Martin-Löf randomness: $f \in 2^\omega$ is Martin-Löf random if it avoids all effective measure 0 sets. The formal definitions follow shortly.

Definition 2.3.9.

1. A sequence of subsets of 2^ω, $\{U_i\}_{i\in\omega}$, is *uniformly* Σ^0_1 if there is a computably enumerable $f \in 2^\omega$ such that

$$U_i = \bigcup_{f(<i,n>)=1} [\sigma_n].$$

2. A *Martin-Löf test (ML test)* is a uniformly Σ^0_1 sequence of subsets of 2^ω, $\{U_i\}_{i\in\omega}$, such that $m(U_i) \leq 2^{-i}$.

3. $f \in 2^\omega$ is *Martin-Löf random (or ML random or 1-random)* if there is no Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $f \in \bigcap_{i\in\omega} U_i$.

4. The collection of ML random elements of 2^ω is denoted by $\mathbf{R}(2^\omega)$.

One crucial fact about Martin-Löf randomness is the following:

Lemma 2.3.10. *There is a Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $f \in \mathbf{R}(2^\omega)$ if and only if $f \notin \bigcap_{i\in\omega} U_i$. (Such a test $\{U_i\}_{i\in\omega}$ is said to be* universal*).*

A direct consequence is the following:

Corollary 2.3.11. *The set $\mathbf{R}(2^\omega)$ is measurable and $m(\mathbf{R}(2^\omega)) = 1$.*

2.4 Random closed sets

In this section we cover the basics of random closed sets of probability theory. Much greater detail can be found in the books of either Matheron [16] or Molchanov [18]. We have tried to follow the notational conventions of these books. Where notation differs we have usually followed Molchanov.

As is often the case in probability we are concerned with the measures that can be put on a particular space. In this case we are looking at the space of all closed subsets of an underlying space and the Borel σ-algebra of the Fell topology on the space of closed sets. In the cases we will consider this is the same σ-algebra as the Effros σ-algebra. Indeed, some presentations of random closed sets skip the Fell topology altogether and simply define the Effros σ-algebra. However, we wish to repeat the development of Martin-Löf randomness and for this reason we will spend some time looking at the Fell topology. The Fell topology is known by a number of names but the most common alternative is the hit-or-miss topology. The basic idea is that a closed set F is described by the open and compact sets which contain points in F.

Definition 2.4.1. Let $\mathbb{E}$ be a topological space.

1. $\mathcal{F}(\mathbb{E})$ is the collection of closed subsets of $\mathbb{E}$.
2. $\mathcal{K}(\mathbb{E})$ is the collection of compact subsets of $\mathbb{E}$.
3. $\mathcal{G}(\mathbb{E})$ is the collection of open subsets of $\mathbb{E}$.

Where the underlying space is clear from context we will omit $\mathbb{E}$ and write only $\mathcal{F}$, $\mathcal{K}$, and $\mathcal{G}$.

$\mathcal{F}$ can be given a variety of topologies. As discussed above we will be interested in a topology based on hitting or missing a given set. The following notation will allow us to describe the topology more easily.

Definition 2.4.2. Let $A \subseteq \mathbb{E}$.

1. $\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \emptyset\}$ (the hitting set of A).

2. $\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \emptyset\}$ (the missing set of A).

Note that $\mathcal{F}^A$ is the complement of $\mathcal{F}_A$ in the space $\mathcal{F}$.

The following is helpful in understanding how the sets $\mathcal{F}_A$ and $\mathcal{F}^A$ combine under unions and intersections.

Proposition 2.4.3. *If $\{A_i : i \in I\}$ is a collection of subsets of $\mathbb{E}$ then:*

1. $\bigcup_{i\in I} \mathcal{F}_{A_i} = \mathcal{F}_{\bigcup_{i\in I} A_i}$;

2. $\bigcap_{i\in I} \mathcal{F}^{A_i} = \mathcal{F}^{\bigcup_{i\in I} A_i}$;

3. $\bigcap_{i\in I} \mathcal{F}_{A_i} \supseteq \mathcal{F}_{\bigcap_{i\in I} A_i}$;

4. $\bigcup_{i\in I} \mathcal{F}^{A_i} \subseteq \mathcal{F}^{\bigcap_{i\in I} A_i}$.

It is important to realize that in general $\mathcal{F}_{A_1} \cap \mathcal{F}_{A_2} \neq \mathcal{F}_{A_1 \cap A_2}$. For example in the space $\mathcal{F}(\mathbb{R})$, $\{0,1\} \in \mathcal{F}_{[0,\frac{1}{2}]} \cap \mathcal{F}_{[\frac{1}{2},1]}$ but $\{0,1\} \notin \mathcal{F}_{\{\frac{1}{2}\}}$. This makes the following notation convenient.

Definition 2.4.4. Let $A, B_1, \ldots, B_n \subseteq \mathbb{E}$. The set $\mathcal{F}^A \cap \mathcal{F}_{B_1} \cap \ldots \mathcal{F}_{B_n}$ will be denoted $\mathcal{F}^A_{B_1,\ldots,B_n}$.

We are now ready to define a topology for $\mathcal{F}$.

Definition 2.4.5. The *Fell topology* on $\mathcal{F}$ is generated by the sub-base of sets of the form $\mathcal{F}^K$ and $\mathcal{F}_G$ where K is compact and G is open. Hence sets of the form $\mathcal{F}^K_{G_1,\ldots,G_n}$, with K compact and $G_1, \ldots, G_n$ open, form a basis for the Fell topology.

The next couple of basic results from the theory of RACS that help us to work with the Fell topology. A proof of lemma 2.4.6 can be found on page 13 of

Molchanov [18] where it is the first step of the first proof of the Choquet capacity theorem. A complete proof of lemma 2.4.7 can be found on page 3 of Matheron [16].

Lemma 2.4.6. *Let $\mathcal{V}$ be a any family of subsets of $\mathbb{E}$. Let $\mathfrak{B}$ be the family containing $\mathcal{F}_V$ and $\mathcal{F}^V$ for $V \in \mathcal{V}$ and closed under finite intersections. Then each $\mathcal{Y} \in \mathfrak{B}$ can be represented as*

$$\mathcal{Y} = \mathcal{F}^{V_{n+1} \cup \cdots \cup V_k}_{V_1, \ldots, V_n}$$

for some $k \geq n \geq 0$ and $V, V_1, \ldots, V_k \in \mathcal{V}$ with $V_i \not\subseteq V_j \cup (V_{n+1} \cup \cdots \cup V_k)$ for $i, j \leq n$ with $i \neq j$ (such a representation is called reduced*). Moreover, if $\mathcal{Y} = \mathcal{F}^{V'_{m+1} \cup \cdots \cup V'_l}_{V'_1, \ldots, V'_m}$ is another reduced representation, then $V_{n+1} \cup \cdots \cup V_k = V'_{m+1} \cup \cdots \cup V'_l$, $n = m$, and for each $i \in \{1, \ldots, n\}$ there is $j_i \in \{1, \ldots, m\}$ such that $V_i \cup V_{n+1} \cup \cdots \cup V_k = V'_{j_i} \cup V_{n+1} \cup \cdots \cup V_k$.*

It follows from the proof of lemma 2.4.6 that finding a reduced representation is computable. If $\mathcal{F}^V_{V_1, \ldots, V_n}$ is not reduced, then there are $i, j \leq n$ such that $V_i \subseteq V \cup V_j$. It then follows that removing V_i does not change the set, i.e.

$$\mathcal{F}^V_{V_1, \ldots, V_n} = \mathcal{F}^V_{V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n}.$$

Thus it is easy to compute the reduced representation of any element of $\mathfrak{B}$. Ultimately we will use this to show that it is possible to computably determine if two basic open sets of the Fell topology are equal in certain situations (to be made more precise later). This will be an important part of our use of the Fell topology for computable analysis. The next proposition is also a key result for this purpose.

Lemma 2.4.7. *Let be $\mathbb{E}$ a locally compact, Hausdorff, second countable (LCHS) space and let $\{B_0, B_1, \dots\}$ be a basis for $\mathbb{E}$ such that $\overline{B_i}$ is compact for each $i \in \omega$. Then $\mathcal{F}(\mathbb{E})$ is compact, Hausdorff, and second countable with a sub-basis consisting of the sets $\mathcal{F}_{B_i}$ and $\mathcal{F}^{\overline{B_j}}$, $i, j \in \omega$.*

We omit the proof that $\mathcal{F}$ is compact.

Proof. First we show that the sets $\mathcal{F}_{B_i}$ and $\mathcal{F}^{\overline{B_j}}$ form a sub-basis for $\mathcal{F}$. Given a closed set F and a neighborhood of F we must produce a finite intersection of sets from our proposed sub-basis that is contained in this neighborhood but still contains F. Note that this means finding smaller hitting sets and larger missing sets. We have two cases.

Case 1: $F = \emptyset$. The only open neighborhoods of $\emptyset$ are of the form $\mathcal{F}^K$ with $K \in \mathcal{K}$. Take N large enough so that $K \subseteq \bigcup_{i \leq N} B_i$. Such an $N \in \omega$ exists because K is compact. Then $F \in \bigcap_{i \leq N} \mathcal{F}^{\overline{B_i}} \subseteq \mathcal{F}^K$.

Case 2: $F \neq \emptyset$. In this case a neighborhood of F is of the form $\mathcal{F}^K_{G_1, \dots, G_n}$ for $G_1, G_2, \dots, G_n \in \mathcal{G}$ and $K \in \mathcal{K}$. Then for each $i \in \{1, 2, \dots, n\}$ we chose a point $x_i \in G_i \cap F$ and a basic open set B_{i_j} such that

$$x_i \in B_{i_j} \subseteq \overline{B_{i_j}} \subseteq G_i \cap K^{\complement}.$$

Then take $B_{k_1}, B_{k_2}, \dots, B_{k_m}$ such that $K \subseteq B_{k_1} \cup B_{k_2} \cup \dots \cup B_{k_m}$ and

$$\left(\overline{B_{k_1}} \cup \overline{B_{k_2}} \cup \dots \cup \overline{B_{k_m}}\right) \cap (B_{i_1} \cup B_{i_2} \cup \dots \cup B_{i_n}) = \emptyset.$$

This is all possible because $\mathbb{E}$ is Hausdorff. Now we have

$$F \in \mathcal{F}^{\overline{B_{k_1}} \cup \overline{B_{k_2}} \cup \dots \cup \overline{B_{k_m}}}_{B_{i_1}, B_{i_2}, \dots, B_{i_n}} \subseteq \mathcal{F}^K_{G_1 \cup G_2 \cup \dots \cup G_n}.$$

To prove that $\mathcal{F}$ is Hausdorff let F and E be closed sets such that $F \neq E$. Without loss of generality assume that $F \neq \emptyset$. Then let $x \in F \setminus E$. Because $\mathbb{E}$ is Hausdorff there is a basic open set B_i such that $x \in B_i$ and $\overline{B_i} \cap E = \emptyset$. Therefore $F \in \mathcal{F}_{B_i}$, $E \in \mathcal{F}^{\overline{B_i}}$, and $\mathcal{F}_{B_i} \cap \mathcal{F}^{\overline{B_i}} = \emptyset$. □

Proposition 2.4.8. *Let $\mathbb{E}$ be an LCHS space. The space $\mathcal{F}(\mathbb{E})$ is the one-point compactification of $\mathcal{F}(\mathbb{E}) \setminus \{\emptyset\}$ and $\mathcal{F}(\mathbb{E}) \setminus \{\emptyset\}$ is compact if and only if E is compact.*

Definition 2.4.9. A *random closed set (or RACS)* is a measurable map from a probability space to $\mathcal{F}(\mathbb{E})$ (where $\mathcal{F}(\mathbb{E})$ is equipped with the Borel σ-algebra of the Fell topology, $\mathcal{B}(\mathcal{F})$).

Because Barmpalias et al. [1] use the phrase "random closed set" to describe a very different object we will usually use the abbreviation "RACS" to refer to such a measurable map. This notation is standard in the literature (see, e. g. Matheron [16] and Molchanov [18]) and should help to avoid confusion. To further help make the distinction between these two very different definitions we will use "BBCDW-random closed set" to describe the objects studied in Barmpalias et al. [1].

In general any measurable map from a probability space to $\mathbb{R}$ is called a random variable. Such a map can be thought of as an assignment of probability to each measurable subset of $\mathbb{R}$. These objects are central to the study of probability and statistics. In the case of a RACS we have something like a random variable except that the measurable map takes values in $\mathcal{F}$ instead of $\mathbb{R}$. Hence the name "random closed set". A RACS can, like a random variable, be thought of as an assignment of probability to each measurable subset of $\mathcal{F}$. The following describes how.

Definition 2.4.10. Let $(\Omega, \mathcal{A}, P)$ be a probability space (with σ-algebra $\mathcal{A}$ and measure P). Let $X : \Omega \to \mathcal{F}$ be a RACS. X induces a probability measure $\mathbf{P}_X$ on $\mathcal{F}$ which is given by $\mathbf{P}_X(\mathcal{S}) = P(X^{-1}(\mathcal{S}))$.

2.4.1 Robbins' theorem

A number of results in the theory of RACS are useful in dealing with Martin-Löf random closed sets. This section describes one such useful theorem which provides a way of finding the average size of a randomly chosen set. This theorem was originally proved in Robbins [24]. We provide the proof here because it is a good example of the kind of analysis that the probability theoretic framework allows us to perform.

The expected value of a random variable is the "average value" of the random variable. The formulation of expected value that most readers will be familiar with is given in the review of probability in section 2.2. We recall if $Y : \Omega \to \mathbb{R}$ is a random variable with density function $\rho : \mathbb{R} \to [0, 1]$ then the expected value of Y is $E(Y) := \int_{\mathbb{R}} x\rho(x)dx$. We will use an alternative formulation of the expectation as an integral in Ω (which we recall is equipped with probability measure P). Let Y be a random variable as before, then the expected value of Y is

$$E(Y) = \int_{\Omega} Y(\alpha)dP(\alpha). \tag{2.1}$$

This formulation can be arrived at from the standard formulation by a change of variables: $Y(\alpha) = x$ and $\int_{\mathbb{R}} \rho(x)dx = \int_{\Omega} dP$. This expression of the expected value of a random variable along with the Fubini-Tonelli theorem are the keys to the proof of Robbins' theorem below. The Fubini-Tonelli theorem can be found in any real analysis textbook, in particular Folland [7] would be a good choice.

A measurable functional $\rho : \mathcal{F}(\mathbb{E}) \to \mathbb{R}$ can be composed with a RACS X :

$\Omega \to \mathcal{F}(\mathbb{E})$ to give a random variable $\rho \circ X : \Omega \to \mathbb{R}$. We can then investigate the expectation of this random variable as a way of gaining information about the RACS X. The expectation $E(\rho \circ X)$ tells us about the behavior of "average" closed sets in $\mathcal{F}(\mathbb{E})$ under the measure $\mathbf{P}_X$.

The following theorem, due to Robbins [24], shows that a measure ν on $\mathbb{E}$ is a measurable functional $\mathcal{F}(\mathbb{E}) \to \mathbb{R}$ and hence gives rise to a random variable by composition with a RACS X. The theorem also provides an easy formula for calculating $E(\nu \circ X)$. Once we are able to calculate $E(\nu \circ X)$ we will know something about the average ν-measure of the members of $\mathcal{F}(\mathbb{E})$. For example, if $E(\nu \circ X) = 0$ then we know that $\mathbf{P}_X$-almost every closed set has ν-measure 0.

First a note on notation. Let $X : \Omega \to \mathcal{F}(\mathbb{E})$ be a RACS. Let $x \in \mathbb{E}$. It is standard in probability to use the following shorthand when dealing with random variables:

$$P(x \in X) := P(\{\alpha \in \Omega : x \in X(\alpha)\} = \mathbf{P}_X(\mathcal{F}_{\{x\}}).$$

Theorem 2.4.11 (Robbins' Theorem)**.** *If ν is a σ-finite Borel measure on Polish space $\mathbb{E}$ and $X : \Omega \to \mathcal{F}(\mathbb{E})$ is a RACS, then $\nu \circ X : \Omega \to \mathbb{R}$ is a random variable and*

$$E\left(\nu \circ X\right) = \int_{\mathbb{E}} P(x \in X) d\nu(x).$$

Note: it is possible that $E(\nu \circ X) = \infty$ and in that case $\int_{\mathbb{E}} P(x \in X) d\nu(x) = \infty$.

Proof. First we make one simplification. If we show that $\nu : \mathcal{F}(\mathbb{E}) \to \mathbb{R}$ is a measurable map then the composition $\nu \circ X$ will be measurable. Furthermore $\mathbf{P}_X$ is induced by X and so the $\mathbf{P}_X$-expectation of $\nu(F)$ for F ranging over $\mathcal{F}(\mathbb{E})$ is equal to the P-expectation of $\nu \circ X(\alpha)$ for α ranging over Ω:

$$E(\nu) = \int_{\mathcal{F}} \nu(F) d\mathbf{P}_X(F) = \int_{\Omega} \nu \circ X(\alpha) dP(\alpha) = E(\nu \circ X).$$

Consequently it suffices to consider only $\nu : \mathcal{F}(\mathbb{E}) \to \mathbb{R}$ and prove the result for the $\mathbf{P}_X$-expectation of $\nu(F)$ with F ranging over $\mathcal{F}(\mathbb{E})$.

The goal now is to apply the Fubini-Tonelli theorem. Since $\mathbf{P}_X$ is a probability measure, and hence finite, and ν is σ-finite, the Fubini-Tonelli theorem will apply to integrals in the product space $\mathcal{F}(\mathbb{E}) \times \mathbb{E}$. All that remains to be done is to express $E \circ \nu$ as the integral of some function measurable in this product space.

Define $\mathbf{I} : \mathcal{F}(\mathbb{E}) \times \mathbb{E} \to \{0, 1\}$ by

$$\mathbf{I}(F, x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \notin F \end{cases}$$

Observe that for any $F \subset \mathbb{E}$,

$$\nu(F) = \int_{\mathbb{E}} \mathbf{I}(F, x) d\nu(x). \tag{2.2}$$

We will show that the function $\mathbf{I}$ is jointly measurable in $\mathcal{F}(\mathbb{E}) \times \mathbb{E}$ and then apply Fubini-Tonelli to show that $\nu : \mathcal{F}(\mathbb{E}) \to [0, 1]$ is measurable. Once we have shown that $\nu : \mathcal{F} \to \mathbb{R}$ is a measurable function we will know that $E(\nu) = \int_{\mathcal{F}} \left(\int_{\mathbb{E}} \mathbf{I}(F, x) d\nu(x) \right) d\mathbf{P}_X(F)$, using the formulation of equation 2.1. Applying Fubini-Tonelli again will allow us to swap the order of integration and obtain the desired result.

Claim: $\mathbf{I}$ is jointly measurable with respect to the product σ-algebra. Because the σ-algebra on $\{0, 1\}$ is generated by the set $\{0\}$ it will suffice to prove that

$\mathbf{I}^{-1}(\{0\}) = \{(F,x) \in \mathcal{F} \times \mathbb{E} : \mathbf{I}(F,x) = 0\}$ is jointly measurable. For the following let $\mathcal{B}$ be a countable base for the topology on $\mathbb{E}$.

$$\begin{aligned}
\{(F,x) \in \mathcal{F} \times \mathbb{E} : \mathbf{I}(F,x) = 0\} &= \{(F,x) : x \notin F\} \\
&= \bigcup_{B \in \mathcal{B}} \{(F,x) : x \in B \;\&\; B \cap F = \emptyset\} \\
&= \bigcup_{B \in \mathcal{B}} \left(\mathcal{F}^B \times B\right)
\end{aligned}$$

$\mathcal{F}^B \times B$ is clearly in the product σ-algebra and therefore the map $\mathbf{I}$ is jointly measurable.

We now apply Fubini-Tonelli. First, $\mathbf{I}$ is jointly measurable and so by Fubini-Tonelli $\nu(F) = \int_{\mathbb{E}} \mathbf{I}(F,x) d\nu(x)$ is measurable in $\mathcal{F}$. Second, we use Fubini-Tonelli to swap the order of integration for $E(\nu)$.

$$\begin{aligned}
E(\nu) &= \int_{\mathcal{F}} \nu(F) d\mathbf{P}_X(F) \\
&= \int_{\mathcal{F}} \left(\int_{\mathbb{E}} \mathbf{I}(F,x) d\nu(x) \right) d\mathbf{P}_X(F) \text{ by equation 2.2} \\
&= \int_{\mathbb{E}} \left(\int_{\mathcal{F}} \mathbf{I}(F,x) d\mathbf{P}_X(F) \right) d\nu(x) \text{ by Fubini-Tonelli} \\
&= \int_{\mathbb{E}} \mathbf{P}_X(\mathcal{F}_{\{x\}}) d\nu(x) \\
&= \int_{\mathbb{E}} P(x \in X) d\nu(x) \text{ by the definition of } \mathbf{P}_X
\end{aligned}$$

□

Frequently in our application of Robbins' theorem we wish to compute an integral over the space 2^ω (with the Lebesgue measure). By identifying each $f \in 2^\omega$ with the binary expansion of a real number $x \in [0,1]$ we can convert such integrals into integrals over the real unit interval $[0,1]$. We will make use of this identification freely in the following.

2.4.2 The Choquet capacity theorem

This section presents what is in some sense the fundamental theorem of the study of random closed sets in probability, the Choquet capacity theorem. This theorem completely characterizes all possible measures on $\mathcal{F}(\mathbb{E})$ as functions from the space of compact sets of $\mathbb{E}$ to $[0, 1]$. From the probability perspective the interesting part of a RACS is the measure it gives rise to and so the Choquet theorem is thought of as a characterization of all RACS. A much more detailed exposition of the following can be found in Matheron [16] or Molchanov [18]. Choquet's original paper [3] is also a good source.

For all of the following suppose that $\mathbb{E}$ is locally compact, Hausdorff, and second countable (LCHS). In this case we find particularly nice behavior in the Borel σ-algebra generated by the Fell topology (which we recall is denoted by $\mathcal{B}(\mathcal{F})$).

Proposition 2.4.12. *If $\mathbb{E}$ is LCHS then the Borel σ-algebra on $\mathcal{F}(\mathbb{E})$ is generated by the sets $\mathcal{F}_K$ for $K \in \mathcal{K}$.*

This proposition suggests that any measure on $\mathcal{F}$ could be fully characterized by its behavior on the sets $\mathcal{F}_K$ for $K \in \mathcal{K}$. This is, in fact, the case but we do not prove it directly. Instead we appeal to the Choquet capacity theorem which provides a much more detailed characterization of the measures on $\mathcal{F}$. First we investigate the properties of $\mathbf{P}_X(\mathcal{F}_K)$ for an arbitrary RACS.

Definition 2.4.13. Suppose that $X : \Omega \to \mathcal{F}(\mathbb{E})$ is a RACS. Then X induces a functional $T_X : \mathcal{K} \to [0, 1]$ given by $T_X(K) = P(X \cap K \neq \emptyset) = \mathbf{P}_X(\mathcal{F}_K)$. T_X is called the *capacity functional* of X. Where the RACS X is clear from context we use only T instead of T_X.

Note that T_X does not actually depend on the map X but instead on the measure $\mathbf{P}_X$. That is, if X_1 and X_2 are two RACS such that $\mathbf{P}_{X_1} = \mathbf{P}_{X_2}$ then $T_{X_1} = T_{X_2}$.

Proposition 2.4.14. *Let T be the capacity functional of a RACS with measure $\mathbf{P}$.*

1. $T(\emptyset) = 0$.

2. *T is monotone: if $K_1, K_2 \in \mathcal{K}$ and $K_1 \subseteq K_2$ then $T(K_1) \leq T(K_2)$.*

3. *T is sub-additive: if $K_2, K_2 \in \mathcal{K}$ then $T(K_1 \cup K_2) \leq T(K_1) + T(K_2)$.*

4. *T is upper semi-continuous: if $K_0 \supseteq K_1 \supseteq \cdots \in \mathcal{K}$ such that $\bigcap_{i\in\omega} K_i = K \in \mathcal{K}$ then $\lim_{i\to\infty} T(K_i) = T(K)$.*

Proof.

1. $T(\emptyset) = \mathbf{P}(X \cap \emptyset \neq \emptyset) = 0$.

2. $T(K_1) = \mathbf{P}(X \cap K_1 \neq \emptyset) \leq \mathbf{P}(X \cap K_2 \neq \emptyset) = T(K_2)$.

3.

$$
\begin{aligned}
T(K_1 \cup K_2) &= \mathbf{P}(X \cap (K_1 \cup K_2) \neq \emptyset) \\
&= \mathbf{P}(X \cap K_1 \neq \emptyset) + \mathbf{P}(X \cap K_2 \neq \emptyset) \\
&\quad - \mathbf{P}(X \cap K_1 \neq \emptyset \ \&\ X \cap K_2 \neq \emptyset) \\
&\leq \mathbf{P}(X \cap K_1 \neq \emptyset) + \mathbf{P}(X \cap K_2 \neq \emptyset) \\
&= T(K_1) + T(K_2)
\end{aligned}
$$

4. Clearly $\mathcal{F}_{K_0} \supseteq \mathcal{F}_{K_1} \supseteq \dots$ and so $\mathbf{P}\left(\bigcap_{i\in\omega} \mathcal{F}_{K_i}\right) = \lim_{i\to\infty} \mathbf{P}\left(\mathcal{F}_{K_i}\right)$. It thus suffices to show that $\bigcap_{i\in\omega} \mathcal{F}_{K_i} = \mathcal{F}_K$ for then

$$\begin{aligned}
\lim_{i\to\infty} T(K_i) &= \lim_{i\to\infty} \mathbf{P}\left(\mathcal{F}_{K_i}\right) \\
&= \mathbf{P}\left(\bigcap_{i\in\omega} \mathcal{F}_{K_i}\right) \\
&= \mathbf{P}\left(\mathcal{F}_K\right) \\
&= T(K)
\end{aligned}$$

Because $\bigcap_{i\in\omega} K_i = K$ we know that $\mathcal{F}_K \subseteq \mathcal{F}_{K_i}$ for every $i \in \omega$. Hence $\mathcal{F}_K \subseteq \bigcap_{i\in\omega} \mathcal{F}_{K_i}$.

To show the other containment suppose that $F \in \left(\bigcap_{i\in\omega} \mathcal{F}_{K_i} \setminus \mathcal{F}_K\right)$. Define $\hat{F} = F \cap K_0$. Then $\hat{F} \neq \emptyset$, $\hat{F}$ is compact, and

$$\hat{F} \subseteq K^{\complement} = \bigcup_{i\in\omega} K_i^{\complement}.$$

By compactness there is $N \in \omega$ such that $\hat{F} \subseteq \bigcup_{i<N} K_i^{\complement}$. Taking complements gives $\left(\hat{F}\right)^{\complement} \supseteq \bigcap_{i<N} K_i$. Thus $F \cap K_N = \emptyset$. This is a contradiction and therefore no such set F exists. Therefore $\bigcap_{i\in\omega} \mathcal{F}_{K_i} = \mathcal{F}_K$.

□

The Choquet capacity theorem depends on developing the following operations.

Definition 2.4.15. Let $K, K_1, K_2, \dots, K_n \in \mathcal{K}$. For $n \geq 2$ define $\Delta_{K_1} T(K) = T(K \cup K_1) - T(K)$. Define (by way of induction on n)

$$\Delta_{K_n} \dots \Delta_{K_1} T(K) = \Delta_{K_{n-1}} \dots \Delta_{K_1} T(K) - \Delta_{K_{n-1}} \dots \Delta_{K_1} T(K \cup K_n).$$

Definition 2.4.16. A functional $T : \mathcal{K} \to \mathbb{R}$ is *completely alternating* if for every $K, K_1, \ldots, K_n \in \mathcal{K}$, $\Delta_{K_n} \ldots \Delta_{K_1} T(K) \geq 0$

Proposition 2.4.17 is the key to understanding the completely alternating property.

Proposition 2.4.17. *If T is the capacity functional of a RACS $X : \Omega \to \mathcal{F}$, then T is completely alternating. In fact, for every $K_1, \ldots, K_n, K \in \mathcal{K}$*

$$\Delta_{K_n} \ldots \Delta_{K_1} T(K) = \mathbf{P}_X\left(\mathcal{F}^K_{K_1,\ldots,K_n}\right).$$

Proof. We first consider $\Delta_{K_1} T(K) = T(K \cup K_1) - T(K)$ as a base case. By definition

$$\begin{aligned} T(K \cup K_1) - T(K) &= \mathbf{P}_X\left(\mathcal{F}_{K \cup K_1}\right) - \mathbf{P}_X\left(\mathcal{F}_K\right) \\ &= \mathbf{P}_X\left(\mathcal{F}^K_{K_1}\right) \text{ by the additivity of measures} \\ &\geq 0. \end{aligned}$$

Now suppose that

$$\Delta_{K_n} \ldots \Delta_{K_1} T(K) = \mathbf{P}_X\left(\mathcal{F}^K_{K_1,\ldots,K_n}\right).$$

Consider $\Delta_{K_{n+1}} \ldots \Delta_{K_1} T(K)$. By definition

$$\begin{aligned} \Delta_{K_{n+1}} \ldots \Delta_{K_1} T(K) &= \Delta_{K_n} \ldots \Delta_{K_1} T(K) - \Delta_{K_n} \ldots \Delta_{K_1} T(K \cup K_{n+1}) \\ &= \mathbf{P}_X\left(\mathcal{F}^K_{K_1,\ldots,K_n}\right) - \mathbf{P}_X\left(\mathcal{F}^{K \cup K_{n+1}}_{K_1,\ldots,K_n}\right) \\ &= \mathbf{P}_X\left(\mathcal{F}^K_{K_1,\ldots,K_n,K_{n+1}}\right) \text{ by the additivity of measures} \\ &\geq 0. \end{aligned}$$

□

We are now ready to state the Choquet capacity theorem. This theorem characterizes those functionals $T : \mathcal{K} \to [0,1]$ that are the capacities of RACS by determining which of the preceding necessary conditions are sufficient. This is the foundational result in the study of RACS. Proofs can be found in either Matheron [16] or Molchanov [18].

Theorem 2.4.18 (Choquet capacity theorem)**.** *Let $\mathbb{E}$ be an LCHS space and let $T : \mathcal{K} \to [0,1]$. Then T gives rise to a Borel probability measure $\mathbf{P}$ on $\mathcal{F}(\mathbb{E})$ such that $\mathbf{P}(\mathcal{F}_K) = T(K)$ for $K \in \mathcal{K}$ if and only if T satisfies the following conditions:*

1. *$T(\emptyset) = 0$;*

2. *T is upper semi-continuous on $\mathcal{K}$: if $K_0 \supseteq K_1 \supseteq \cdots \in \mathcal{K}$ such that $\bigcap_{i\in\omega} K_i = K \in \mathcal{K}$ then $\lim_{i\to\infty} T(K_i) = T(K)$;*

3. *T is completely alternating on $\mathcal{K}$:*

$$(\forall K, K_1, \ldots, K_n \in \mathcal{K})\ \ \Delta_{K_n} \ldots \Delta_{K_1} T(K) \geq 0.$$

Moreover $\mathbf{P}_T$ is unique.

The proof of the "only if" direction of this theorem is in propositions 2.4.14 and 2.4.17. We outline briefly the proof of the other direction. First one argues that T can be extended to the class of open subsets of $\mathbb{E}$ by setting

$$T(G) = \sup\{T(K) : K \in \mathcal{K}\ \&\ K \subseteq G\}$$

for $G \in \mathcal{G}$. The probability measure $\mathbf{P}$ on $\mathcal{F}$ is then defined by assigning

$\mathbf{P}(\mathcal{F}^K_{G_1,\ldots,G_n}) = \Delta_{G_n} \ldots \Delta_{G_1} T(K)$ for $G_1, \ldots, G_n \in \mathcal{G}$ and $K \in \mathcal{K}$. This produces a RACS on $\mathcal{F}$ in the sense that $\mathcal{F}$ becomes a probability space with the measure $\mathbf{P}$ and consequently the identity map on $\mathcal{F}$ is a RACS.

Definition 2.4.19. If $T : \mathcal{K} \to [0, 1]$ satisfies the conditions of the Choquet Capacity Theorem then we call T a *Choquet capacity.*

In general we cannot computably pass from a capacity T to a measure μ_T because this involves taking the supremum of an infinite set. However, there are many cases in which we can get around this. In particular when $\mathbb{E} = 2^\omega$ we can prove a computable version of the Choquet theorem. In fact this computable version is a corollary of proposition 2-3-1 in Matheron [16]. We simply apply this proposition to the case of $\mathbb{E} = 2^\omega$, taking as our base for the topology on 2^ω the collection of all clopen subsets of 2^ω. Matheron's proposition is proved by extending the functional from this class to the (larger) class of all compact subsets of $\mathbb{E}$ and then proving that this extension satisfies the hypotheses of the Choquet capacity theorem (2.4.18).

Proposition 2.4.20 (Matheron's 2-3-1)**.** *Let $\mathbb{E}$ be an LCHS space. Let $\mathcal{B}$ be a countable basis for $\mathbb{E}$ that is closed under finite unions and such that for any $B \in \mathcal{B}$ $\overline{B}$ is compact. Let $\mathcal{B}' := \{\overline{B} : B \in \mathcal{B}\}$ and $T : \mathcal{B}' \to [0, 1]$. Then T gives rise to a unique probability measure $\mathbf{P}$ on $\mathcal{F}$ such that for any $B \in \mathcal{B}$ $\mathbf{P}(\mathcal{F}_{\overline{B}}) = T(\overline{B})$ if and only if T satisfies the following conditions:*

1. *$T(\emptyset) = 0$;*

2. *T is upper semi-continuous on $\mathcal{B}'$: if $B_0 \supseteq B_1 \supseteq \cdots \in \mathcal{B}$ such that there is $B \in \mathcal{B}$ with $\bigcap_{i\in\omega} \overline{B_i} = \overline{B}$, then $\lim_{i\to\infty} T(\overline{B_i}) = T(\overline{B})$;*

3. T is completely alternating on $\mathcal{B}'$:

$$(\forall B, B_1, \ldots, B_n \in \mathcal{B}) \;\; \Delta_{\overline{B_n}} \ldots \Delta_{\overline{B_1}} T(\overline{B}) \geq 0.$$

Corollary 2.4.21 (Computable Choquet capacity theorem)**.** *Let $\mathcal{C}$ be the collection of clopen subsets of 2^ω. A functional $T : \mathcal{C} \to [0,1]$ gives rise to a (unique) probability measure $\mathbf{P}$ on $\mathcal{F}(2^\omega)$ such that $\mathbf{P}(\mathcal{F}_C) = T(C)$ for every $C \in \mathcal{C}$ if and only if T satisfies the following conditions:*

1. $T(\emptyset) = 0$;

2. T is upper semi-continuous on $\mathcal{C}$;

3. T is completely alternating on $\mathcal{C}$.

Proof. Let $\mathcal{B} = \mathcal{C}$ in proposition 2.4.20. Note that for C clopen $\overline{C} = C$ and so $\mathcal{B}' = \mathcal{B} = \mathcal{C}$. □

Later (definition 3.0.4) we will define formally what it means to compute a measure. For now we will say that we can compute a measure if we can compute the measure of every basic open set. Recall that sets of the form $\mathcal{F}^C_{C_1,\ldots,C_n}$, where $C, C_1, \ldots, C_n \subseteq 2^\omega$ are clopen, form a basis for $\mathcal{F}(2^\omega)$. Hence the computable Choquet capacity theorem shows how to compute the measure $\mathbf{P}_T$ from T. More concretely, if we know $T(C)$ for every clopen set C, then we can compute $\mathbf{P}\left(\mathcal{F}^C_{C_1,\ldots,C_n}\right)$ for any clopen sets $C, C_1, \ldots, C_n$. This is possible because $\mathbf{P}\left(\mathcal{F}^C_{C_1,\ldots,C_n}\right) = \Delta_{C_n} \ldots \Delta_{C_1} T(C)$ and $Delta_{C_n} \ldots \Delta_{C_1} T(C)$ has a simple recursive definition. Thus the computability theoretic strength of the measure on $\mathcal{F}$ given by the computable Choquet theorem is no higher than that of the capacity of the measure.

In lemma 4.1.4 we establish the existence of a homeomorphism, Z between $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ and 3^ω that preserves computability properties. Hence we are able to characterize the Borel measures of $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ as exactly the measures induced on $\mathcal{F}(2^\omega)$ by Z from Borel measures on 3^ω. In other words, for every Borel measure $\mathbf{P}$ on $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ there is a Turing equivalent Borel measure P on 3^ω such that for every measurable $\mathcal{H} \subseteq \mathcal{F}(2^\omega)$ $\mathbf{P}(\mathcal{H}) = P(Z^{-1}(\mathcal{H}))$. This means that everything to be learned about Borel measures on $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ can be learned by studying Borel measures on 3^ω and Z.

2.4.3 Examples of Choquet capacities

In general it can be difficult to determine if a functional $T : \mathcal{K} \to [0, 1]$ satisfies the hypotheses of the Choquet capacity theorem. The completely alternating property in particular can be difficult to work with. The examples in this section will be used later on and so we give some proofs or at least ideas of proofs that they do in fact correspond to RACS.

Throughout the following $\mathbb{E}$ is an LCHS topological space and as usual $\mathcal{F}$ and $\mathcal{K}$ are the collection of closed subsets of $\mathbb{E}$ and compact subsets of $\mathbb{E}$, respectively.

Proposition 2.4.22. *The functional $T : \mathcal{K} \to [0, 1]$ is a measure on $\mathbb{E}$ (meaning that $T\left(\bigcup_{i\in\omega} K_i\right) = \sum_{i\in\omega} T(K_i)$ for $K_0, K_1, \dots \in \mathcal{K}$ pairwise disjoint) if and only if T is the capacity induced by the RACS $X : \mathbb{E} \to \mathcal{F}(\mathbb{E})$ given by $X(x) = \{x\}$.*

Proof. We first show that $X(x) = \{x\}$ is measurable. The sets $\mathcal{F}_K$ for $K \in \mathcal{K}(\mathbb{E})$ generate the Borel σ-algebra on $\mathcal{F}(\mathbb{E})$. Thus it suffices to show that for K compact $X^{-1}(\mathcal{F}_K)$ is a Borel measurable subset of $\mathbb{E}$. But $X^{-1}(\mathcal{F}_K) = K$ so this follows easily.

Let $\mathbf{P}$ be a Borel probability measure on $\mathbb{E}$. By the preceding calculations we see that in this case $T_X(K) = \mathbf{P}\left(X \cap K \neq \emptyset\right) = \mathbf{P}(K)$. Therefore T_X is a measure on $\mathbb{E}$.

The same calculations also show that if T is a measure on $\mathbb{E}$, then $T_X(K) = T(X \cap K \neq \emptyset) = T(K)$. This completes the proof. □

The next example was originally discovered by Matheron [16] and is a mainstay of the RACS literature. The proof that the given functional is a capacity does not use the Choquet capacity theorem but instead relies on constructing a RACS that gives rise to the capacity. The full details of this construction are beyond our scope but we will prove the key lemma and sketch the idea of the proof. This capacity is used in sections 5.1 and 5.2.

Proposition 2.4.23. *Let Λ be a Borel measure on $\mathbb{E}$ such that $\forall K \in \mathcal{K}\ \Lambda(K) < \infty$. Then $T(K) = 1 - e^{-\Lambda(K)}$ is a Choquet capacity (called a* Poisson point process with intensity Λ *or* generalized Poisson process*).*

First the key lemma.

Lemma 2.4.24. *Let T be a Choquet capacity and $G(s) = \sum_{i \in \omega} p_i s^i$ be the generating function of a probability measure on the set ω. Then $1 - G(1 - T)$ is a Choquet capacity.*

Proof. By the Choquet capacity theorem the capacity T is associated with a RACS. Let $A_1, A_2, \ldots$ be i.i.d. instances of this RACS. Let N be a random variable on ω with $\mathbf{P}(N = i) = p_i$. Then the mapping $\omega \times \mathcal{F}^\omega \to \mathcal{F}$ defined by $(N, A_1, A_2, \ldots) \mapsto A_1 \cup \cdots \cup A_N$ is measurable. Moreover, for $K \in \mathcal{K}$

$$\mathbf{P}\left(\left(A_1 \cup \cdots \cup A_N\right) \cap K = \emptyset\right) = \sum_{i \in \omega} p_i \left(1 - T(K)\right)^i = G\left(1 - T(K)\right).$$

The lemma then follows from the Choquet capacity theorem. □

Proof sketch for proposition 2.4.23. Because Λ is a measure it can easily be seen that $T(\emptyset) = 0$ and T is upper semi-continuous. Showing that T is completely alternating is an obstacle, however. Instead of directly showing that T is completely alternating we use the preceding lemma to show that T is the capacity functional of a RACS.

Let $B_0 \subseteq B_1 \subseteq \cdots \in \mathcal{G}$ be such that $\bigcup_{i\in\omega} B_i = \mathbb{E}$ and $\overline{B_i} \in \mathcal{K}$ for each $i \in \omega$. For any $K, K_1, \ldots, K_n \in \mathcal{K}$ there is $i \in \omega$ such that $B_i \supseteq K \cup K_1 \cup \cdots \cup K_n$. Then $\Lambda(\overline{B_i}) = \lambda < \infty$ and so $\frac{1}{\lambda}\Lambda$ is a Choquet capacity on the subspace B_i. Let $G(s) = 1 - e^{-\lambda(1-s)}$. By the lemma

$$G\left(1 - \frac{1}{\lambda}\Lambda\right) = 1 - e^{-\Lambda}$$

is a Choquet capacity. By the Choquet capacity theorem this capacity corresponds to a RACS A_i on $\mathcal{F}\left(\overline{B_i}\right)$. It can now be shown that there is an inductive limit of the sequence of RACS $A_0, A_1, \ldots$ that is defined on $\mathcal{F}(\mathbb{E})$. This inductive limit is then a RACS with the capacity functional $T(K) = 1 - e^{-\Lambda(K)}$. Therefore T is a Choquet capacity. □

The next example is another staple of the RACS literature. This example was originally discovered by Choquet [3]. We examine this capacity in section 4.5.

Definition 2.4.25. A function $\varphi : \mathbb{E} \to \mathbb{R}$ is *upper semi-continuous* if the set

$$\{x \in \mathbb{E} : \varphi(x) \geq r\}$$

is closed for every $r \in \mathbb{R}$.

Proposition 2.4.26 (Choquet)**.** *Let $\mathbb{E}$ be an LCHS space and let $\varphi : \mathbb{E} \to [0, 1]$*

be an upper semi-continuous function. Then the map $T : \mathcal{K}(\mathbb{E}) \to \mathbb{R}$ *given by*

$$T(K) = \begin{cases} \sup_{x \in K} \varphi(x) & \text{if } K \neq \emptyset \\ 0 & \text{if } K = \emptyset \end{cases}$$

is a Choquet capacity.

Proof. We must show that T satisfies the three conditions of the Choquet capacity theorem (2.4.18).

(1) By definition $T(\emptyset) = 0$.

(2) Now we must prove that T is upper semi-continuous. This means that for $K_0 \supseteq K_1 \supseteq \cdots \in \mathcal{K}$ such that $\bigcap_{i \in \omega} K_i = K \in \mathcal{K}$ we must show that

$$\lim_{i \to \infty} T(K_i) = T(K).$$

We have two cases to deal with.

Case 1: $K = \emptyset$. Then by compactness there is $n \in \omega$ such that for $i \geq n$ $K_i = \emptyset$. Hence $\lim_{i \to \infty} T(K_i) = 0 = T(K)$.

Case 2: $K \neq \emptyset$. The sequence $\{T(K_n)\}_{n \in \omega}$ is non-increasing and bounded below by 0 and thus has a limit t. Clearly $T(K) \leq t$. Let $\epsilon > 0$ and define

$$L_n := \{x \in \mathbb{E} : \varphi(x) \geq t - \epsilon\} \cap K_n.$$

Each L_n is nonempty because $t \leq T(K_n)$. Because f is upper semi-continuous L_n is a closed subset of K_n and hence compact. Therefore $\bigcap_{n \in \omega} L_n \neq \emptyset$. But $\bigcap_{n \in \omega} L_n \subseteq K$ and therefore $T(K) \geq t - \epsilon$. This holds for each $\epsilon > 0$ and therefore $T(K) = t$.

(3) Now we must show that T is completely alternating, i.e.

$$(\forall K, K_1, \ldots, K_n \in \mathcal{K}) \;\; \Delta_{K_n} \ldots \Delta_{K_1} T(K) \geq 0.$$

We actually prove by induction (on n) that

$$\Delta_{K_n} \ldots \Delta_{K_1} T(K) = \min\left(T\left(K_1\right), \ldots, T\left(K_n\right)\right) - \min\left(T\left(K_1\right), \ldots, T\left(K_n\right), T(K)\right).$$

That T is completely alternating follows.

Base case: $n = 1$

$$\begin{aligned} \Delta_{K_1} T(K) &= T\left(K \cup K_1\right) - T(K) \\ &= \max\left(T(K), T\left(K_1\right)\right) - T(K) \\ &= T\left(K_1\right) - \min\left(T(K), T\left(K_1\right)\right) \end{aligned}$$

where the last equality follows from the identity

$$\min(u, v) + \max(u, v) = u + v.$$

Now suppose that

$$\Delta_{K_n} \ldots \Delta_{K_1} T(K) = \min\left(T\left(K_1\right), \ldots, T\left(K_n\right)\right) - \min\left(T\left(K_1\right), \ldots, T\left(K_n\right), T(K)\right).$$

Then

$$
\begin{aligned}
\Delta_{K_{n+1}} \dots \Delta_{K_1} T(K) &= \Delta_{K_n} \dots \Delta_{K_1} T(K) - \Delta_{K_n} \dots \Delta_{K_1} T\left(K \cup K_{n+1}\right) \\
&= \min\left(T\left(K_1\right), \dots, T\left(K_n\right)\right) \\
&\quad - \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T(K)\right) \\
&\quad\quad - \Big[\min\left(T\left(K_1\right), \dots, T\left(K_n\right)\right) \\
&\quad\quad\quad - \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T\left(K \cup K_{n+1}\right)\right)\Big] \\
&= \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T\left(K \cup K_{n+1}\right)\right) \\
&\quad - \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T(K)\right)
\end{aligned}
$$

We know that $T\left(K \cup K_{n+1}\right) = \max\left(T(K), T(K_{n+1})\right)$. If $T(K_{n+1}) < T(K)$, then we find that

$$
\begin{aligned}
&\min\left(T\left(K_1\right), \dots, T\left(K_n\right), T\left(K \cup K_{n+1}\right)\right) - \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T(K)\right) \\
&\quad = \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T\left(K\right)\right) - \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T(K)\right) \\
&\quad = 0.
\end{aligned}
$$

On the other hand, if $T(K_{n+1}) \geq T(K)$, then we have

$$
\min\left(T\left(K_1\right), \dots, T\left(K_n\right), T\left(K \cup K_{n+1}\right)\right) = \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T\left(K_{n+1}\right)\right)
$$

and

$$
\min\left(T\left(K_1\right), \dots, T\left(K_n\right), T(K)\right) = \min\left(T\left(K_1\right), \dots, T\left(K_n\right), T\left(K_{n+1}\right), T(K)\right).
$$

In both cases we have the desired result and so we have completed the proof. $\square$

In general a capacity T such that $T(K_1 \cup K_2) = \max\{T(K_1), T(K_2)\}$ is said to be *maxitive*. Choquet proves in [3] that all maxitive Choquet capacities are of the form given in proposition 2.4.26.

2.5 Fractal geometry

Much of this draws on Falconer [6] which is a good general introduction to fractal geometry. Chapter 15 of Falconer discusses but does not prove the main theorem in this section. Good sources with complete proofs of the main theorem are Graf et al. [10] and Mauldin and Williams [17]. It should also be noted that Falconer develops the theory of fractals only for subsets of $\mathbb{R}^n$. In fact fractal geometry naturally lives in metric spaces, including Polish spaces like 2^ω.

For the following we assume that $\mathbb{E}$ is a metric space with metric d.

We recall briefly the definition of Hausdorff dimension.

Definition 2.5.1.

1. Let $U \subseteq \mathbb{E}$. The *diameter* of U is defined by $|U| = \sup\{d(x, y) : x, y \in U\}$.

2. Let $F \subseteq \mathbb{E}$ and let $\delta \in (0, \infty)$. Then we say that a collection $\{U_i\}_{i \in \omega}$ of subsets of $\mathbb{E}$ is a *δ-cover of F* if $F \subseteq \bigcup_{i \in \omega} U_i$ and for each $i \in \omega$ $|U| \leq \delta$.

3. Let $F \subseteq \mathbb{E}$, $\delta \in (0, \infty)$, and $s \in [0, \infty)$. The we define
$$\mathcal{H}^s_\delta = \inf \left\{ \sum_{i \in \omega} |U_i|^s : \{U_i\}_{i \in \omega} \text{ is a } \delta\text{-cover of } F \right\}.$$

4. Let $F \subseteq \mathbb{E}$ and let $s \in [0, \infty)$. The *s-dimensional Hausdorff measure of F* is defined as
$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

5. Let $F \subseteq \mathbb{E}$. The *Hausdorff dimension of* F is defined as

$$\dim_H(F) = \inf\{s \in [0,\infty) : \mathcal{H}^s(F) = 0\} = \sup\{s \in [0,\infty) : \mathcal{H}^s(F) = \infty\}.$$

For random fractal constructions we wish to discuss self-similar sets. We now provide the necessary definitions.

Definition 2.5.2. A function $\varphi : \mathbb{E} \hookrightarrow \mathbb{E}$ is a *similarity* if φ is an injection and there is $s \in (0,1]$ such that for every $x, y \in \mathbb{E}$ $d(\varphi(x), \varphi(y)) = sd(x,y)$. In this case s is called the *ratio* of the similarity. If $V, U \subseteq \mathbb{E}$ then we say that U is *similar* to V if there is a similarity φ such that $\varphi(V) = U$.

Many fractals are produced as self similar sets by iterating a collection similarities repeatedly. For example the Cantor middle thirds set is produced by iterating two similarities: $\varphi_1(x) = \frac{1}{3}x$ and $\varphi_2(x) = \frac{1}{3}x + \frac{2}{3}$. Let $C_0 = [0,1]$. Let $C_{n+1} = \varphi_1(C_n) \cup \varphi_2(C_n)$. For example, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}], \cup [\frac{8}{9}, 1]$. Then the Cantor middle thirds set is $\bigcap_{i \in \omega} C_i$.

Random self similar sets are constructed in much the same way as the Cantor middle thirds set. The main difference is that we allow the ratio of similarity to change in a random manner over the course of the construction.

Let $V \subseteq \mathbb{E}$ be open with closure $\overline{V}$. Let $m \geq 1$ be an integer and let $s_0, s_1, \ldots, s_{(m-1)}$ be random variables in $[0,1]$. These random variables are our random ratios of similarity. Let $V_0, V_1, \ldots, V_{(m-1)}$ be pairwise disjoint subsets of V such that for $i = 0, 1, \ldots, m-1$ V_i is similar to V with ratio of similarity s_i. If $s_i = 0$ we take $V_i = \emptyset$. Note that we do not assume that the variables $s_0, s_1, \ldots, s_{(m-1)}$ are independent. Indeed dependence may be necessary to insure that it is possible to produce m mutually disjoint subsets of V with the correct ratios of similarity. Let $E_1 = \bigcup_{i<m} \overline{V}_i$.

We iterate this procedure just as we did in the construction of the Cantor middle thirds set. Suppose that we have

$$E_k = \bigcup_{\sigma \in m^k} \overline{V}_\sigma.$$

We now produce $E_{k+1} \subseteq E_k$ by applying our random similarities to E_k.

For each $\sigma \in m^k$ let $s_{\sigma^\frown 0}, \ldots, s_{\sigma^\frown (m-1)}$ be random variables in $[0, 1]$ such that $s_{\sigma^\frown j}$ has identical distribution to s_j. This means that $s_{\sigma^\frown 1}, \ldots, s_{\sigma^\frown (m-1)}$ may not be independent. We require, however, that the random variables s_τ for $\sigma \in m^{<\omega}$ are independent wherever else possible.

Let $V_{\sigma^\frown 0}, \ldots, V_{\sigma^\frown (m-1)}$ be pairwise disjoint subsets of V_σ such that $V_{\sigma^\frown j}$ is similar to V_σ with random ratio of similarity $s_{\sigma^\frown j}$. Then

$$E_{k+1} = \bigcup_{\tau \in m^{(k+1)}} \overline{V}_\tau.$$

A random self-similar set F is given by

$$F = \bigcap_k E_k.$$

Let N be the (random) number of the random variables $s_1, s_2, \ldots, s_m$ that is positive.

Theorem 2.5.3. *The random set F produced by the above construction has probability q of being empty where q is the least non-negative solution to*

$$f(t) = \sum_{j=0}^{m} P(N = j)t^j = t. \tag{2.3}$$

With probability $1-q$ F *has Hausdorff and box dimensions given by the solution* d *to*

$$\mathbf{E}\left(\sum_{j=0}^{m} s_j^d\right) = 1. \tag{2.4}$$

This kind of construction is called a random recursive construction. Partly in response to the interest of Barmpalias et al. [1] in random subsets of 2^ω Mauldin and McLinden [4] examined the special case of random recursive constructions for this space. They proved a number of results about the fractal geometry of such constructions. We look at their constructions a little more closely in section 4.3.

CHAPTER 3

MARTIN-LÖF RANDOM CLOSED SETS

Let $\mathbb{E}$ be a locally compact, Hausdorff, second countable topological (LCHS) space. Our goal in this section is to extend the definitions of Martin-Löf randomness to the space $\mathcal{F}(\mathbb{E})$ of closed subsets of $\mathbb{E}$. We have seen already that $\mathcal{F}(\mathbb{E})$ has a countable basis. We will use this basis to define the Σ^0_1 subsets of the space. First, however, we need to work out some technical details. We note that an extension of Martin-Löf randomness to general topological spaces has recently been carried out in [12]. We take a different approach that is somewhat more tailored to our particular situation.

First we need to develop a way of coding measures on second countable spaces. The idea of the definition is to code a measure by recording its value on each basic open set. This can be called a code for the measure because a measure in an LCHS space is uniquely determined by the values it takes on the basic open sets. This definition is standard in the literature, notably in Reimann [21] and Reimann and Slaman [22]).

Definition 3.0.4. Let $\mathbb{X}$ be a topological space with a basis $B_0, B_1, \ldots$ and a Borel measure μ such that $\mu(B_i) \leq 1$ for each $i \in \omega$. We say that $m \in 2^\omega$ is a code for μ if for all $i \in \omega$

$$\mu(B_i) = \sum_{j \in \omega} m(\langle i, j \rangle) 2^{-j-1}.$$

We will abbreviate this by writing $\mu(B_i) = m^{[i]}$, a slight abuse of the notation introduced in section 2.1, item 7.

We now begin our extension of Martin-Löf randomness to more general topological spaces. We start by specifying exactly which spaces we will consider.

Definition 3.0.5. Let $\mathbb{X}$ be a topological space with a Borel measure μ. We say that $\mathbb{X}$ is a *Martin-Löf space* (or ML space) if it has a countable basis $B_0, B_1, \ldots$ meeting the following conditions:

1. $\mu(B_i) \leq 1$ for each $i \in \omega$;
2. $B_0, B_1, \ldots$ is closed under finite intersections;
3. There is an μ-computable intersection function $\mathfrak{g} : \omega^2 \to \omega$ such that

$$B_i \cap B_j = B_{\mathfrak{g}(i,j)}$$

(with μ coded as in definition 3.0.4).

If in addition $\mu(\mathbb{X}) = 1$, then we call $\mathbb{X}$ a Martin-Löf probability space.

Definition 3.0.6. Let $\mathbb{X}$ be a Martin-Löf space with basis $B_0, B_1, \ldots$ and measure μ.

1. $U \subseteq \mathbb{X}$ is $\Sigma_1^{0,\mu}$ if there is $f \in 2^\omega$ c.e in μ such that $U = \bigcup_{f(n)=1} B_n$.

2. A sequence, $\{U_i\}_{i\in\omega}$, of subsets of $\mathbb{X}$ is *uniformly* $\Sigma_1^{0,\mu}$ if there is $f \in 2^\omega$ c.e. in μ such that

$$U_i = \bigcup_{f(\langle i,n\rangle)=1} B_n.$$

3. A *μ-Martin-Löf test* (μ-ML test) is a uniformly $\Sigma_1^{0,\mu}$ sequence of subsets of $\mathbb{X}$, $\{U_i\}_{i\in\omega}$, such that $\mu(U_i) \leq 2^{-i}$.

4. $x \in \mathbb{X}$ is *μ-Martin-Löf random* (μ-ML random) if there is no μ-Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $x \in \bigcap_{i\in\omega} U_i$.

Note: in the case of the Cantor space we will continue say that $f \in 2^\omega$ is Martin-Löf random (or ML random) if it is Martin-Löf random with respect to the fair coin measure. Only when we wish to specify that we are using a different measure, μ, will we write that f is μ-Martin-Löf random. The same will be true for other Cantor spaces, in particular for 3^ω. The "fair coin" measure for 3^ω assigns equal probability to 0, 1, and 2, as a fair 3-sided coin would, if it existed. Then $g \in 3^\omega$ is Martin-Löf random if it is Martin-Löf random with respect to this measure.

At some points it becomes convenient to ignore the oracle μ in the preceding definition. In doing this one defines a weaker notion of algorithmic randomness (more elements of the space are random) called Hippocrates randomness. Note that when the measure μ is computable μ-Martin-Löf randomness is the same as μ-Hippocrates randomness.

Definition 3.0.7. If the oracle μ is eliminated everywhere in definition 3.0.6, then we get the following definitions:

1. $U \subseteq \mathbb{X}$ is Σ_1^0 if there is a c.e $f \in 2^\omega$ such that $U = \bigcup_{f(n)=1} B_n$.

2. A sequence, $\{U_i\}_{i\in\omega}$, of subsets of $\mathbb{X}$ is *uniformly* Σ^0_1 if there is a c.e. $f \in 2^\omega$ such that

$$U_i = \bigcup_{f(<i,n>)=1} B_n.$$

3. A *μ-Hippocrates test* (μ-H test) is a uniformly Σ^0_1 sequence of subsets of $\mathbb{X}$, $\{U_i\}_{i\in\omega}$, such that $\mu(U_i) \leq 2^{-i}$.

4. $x \in \mathbb{X}$ is *μ-Hippocrates random* (μ-H random) if there is no μ-Hippocrates test $\{U_i\}_{i\in\omega}$ such that $x \in \bigcap_{i\in\omega} U_i$.

Probably the most important distinction between Martin-Löf randomness randomness and Hippocrates randomness is the existence of a universal Martin-Löf test.

Lemma 3.0.8. *Let $\mathbb{X}$ is a Martin-Löf space. There is a single Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $x \in \mathbb{X}$ is μ-Martin-Löf random if and only if $x \notin \bigcap_{i\in\omega} U_i$ (we call such a test a universal μ-Martin-Löf test).*

Proof. The standard proof works in this situation. We start with an enumeration of all the uniformly $\Sigma^{0,\mu}_1$ sequences. We modify each $\Sigma^{0,\mu}_1$ sequence $\{V_i\}_{i\in\omega}$ into a Martin-Löf test $\left\{\hat{V}_i\right\}_{i\in\omega}$ in the following way. By definition there is $f \in 2^\omega$ such that

$$V_i = \bigcup_{f(<i,j>)=1} B_j.$$

Then define $\hat{V}_{i,0} := \emptyset$ and

$$\hat{V}_{i,s+1} := \begin{cases} V_{i,s} & \text{if } f(<i,s>) = 0 \text{ or } \mu\left(\hat{V}_{i,s} \cup B_s\right) > 2^{-i} \\ V_{i,s} \cup B_s & \text{otherwise} \end{cases}.$$

We define

$$\hat{V}_i = \bigcup_{s \in \omega} \hat{V}_{i,s}.$$

Clearly $\mu(\hat{V}_i) \leq 2^{-i}$. We must show that $\left\{\hat{V}_i\right\}_{i \in \omega}$ is a uniformly $\Sigma_1^{0,\mu}$ sequence. Our only obstacle is the condition that $\mu\left(\hat{V}_{i,s} \cup B_s\right) > 2^{-i}$. Because there is a μ-computable intersection function $\mathfrak{g}$ we can simply apply the inclusion-exclusion formula to compute the measure of any finite union of basic open sets. Hence $\mu\left(\hat{V}_{i,s} \cup B_s\right) > 2^{-i}$ is μ-computable and therefore our sequence is uniformly $\Sigma_1^{0,\mu}$.

This means that we can μ-computably list all the μ-Martin-Löf tests, $\left\{\hat{V}_i^0\right\}_{i \in \omega}$, $\left\{\hat{V}_i^1\right\}_{i \in \omega}$ .... Diagonalize by defining

$$U_i = \bigcup_{j \in \omega} \hat{V}^j_{j+i+1}.$$

Then $\{U_i\}_{i \in \omega}$ is clearly uniformly $\Sigma_1^{0,\mu}$ and

$$\begin{aligned} \mu\left(U_i\right) &\leq \sum_{j \in \omega} \mu\left(\hat{V}^j_{j+i+1}\right) \\ &\leq \sum_{i \in \omega} 2^{-(j+i+1)} \\ &= 2^{-i}. \end{aligned}$$

Therefore $\{U_i\}_{i \in \omega}$ is a μ-Martin-Löf test.

Now suppose that $x \in \mathbb{X}$ is not μ-Martin-Löf random. By definition there is a μ-Martin-Löf test $\left\{\hat{V}_i^j\right\}_{i \in \omega}$ such that $x \in \bigcap_{i \in \omega} \hat{V}_i^j$. Hence for all $i \in \omega$ $x \in \hat{V}^j_{i+j+1} \subseteq U_i$. In other words $x \in \bigcap_{i \in \omega} U_i$. Therefore $\{U_i\}_{i \in \omega}$ is universal. □

We now show that $\mathcal{F}(\mathbb{E})$ with the Fell topology can be a Martin-Löf space. Our proof of this fact relies on lemma 2.4.7.

Proposition 3.0.9. *If $\mathbb{E}$ is an LCHS space, then $\mathcal{F}(\mathbb{E})$ is a Martin-Löf probability space under any Borel probability measure.*

Proof. The space $\mathbb{E}$ is LCHS and so it has a countable basis $\{B_0, B_1, \dots\}$ such that $\overline{B_i}$ is compact for all $i \in \omega$. By lemma 2.4.7 $\mathcal{F}(\mathbb{E})$ has a basis consisting of the sets

$$\mathcal{F}^{\overline{B}_{j_1} \cup \dots \cup \overline{B}_{j_k}}_{B_{i_1}, \dots, B_{i_n}}.$$

This basis has an obvious enumeration $\mathcal{B}_0, \mathcal{B}_1, \dots$ where $\mathcal{B}_i$ corresponds to

$$\mathcal{F}^{\overline{B}_{j_1} \cup \dots \cup \overline{B}_{j_k}}_{B_{i_1}, \dots, B_{i_n}}$$

exactly when $i = \langle \langle i_1, \dots, i_n \rangle, \langle j_1, \dots, j_k \rangle \rangle$ (with $\langle \ , \ \rangle$ a computable pairing function). This basis is closed under finite intersection and a computable intersection function is then given by

$$\begin{aligned} &\mathfrak{g}\left(\langle \langle i_1, \dots, i_n \rangle, \langle j_1, \dots, j_k \rangle \rangle, \langle \langle i'_1, \dots, i'_m \rangle, \langle j'_1, \dots, j'_l \rangle \rangle \right) \\ &\qquad\qquad = \langle \langle i_1, \dots, i_n, i'_1, \dots, i'_n \rangle, \langle j_1, \dots, j_k, j'_1, \dots, j'_l \rangle \rangle. \end{aligned}$$

Clearly any Borel probability measure $\mathbf{P}$ satisfies $\mathbf{P}(\mathcal{B}_i) \leq 1$. Therefore $\mathcal{F}(\mathbb{E})$ is a Martin-Löf space. □

Definition 3.0.10. Let $\mathbb{E}$ be an LCHS space with a specified basis of sets with compact closures. The enumeration of a basis for $\mathcal{F}(\mathbb{E})$ given in the proof of proposition 3.0.9 is the *canonical enumeration.*

We are also interested in working with measures of $\mathcal{F}(\mathbb{E})$ induced by Choquet capacities. We usually add the measure as an oracle and so we wish to know how much computational power we are adding when we do this. In particular we want to know when it is possible to compute $\mathbf{P}_T$ from T. Here we are representing $\mathbf{P}_T$ in the manner defined in 3.0.4: $p \in 2^\omega$ corresponds to $\mathbf{P}_T$ if $\mathbf{P}_T(\mathcal{B}_i) = p^{[i]}$. We represent T in much the same way: $t \in 2^\omega$ corresponds to T if $T\left(\overline{B}_i\right) = t^{[i]}$.

Lemma 3.0.11. *Let $\mathbb{E}$ be an LCHS space with a basis $\mathcal{B} = B_0, B_1, \dots$ such that:*

1. *$\mathcal{B}$ is closed under finite unions;*
2. *There is a computable union function $\mathfrak{f} : \omega^2 \to \omega$ such that $B_i \cup B_j = B_{\mathfrak{f}(i,j)}$.*
3. *$\overline{B_i}$ is compact for all $i \in \omega$.*

Let $\mathcal{B}' = \{\overline{B}_0, \overline{B}_1, \dots\}$ and let $T : \mathcal{B}' \to [0,1]$ be a Choquet capacity (cf. proposition 2.4.20) such that for all $i, j \in \omega$

$$T\left(\overline{B}_i \cup B_j\right) = T\left(\overline{B}_i \cup \overline{B}_j\right).$$

Then $\mathbf{P}_T$ is computable from T (and hence $\mathbf{P}_T \equiv_T T$).

Proof. By proposition 2.4.7 a basic open set of $\mathcal{F}$ is of the form $\mathcal{F}^{\overline{B}_j}_{B_{i_1},\dots,B_{i_n}}$. By the Choquet capacity theorem $\mathbf{P}_T$ is defined by

$$\mathbf{P}_T\left(\mathcal{F}^{\overline{B}_j}_{B_{i_1},\dots,B_{i_n}}\right) = \Delta_{B_{i_1}} \dots \Delta_{B_{i_n}} T\left(\overline{B}_j\right)$$

We prove by induction that for any $n \geq 1$ $\Delta_{B_{i_1}} \dots \Delta_{B_{i_n}} T\left(\overline{B}_j\right)$ is computable from T.

Base case: $n = 1$.

$$\begin{aligned} \Delta_{B_i} T\left(\overline{B}_j\right) &= T\left(B_i \cup \overline{B}_j\right) - T\left(\overline{B}_j\right) && \text{by definition} \\ &= T\left(\overline{B}_i \cup \overline{B}_j\right) - T\left(\overline{B}_j\right) && \text{by hypothesis} \\ &= T\left(\overline{B}_{\mathfrak{f}(i,j)}\right) - T\left(\overline{B}_j\right) && \text{by hypothesis.} \end{aligned}$$

Therefore $\Delta_{B_i} T\left(\overline{B}_j\right)$ is computable from T. We have also shown that

$$\Delta_{B_i} T\left(\overline{B}_j\right) = \Delta_{\overline{B}_i} T\left(\overline{B}_j\right).$$

The same calculations show that

$$\Delta_{\overline{B}_i} T\left(\overline{B}_j \cup B_k\right) = \Delta_{\overline{B}_i} T\left(\overline{B}_j \cup \overline{B}_k\right).$$

Now suppose that $\Delta_{B_{i_1}} \ldots \Delta_{B_{i_n}} T\left(\overline{B}_j\right)$ is computable from T. Suppose also that

$$\Delta_{B_{i_1}} \ldots \Delta_{B_{i_n}} T\left(\overline{B}_j\right) = \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j\right)$$

and

$$\Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j \cup B_k\right) = \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j \cup \overline{B}_k\right).$$

Then

$$\begin{aligned}
\Delta_{B_{i_1}} \ldots \Delta_{B_{i_{n+1}}} T\left(\overline{B}_j\right) &= -\Delta_{B_{i_1}} \ldots \Delta_{B_{i_n}} T\left(\overline{B}_j\right) - \Delta_{B_{i_1}} \ldots \Delta_{B_{i_n}} T\left(\overline{B}_j \cup B_{i_{n+1}}\right) \\
&= \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j\right) - \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j \cup B_{i_{n+1}}\right) \\
&= \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j\right) - \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j \cup \overline{B}_{i_{n+1}}\right) \\
&= \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_j\right) - \Delta_{\overline{B}_{i_1}} \ldots \Delta_{\overline{B}_{i_n}} T\left(\overline{B}_{\mathfrak{f}(j,i_{n+1})}\right).
\end{aligned}$$

This last line is computable from T by inductive hypothesis. Therefore $\mathbf{P}_T$ is computable from T.

By definition $T(\overline{B}_i) = \mathbf{P}_T\left(\mathcal{F}_{\overline{B}_i}\right) = 1 - \mathbf{P}_T\left(\mathcal{F}^{\overline{B}_i}\right)$ and hence T is always computable from $\mathbf{P}_T$. Therefore, in this situation $\mathbf{P}_T \equiv_T T$. □

This is a very flexible framework in which to study randomness and not much can be said generally. For instance, two different RACS may produce the same measure and thus the same Martin-Löf random closed sets. Most of the following work is an exploration of the possibilities of Martin-Löf random closed sets. This means analyzing a number of new examples as well as translating previous approaches into this framework. In many of these examples we are looking at a RACS $X : \Omega \to \mathcal{F}(\mathbb{E})$ with Ω a Cantor space. In these cases we are interested in when X maps Martin-Löf random elements of Ω to Martin-Löf random elements of $\mathcal{F}(\mathbb{E})$ (and in when Martin-Löf random elements of $\mathcal{F}(\mathbb{E})$ must be the image of Martin-Löf random elements of Ω). The following lemmas address this situation.

We state this first lemma for Hippocrates randomness (or, equivalently, for Martin-Löf randomness under computable measures). Relativizing to one or both of the measures in the lemma converts this into a result about Martin-Löf randomness. This is how we will use the lemma later.

Lemma 3.0.12. *Let Ω be a Martin-Löf probability space with computable measure P. Let $\mathbb{E}$ be an LCHS topological space with basis $B_0, B_1, \dots$ such that $\overline{B}_i$ is compact for every $i \in \omega$. Let $X : \Omega \to \mathcal{F}(\mathbb{E})$ be a map such that*

1. *$X^{-1}(\mathcal{F}_{B_i})$ is uniformly Σ^0_1 over $i \in \omega$;*
2. *$X^{-1}\left(\mathcal{F}^{\overline{B_i}}\right)$ is uniformly Σ^0_1 over $i \in \omega$.*

Then for any P-Hippocrates random $\rho \in \Omega$, $X(\rho)$ is $\mathbf{P}_X$-Hippocrates random. (Equivalently, if $F \in \mathcal{F}$ is not $\mathbf{P}_X$-H random then either $X^{-1}(\{F\})$ consists entirely of non-P-H random elements of Ω or is $\emptyset$.)

Proof. By proposition 3.0.9 $\mathcal{F}$ with $\mathbf{P}_X$ is a Martin-Löf space. We wish to show that if $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}_X$-Hippocrates test, then $\{X^{-1}(\mathcal{V}_i)\}_{i\in\omega}$ is a P-Hippocrates test.

Consider a basic open set of $\mathcal{F}$

$$\mathcal{B} = \mathcal{F}_{B_{i_1},\dots B_{i_n}}^{\overline{B}_{j_1}\cup\cdots\cup\overline{B}_{j_k}}.$$

By definition

$$X^{-1}\left(\mathcal{F}_{B_{i_1},\dots B_{i_n}}^{\overline{B}_{j_1}\cup\cdots\cup\overline{B}_{j_k}}\right)$$
$$= X^{-1}\left(\mathcal{F}_{B_{i_1}}\right) \cap \cdots \cap X^{-1}\left(\mathcal{F}_{B_{i_n}}\right) \cap X^{-1}\left(\mathcal{F}^{\overline{B}_{j_1}}\right) \cap \cdots \cap X^{-1}\left(\mathcal{F}^{\overline{B}_{j_k}}\right).$$

The right hand side of this equation is a finite intersection of Σ_1^0 sets. Such a finite intersection is again Σ_1^0 because Ω has a computable intersection function. Therefore if $\mathcal{B}$ is basic open, then $X^{-1}(\mathcal{B})$ is Σ_1^0. Moreover, this is uniform over the canonical enumeration of basic opens for $\mathcal{F}$.

Let $\mathcal{B}_0, \mathcal{B}_1, \dots$ be the canonical enumeration of a basis for $\mathcal{F}$. Let $\{\mathcal{V}_i\}_{i\in\omega}$ be a $\mathbf{P}_X$-H test. By definition there is a c.e. $f \in 2^\omega$ such that for each $i \in \omega$, $\mathcal{V}_i = \bigcup_{f(\langle i,j\rangle)=1} \mathcal{B}_j$. By the preceding, $X^{-1}(\mathcal{B}_j)$ is uniformly Σ_1^0 over $j \in \omega$. It then follows that

$$X^{-1}\left(\mathcal{V}_i\right) = \bigcup_{f(\langle i,j\rangle)=1} X^{-1}(\mathcal{B}_j)$$

is uniformly Σ_1^0 over $i \in \omega$. Furthermore, by definition

$$P\left(X^{-1}(\mathcal{V}_i)\right) = \mathbf{P}_X(\mathcal{V}_i) \leq 2^{-i}.$$

Therefore $\{X^{-1}(\mathcal{V}_i)\}_{i\in\omega}$ is a P-Hippocrates test.

Suppose $F \in \mathcal{F}$ is not $\mathbf{P}_X$-random. Then there is an $\mathbf{P}_X$-Hippocrates test

$\{\mathcal{V}_i\}_{i\in\omega}$ such that $F \in \bigcap_{i\in\omega} \mathcal{V}_i$. Hence

$$X^{-1}(\{F\}) \subseteq X^{-1}(\bigcap_{i\in\omega} \mathcal{V}_i) = \bigcap_{i\in\omega} X^{-1}(\mathcal{V}_i).$$

But $\{X^{-1}(\mathcal{V}_i)\}_{i\in\omega}$ is a P-Hippocrates test and therefore $X^{-1}(\{F\})$ consists entirely of non-P-H random elements (or is $\emptyset$). □

In general we cannot just use a RACS X to map Martin-Löf tests from Martin-Löf space Ω to the space $\mathcal{F}(\mathbb{E})$ because there is no guarantee that the image of a Σ_1^0 set will be Σ_1^0. Indeed, if X is not an onto map, then the image of the entire space Ω may not even be open. The following lemma gives a sufficient condition for pushing Martin-Löf tests across the RACS X. Once again this theorem can be stated in different ways for non-computable measures. The statement we have chosen is perhaps the simplest.

Lemma 3.0.13. *Let Ω be a Martin-Löf probability space with basis $B_0, B_1, \ldots$ and measure P. Let $X : \Omega \to \mathcal{F}(\mathbb{E})$ be a measurable map such that uniformly in P for each $i \in \omega$ there is a $\Sigma_1^{0,P}$ set $\mathcal{H}_i \subseteq \mathcal{F}(\mathbb{E})$ such that $X^{-1}(\mathcal{H}_i) = B_i$. In this situation if F is $\mathbf{P}_X$-H random with oracle P, then $X^{-1}(\{F\})$ consists entirely of P-H random elements of Ω (or is $\emptyset$).*

Proof. For each $i \in \omega$ let $\mathcal{H}_i \subseteq \mathcal{F}(\mathbb{E})$ be the $\Sigma_1^{0,P}$ set such that $X^{-1}(\mathcal{H}_i) = B_i$.

Let $\{U_i\}_{i\in\omega}$ be a P-Martin-Löf test. By definition there is a c. e. in P $f \in 2^\omega$ such that $U_i = \bigcup_{f(\langle i,j\rangle)=1} B_j$. Define for each $i \in \omega$

$$\mathcal{V}_i := \bigcup_{f(\langle i,j\rangle)=1} \mathcal{H}_j.$$

Our hypotheses ensure that $\{\mathcal{V}_i\}_{i\in\omega}$ is uniformly $\Sigma_1^{0,P}$. Furthermore $X^{-1}(\mathcal{V}_i) = U_i$

and thus,

$$\mathbf{P}_X\left(\mathcal{V}_i\right) = P\left(X^{-1}\left(\mathcal{V}_i\right)\right) = P\left(U_i\right) \leq 2^{-i}.$$

Therefore $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}_X$-ML test over oracle P.

Now suppose that $F \in \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_X$-ML random over oracle P. Then $F \notin \bigcap_{i\in\omega} \mathcal{V}_i$. Equivalently, $\{F\} \cap \left(\bigcap_{i\in\omega} \mathcal{V}_i\right) = \emptyset$. By applying X^{-1} to this equation we find that $X^{-1}(\{F\}) \cap \left(\bigcap_{i\in\omega} U_i\right) = \emptyset$. Since this is true for every P-ML test $\{U_i\}_{i\in\omega}$ we therefore conclude that $X^{-1}(\{F\})$ consists entirely of P-ML random elements of Ω (or is $\emptyset$). □

In the classical theory of RACS it is easy to show that if X and Y are RACS, then $X \cup Y$ is also a RACS. We can prove an effective version of this theorem. First we state a proposition that is not directly related to random closed sets but which relates to the following lemmas.

Proposition 3.0.14. *If Ω_1 and Ω_2 are Martin-Löf probability spaces with measures P_1 and P_2, respectively, then $\Omega_1 \times \Omega_2$ is a Martin-Löf probability space under the measure $P_1 \times P_2$ and if $(\rho_1, \rho_2) \in \Omega_1 \times \Omega_2$ is $P_1 \times P_2$-ML random, then ρ_1 is P_1-ML random and ρ_2 is P_2-ML random.*

Proof. Take as the basis for $\Omega_1 \times \Omega_2$ the product of the bases for Ω_1 and Ω_2. This is a countable basis with an obvious enumeration. The intersection function for this basis is easily computed from the intersection functions for the bases for Ω_1 and Ω_2. The space $\Omega_1 \times \Omega_2$ clearly has $P_1 \times P_2$-measure one. Therefore $\Omega_1 \times \Omega_2$ is a Martin-Löf probability space.

Now suppose that $\rho_1 \in \Omega_1$ is not P_1-ML random. Then there is a P_1-ML test $\{U_i\}_{i\in\omega}$ such that $\rho_1 \in \bigcap_{i\in\omega} U_i$. Define

$$V_i = U_i \times \Omega_2.$$

It is clear that $\{V_i\}_{i\in\omega}$ is a uniformly Σ^0_1 sequence of subsets of $\Omega_1 \times \Omega_2$. Furthermore, $(P_1 \times P_2)(V_i) = P_1(U_i) \times P_2(\Omega_2) = P_1(U_i) \leq 2^{-i}$. Therefore $\{V_i\}_{i\in\omega}$ is a $P_1 \times P_2$-ML test and for all $\rho_2 \in \Omega_2$ $(\rho_1, \rho_2) \in \bigcap_{i\in\omega} V_i$.

We have shown that if (ρ_1, ρ_2) is $P_1 \times P_2$-ML random, then ρ_1 must be P_1-ML random. By symmetry ρ_2 must also be P_2-ML random. □

Note that in the special case of $2^\omega \times 2^\omega$ we can prove that (f_1, f_2) is $m \times m$-ML random if and only if $f_1 \oplus f_2$ is ML random. The proof we have in mind here uses the fact that $(f_1, f_2) \mapsto f_1 \oplus f_2$ is a measure-preserving computable homeomorphism. Van Lambalgen's theorem then tells us that $f_1 \oplus f_2$ is ML random if and only if f_1 is ML random relative to f_2. Unfortunately, this technique does not seem to extend to more general situations.

Lemma 3.0.15. *Let Ω be a probability space and let $\mathbb{E}$ be an LCHS topological space. Let $X : \Omega \to \mathcal{F}(\mathbb{E})$ and $Y : \Omega \to \mathcal{F}(\mathbb{E})$ be RACS. Then the map $X \cup Y : \Omega \times \Omega \to \mathcal{F}(\mathbb{E})$ given by $(X \cup Y)(\rho_1, \rho_2) = X(\rho_1) \cup Y(\rho_2)$ is a RACS. Moreover, if $(F_1, F_2) \in \mathcal{F}(\mathbb{E}) \times \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_X \times \mathbf{P}_Y$-ML random, then $F_1 \cup F_2 \in \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_{X\cup Y}$-ML random.*

Proof. It is clear that $(X, Y) : \Omega_1 \times \Omega_2 \to \mathcal{F}(\mathbb{E}) \times \mathcal{F}(\mathbb{E})$ is measurable. Consequently, to prove that $X \cup Y$ is a RACS it will suffice to prove that $Z : \mathcal{F}(\mathbb{E}) \times \mathcal{F}(\mathbb{E}) \to \mathcal{F}(\mathbb{E})$ given by $Z(F_1, F_2) = F_1 \cup F_2$ is measurable.

To prove that Z is measurable it is sufficient to show that $Z^{-1}\left(\mathcal{F}^K\right)$ is a measurable subset of $\mathcal{F}(\mathbb{E}) \times \mathcal{F}(\mathbb{E})$. However, we will actually be able to prove the stronger result that Z meets the hypotheses of lemma 3.0.12. Let B be an open subset of $\mathbb{E}$ such that $\overline{B}$ is compact. First,

$$Z^{-1}\left(\mathcal{F}^{\overline{B}}\right) = \mathcal{F}^{\overline{B}} \times \mathcal{F}^{\overline{B}}.$$

Second,

$$Z^{-1}(\mathcal{F}_B) = (\mathcal{F}_B \times \mathcal{F}(\mathbb{E})) \cup (\mathcal{F}(\mathbb{E}) \times \mathcal{F}_B).$$

We note that these calculations show that $\mathbf{P}_Z \leq_T (\mathbf{P}_X \times \mathbf{P}_Y)$. Consequently if $\{\mathcal{U}_i\}_{i\in\omega}$ is a $\mathbf{P}_Z$-ML test, then $\{Z^{-1}(\mathcal{U}_i)\}_{i\in\omega}$ is a $\mathbf{P}_X \times \mathbf{P}_Y$-ML test. In other words, we can apply lemma 3.0.12 relativized to $\mathbf{P}_Z$ to find that if $F \in \mathcal{F}(\mathbb{E})$ is not $\mathbf{P}_Z$-ML random and $E_1 \cup E_2 = F$, then (E_1, E_2) is not $(\mathbf{P}_X \times \mathbf{P}_Y)$-ML random.

This is almost the statement we want: we need only to prove that $\mathbf{P}_Z = \mathbf{P}_{X\cup Y}$.

$$\begin{aligned}
\mathbf{P}_Z(\mathcal{H}) &= (\mathbf{P}_X \times \mathbf{P}_Y)\left(Z^{-1}(\mathcal{H})\right) \\
&= (P_1 \times P_2)\left((X \cup Y)^{-1}(\mathcal{H})\right) \\
&= \mathbf{P}_{X\cup Y}(\mathcal{H}).
\end{aligned}$$

Therefore if (E_X, E_Y) is $(\mathbf{P}_X \times \mathbf{P}_Y)$-ML random, then $E_x \cup E_Y \in \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_{X\cup Y}$-ML random. □

We note that in the situation of lemma 3.0.15 we can actually prove stronger conclusion that $F_1 \cup F_2$ is $\mathbf{P}_{X\cup Y}$-ML random over oracle $\mathbf{P}_X \times \mathbf{P}_Y$. To achieve this we simply relativize everywhere to the oracle $\mathbf{P}_X \times \mathbf{P}_Y$.

3.1 Random subsets of the natural numbers

We begin with what might be considered a test-case for the preceding framework. In this example we consider the space $\mathbb{N}$ with the discrete topology. Because $\mathbb{N}$ is endowed with the discrete topology $\mathcal{F}(\mathbb{N})$ consists of all subsets of $\mathbb{N}$. We can thus identify the class $\mathcal{F}(\mathbb{N})$ with $2^{\mathbb{N}} = 2^{\omega}$. Of course we already have many notions of algorithmic randomness for the space 2^{ω}. Working from the perspective of the Fell topology on $2^{\omega} = \mathcal{F}(\mathbb{N})$ is a potentially new notion. As it turns out, however, the Fell topology for $\mathcal{F}(\mathbb{N})$ is exactly the standard topology for 2^ω. Consequently we arrive at the same Martin-Löf random elements of 2^ω whether we proceed by the usual route (as described in section 2.3) or by the novel route described above 3.

Proposition 3.1.1. *The topology generated by the sets $[\sigma]$, $\sigma \in 2^{<\omega}$, (the standard topology) is the same as the Fell topology for $\mathcal{F}(\mathbb{N}) = 2^\omega$.*

Proof. We begin by proving that the standard topology contains the Fell topology. By theorem 2.4.7 the Fell topology for $\mathcal{F}(\mathbb{N}) = 2^\omega$ has a sub-basis consisting of sets of the form $\mathcal{F}^{\{a\}}$ and $\mathcal{F}_{\{b\}}$ where $a, b \in \mathbb{N}$. To prove that the standard topology contains the Fell topology it will suffice, to prove that $\mathcal{F}_{\{b\}}$ is clopen in the standard topology. This is sufficient because $\mathcal{F}^{\{a\}} = \left(\mathcal{F}_{\{a\}}\right)^{\complement}$. Hence if $\mathcal{F}_{\{b\}}$ is clopen in the standard topology then the sub-basis for the Fell topology consists entirely of clopen subsets in the standard topology and the containment is clear.

$$\mathcal{F}_{\{a\}} = \bigcup \{[\sigma] : |\sigma| = a + 1 \;\&\; \sigma(a) = 1\}.$$

To show that the standard topology is contained in the Fell topology we show that for any $\sigma \in 2^{<\omega}$ the cylinder $[\sigma]$ is open in the Fell topology.

$$[\sigma] = \left(\bigcap \{\mathcal{F}^{\{n\}} : \sigma(n) = 0\}\right) \cap \left(\bigcap \{\mathcal{F}_{\{n\}} : \sigma(n) = 1\}\right).$$

This set is (basic) open in the Fell topology.

Therefore the two topologies coincide. $\square$

A closer look at the proof reveals that we have shown that $[\sigma]$ is a basic open set in the Fell topology and that any basic open in the Fell topology is clopen in the standard topology. Moreover the correspondence is computable (by following the calculations above). Hence any subset of 2^ω is Σ_1^0 in the standard topology if and only if it is Σ_1^0 in the Fell topology. This means that Martin-Löf tests in the Fell topology are exactly the same as Martin-Löf tests in the standard topology.

CHAPTER 4

MARTIN-LÖF RANDOM CLOSED SUBSETS OF CANTOR SPACE

In this chapter we consider Martin-Löf random closed subsets of 2^ω. We begin by addressing three other approaches to defining algorithmically random closed subsets of 2^ω: the approach of Barmpalias et al. [1]; the approach of Kjos-Hanssen and Diamondstone [14]; and the approach of Mauldin and McLinden [4]. We show that each of these approaches produces a RACS and hence is compatible with the framework of Martin-Löf random closed sets developed in chapter 3. All three of these approaches are similar in that they make use of the connection between infinite binary trees and closed subsets of 2^ω. This is not the only way of producing a RACS for 2^ω. In the final sections of this chapter we explore some very different RACS and the Martin-Löf random closed sets they give rise to.

4.1 BBCDW-random closed sets

The RACS in this section comes from Barmpalias, Brodhead, Cenzer, Dashti, and Weber and can be found in “Algorithmic randomness of closed sets” [1]. However, this paper does not use the framework of Martin-Löf tests in the Fell topology. Instead they code closed subsets of 2^ω as ternary reals and then define a random closed set to be any closed set that has a Martin-Löf random code. This definition of “random closed set” is in conflict with the definition of “random closed set" used by probability theory (which we have given in section 2). We have chosen to respect the prior claim of probability theory and so we call the random closed sets of Barmpalias et al. [1] BBCDW-random closed sets.

We provide a proof that the coding of closed sets as ternary reals use in Barmpalias et al. [1] is compatible with the Fell topology and in fact gives rise to a RACS which we call the "canonical decoding" and denote by Z. This basic result was independently proved by Brodhead and can be found in his thesis [2], although that proof differs from ours (indeed, a proof of lemma 4.1.4 is implicit in [2]). We use this result to prove that a closed set $F \subseteq 2^\omega$ is $\mathbf{P}_Z$-Martin-Löf random if and only if it is BBCDW-random. This allows us to explore the BBCDW-random closed sets using the tools of the theory of RACS.

The development of BBCDW-random closed sets depends on the characterization of the closed sets of Cantor space as the sets of paths through trees (proposition 2.3.8). In particular we use the one-to-one correspondence of extensible trees and closed subsets of Cantor space.

Definition 4.1.1 (Barmpalias et al. [1])**.** Let $F \subseteq 2^\omega$ be nonempty and closed and let T_F be the unique extensible tree such that $F = [T_F]$. We code T_F as a ternary real, Z_F, as follows. Enumerate the nodes of T_F in order (by length and then lexicographically), starting with the empty string, $\lambda = \tau_0, \tau_1, \tau_2, \ldots$.

$$Z_F(n) = \begin{cases} 2 & \text{if } \tau_n{}^\frown 0 \in T_F \ \& \ \tau_n{}^\frown 1 \in T_F \\ 1 & \text{if } \tau_n{}^\frown 0 \notin T_F \ \& \ \tau_n{}^\frown 1 \in T_F \\ 0 & \text{if } \tau_n{}^\frown 0 \in T_F \ \& \ \tau_n{}^\frown 1 \notin T_F \end{cases}$$

This coding is called the *canonical coding* of F as a ternary real.

Proposition 4.1.2 (Barmpalias et al. [1])**.** *The canonical coding is a bijection between the nonempty closed sets of* 2^ω *and* 3^ω*.*

The central definition of Barmpalias et al. [1] is the following.

Definition 4.1.3 (Barmpalias et al. [1])**.** A nonempty closed set $F \subseteq 2^\omega$ is *BBCDW-random* if its canonical code $Z_F \in 3^\omega$ is Martin-Löf random (with respect to the "fair coin" measure on 3^ω).

To bring the canonical coding of a closed set into the probability framework let

$$Z : 3^\omega \to \mathcal{F}(2^\omega)$$

be the inverse map of the canonical coding (the *canonical decoding*). We wish to prove that Z is a RACS. In fact we can do better: we prove the stronger result that Z is a homeomorphism between 3^ω and $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ and moreover, Z and Z^{-1} both preserve the complexity of sets.

By proposition 3.0.9 $\mathcal{F}(2^\omega)$ is a Martin-Löf space. In particular, by lemma 2.4.7, the Fell topology has a basis consisting of the sets

$$\mathcal{F}^{[\sigma_{n+1}],\ldots,[\sigma_k]}_{[\sigma_1],\ldots,[\sigma_n]}$$

for $\sigma_1, \ldots, \sigma_n, \sigma_{n+1}, \ldots, \sigma_k \in 2^{<\omega}$. Let $\mathcal{B}_0, \mathcal{B}_1, \ldots$ be the canonical enumeration (cf. definition 3.0.10) of this basis.

Lemma 4.1.4. *$Z : 3^\omega \to \mathcal{F}(2^\omega) \setminus \{\emptyset\}$ is a homeomorphism that preserves the complexity of sets, i.e. if $A \subseteq 3^\omega$ is Σ^0_1 (Π^0_1), then $Z(A) \subseteq \mathcal{F}(2^\omega)$ is Σ^0_1 (Π^0_1) and if $\mathcal{A} \subseteq \mathcal{F}$ is Σ^0_1 (Π^0_1), then $Z^{-1}(\mathcal{A})$ is (Π^0_1).*

Proof. We already know that Z is a bijection (proposition 4.1.2). It remains to be shown that Z is a homeomorphism and that it preserves complexity. We will prove this by looking at the effect of the map Z and Z^{-1} on basic open sets.

We first prove that for any $\sigma \in 2^{<\omega}$ $Z^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ is clopen. The map Z is given by decoding a real, f, into an extensible tree, T_f, and then taking the paths through that tree. That is, $f \mapsto [T_f]$. The key fact is that T_f is an extensible tree. By definition

$$Z^{-1}(\mathcal{F}_{[\sigma]}) = \{f \in 3^\omega : \sigma \in T_f\}.$$

But $\sigma \in T_f$ if and only if $\forall n \leq |\sigma|$ if k is the coding location for $\sigma \restriction n$, then $f(k) = \sigma(n)$ or $f(k) = 2$.

Claim: The coding locations for $\sigma \restriction n$ with $n \leq |\sigma|$ all occur in the first $2^{|\sigma|+1}$ bits of f. This must be the case because each bit of f corresponds to a unique element of $2^{<\omega}$. The number of elements of $2^{<\omega}$ of length no more than $|\sigma|$ is $2^{|\sigma|+1} - 1$.

The claim is true and so if $g \in Z^{-1}(\mathcal{F}_{[\sigma]})$, then

$$[g \restriction 2^{|\sigma|+1}] \subseteq Z^{-1}\left(\mathcal{F}_{[\sigma]}\right).$$

Consequently there are $\tau_1, \ldots, \tau_m \in 2^{|\sigma|+1}$ such that

$$Z^{-1}\left(\mathcal{F}_{[\sigma]}\right) = [\tau_1] \cup \cdots \cup [\tau_m].$$

Therefore $Z^{-1}(\mathcal{F}_{[\sigma]})$ is clopen. Moreover, finding $\tau_1, \ldots, \tau_m \in 2^{|\sigma|+1}$ is uniformly computable (over $\sigma \in 2^{<\omega}$). We simply decode each $\tau \in\in 2^{|\sigma|+1}$ into a (finite) tree and check to see if σ is in that tree.

Now $Z^{-1}\left(\mathcal{F}^{[\sigma]}\right) = \left[Z^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right]^{\mathsf{C}}$ and hence is also clopen and uniformly computable from σ. Therefore if $\mathcal{B} = \mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}^{[\sigma_{n+1}]\cup\cdots\cup[\sigma_k]}$ is a basic open set of $\mathcal{F}$, then

$$Z^{-1}(\mathcal{B}) = Z^{-1}\left(\mathcal{F}_{[\sigma_1]}\right) \cap \cdots \cap Z^{-1}\left(\mathcal{F}_{[\sigma_n]}\right) \cap Z^{-1}\left(\mathcal{F}^{[\sigma_{n+1}]}\right) \cap \cdots \cap Z^{-1}\left(\mathcal{F}^{[\sigma_k]}\right).$$

A finite intersection of uniformly computable clopen sets in 3^ω is again uniformly computably clopen. Therefore for any basic open set $\mathcal{B} \subseteq \mathcal{F}$ $Z^{-1}(\mathcal{B})$ is clopen and moreover, this set is uniformly computable from the canonical index for $\mathcal{B}$. This means that Z is continuous and that if $\mathcal{A} \subseteq \mathcal{F}$ is Σ_1^0 (Π_1^0), then $Z^{-1}(\mathcal{A})$ is uniformly Σ_1^0 (Π_1^0).

We now consider $Z([\tau])$ for $\tau \in 3^{<\omega}$. We prove by induction on τ that $Z([\tau])$ is a basic open of $\mathcal{F}$.

Base case: $\tau = \lambda$ (the empty string).

$$Z([\lambda]) = \mathcal{F} \setminus \{\emptyset\} = \mathcal{F}_{2^\omega}.$$

Suppose now that

$$Z([\tau]) = \mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}^{[\sigma_{n+1}]\cup\cdots\cup[\sigma_k]}$$

where $\sigma_1, \ldots, \sigma_n, \sigma_{n+1}, \ldots, \sigma_k \in 2^{<\omega}$. Then $\tau^\frown i$ codes for the branching at some node $\sigma \in 2^{<\omega}$ which we can compute uniformly from τ. We the have the following:

$$Z([\tau^\frown 0]) = \mathcal{F}_{[\sigma_1],\ldots,[\sigma_n],[\sigma^\frown 0]}^{C\cup[\sigma^\frown 1]}$$
$$Z([\tau^\frown 1]) = \mathcal{F}_{[\sigma_1],\ldots,[\sigma_n],[\sigma^\frown 1]}^{C\cup[\sigma^\frown 0]}$$
$$Z([\tau^\frown 2]) = \mathcal{F}_{[\sigma_1],\ldots,[\sigma_n],[\sigma^\frown 0],[\sigma^\frown 1]}^{C}.$$

Hence $Z([\tau])$ is a basic open for any $\tau \in 3^\omega$ and moreover, the canonical index for this basic open is uniformly computable from τ. Therefore Z^{-1} is continuous and so we have shown that Z is a homeomorphism. In addition we have shown that if $A \subseteq 3^\omega$ is Σ^0_1 (Π^0_1), then $Z(A)$ is uniformly Σ^0_1 (Π^0_1). □

Corollary 4.1.5. *A element $f \in 3^\omega$ is Martin-Löf random if and only if $Z(f)$ is $\mathbf{P}_Z$-Martin-Löf random. (That is, a closed set is BBCDW-random if and only if it is $\mathbf{P}_Z$-ML random).*

Proof. We first note that a direct consequence of lemma 4.1.4 is that the measure $\mathbf{P}_Z$ is computable. This means in particular that $\mathbf{P}_Z$-H randomness is the same as $\mathbf{P}_Z$-ML randomness in this case (see definitions 3.0.6 and 3.0.7). This simplifies the computability concerns and so applying lemmas 3.0.12 and 3.0.13 becomes straightforward.

(⇒) By lemma 4.1.4 $Z^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ and $Z^{-1}\left(\mathcal{F}^{[\sigma]}\right)$ are uniformly Σ^0_1 for $\sigma \in 2^{<\omega}$. Therefore by lemma 3.0.12, if $f \in 3^\omega$ is ML random, then $Z(f)$ is $\mathbf{P}_Z$-ML random.

(⇐) Suppose that $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-ML random.

First we show that $F \neq \emptyset$. Note that $\mathcal{F}^{2^\omega} = \{\emptyset\}$ is a basic open set in $\mathcal{F}$. Furthermore $Z^{-1}\left(\mathcal{F}^{2^\omega}\right) = \emptyset$ and so $\mathbf{P}_Z\left(\mathcal{F}^{2^\omega}\right) = 0$. Therefore $\emptyset$ is not Z-random and hence $F \neq \emptyset$.

Now that we know that $Z^{-1}(F)$ exists we now wish to show that we can apply lemma 3.0.13 to show that $Z^{-1}(F)$ is ML random. Suppose that $\tau \in 3^{<\omega}$. Then τ codes for a finite tree with terminal nodes $\sigma_1, \ldots, \sigma_n \in 2^{<\omega}$ (and these nodes are uniformly computable from τ). It is clear from the definition that

$$Z([\tau]) = \mathcal{F}_{\sigma_1,\ldots,\sigma_n}^{([\sigma_1]\cup\cdots\cup[\sigma_n])^\complement}.$$

But Z is a bijection and thus

$$[\tau] = Z^{-1}\left(\mathcal{F}_{\sigma_1,\ldots,\sigma_n}^{([\sigma_1]\cup\cdots\cup[\sigma_n])^\complement}\right).$$

In other words, for each $\tau \in 3^{<\omega}$ there is a basic open set $\mathcal{B}$ such that $Z^{-1}(\mathcal{B}) = [\tau]$ and the index of this basic open set is uniformly computable from τ. Therefore by lemma 3.0.13 $Z^{-1}(F)$ is ML random. $\square$

By lemma 4.1.4 corollary and 4.1.5 any topological property of the random reals must be shared by the $\mathbf{P}_Z$-ML random closed sets. For example:

Corollary 4.1.6. *The class of $\mathbf{P}_Z$-ML random closed sets is dense in $\mathcal{F}(2^\omega)$.*

This means that for all $\sigma_1, \ldots \sigma_n, \tau_1, \ldots, \tau_k \in 2^{<\omega}$ such that $\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}^{[\tau_1]\cup\cdots\cup[\tau_k]} \neq \emptyset$, there is a $\mathbf{P}_Z$-ML random closed set $F \in \mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}^{[\tau_1]\cup\cdots\cup[\tau_k]}$. In other words, we can specify finitely much information about a closed set (finitely many initial segments it must have extensions of and finitely many that it must not have extensions of) and know that there is a $\mathbf{P}_Z$-ML random closed set with these properties. For example, we know that for each $\sigma \in 2^{<\omega}$ there is a $\mathbf{P}_Z$-ML random closed set $F \subseteq [\sigma]$.

Lemma 4.1.4 also means that any notion of randomness based on Martin-Löf tests is the same for the space 3^ω and the space $\mathcal{F}(2^\omega) \setminus \{\emptyset\}$ with measure $\mathbf{P}_Z$. In particular, relativizing corollary 4.1.5 will show that $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-n-ML random relative to oracle g if and only if $Z^{-1}(F)$ is n-ML random relative to g. If we wish to consider a different measure P on 3^ω, then we can take the oracle to be P (and $n = 1$ to find the following corollaries.

Corollary 4.1.7. *Let P be a measure on 3^ω and let $\mathbf{P}_Z$ be defined by $\mathbf{P}_Z(\mathcal{A}) = P(Z^{-1}(\mathcal{A}))$ for measurable $\mathcal{A} \subseteq \mathcal{F}(2^\omega)$. Then $P \equiv_T \mathbf{P}_Z$.*

Proof. This follows directly from the definition of $\mathbf{P}_Z$ and the fact that Z preserves complexity. $\square$

Corollary 4.1.8. *Let P be any probability measure on 3^ω and let $\mathbf{P}_Z$ be defined by $\mathbf{P}_Z(\mathcal{A}) = P\left(Z^{-1}(\mathcal{A})\right)$ for measurable $\mathcal{A} \subseteq \mathcal{F}(2^\omega)$. Then $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-Martin-Löf random if and only if $Z^{-1}(F)$ is P-Marin-Löf random.*

Proof. Simply relativize the proof of corollary 4.1.5 to P. By the preceding lemma (4.1.7) $P \equiv_T \mathbf{P}_Z$ and so the corollary follows. □

The following theorem about the measure of $\mathbf{P}_Z$-ML random closed sets was proved for the fair coin measures on 2^ω and 3^ω in Barmpalias et al. [1]. We give a different proof in order to illustrate the utility of our approach to algorithmically random closed sets. In particular we note that the use of existing theorems in the literature of RACS is possible using this approach. The following proof also works naturally for non-computable measures and so we state the theorem in some generality.

Proposition 4.1.9. *Let μ be any σ-finite Borel measure on 2^ω and let P be any computable, Borel, probability measure on 3^ω such that for μ-almost every $f \in 2^\omega$ $P\left(\{g \in 3^\omega : f \in Z(g)\}\right) = 0$. If $F \in \mathcal{F}\left(2^\omega\right)$ is $\mathbf{P}_Z$-ML random over oracle μ, then $\mu(F) = 0$.*

Proof. By Robbins' theorem (2.4.11):

$$\begin{aligned} E\left(\mu \circ Z\right) &= \int_{2^\omega} P\left(\{g \in 3^\omega : f \in Z(g)\}\right) df \\ &= \int_{2^\omega} 0 \; df \\ &= 0. \end{aligned}$$

Consequently $\mathbf{P}_Z$-almost every closed set has measure 0. Equivalently, by lemma 4.1.4, P-almost every $g \in 3^\omega$ maps to a closed set with measure 0. We will use this fact to build P-ML tests that catch each $g \in 3^\omega$ such that $Z(g)$ has positive measure. Then corollary 4.1.8 shows that if F has positive measure, then F is not $\mathbf{P}_Z$-ML random.

As in the proof of lemma 4.1.4, let T_g be the extensible tree coded by $g \in 3^\omega$. For $n \in \omega$ define the level n approximation of $[T_g] = Z(g)$:

$$A_{g,n} := \bigcup_{\sigma \in T_g \cap 2^n} [\sigma].$$

Then $A_{g,n} \subseteq 2^\omega$ is clopen for each $n \in \omega$ and $\bigcap_{n\in\omega} A_{g,n} = [T_g]$. Moreover, $\mu(A_{g,n})$ is computable from μ and

$$\mu\left([T_g]\right) = \lim_{n\to\infty} \mu\left(A_{g,n}\right).$$

Fix $k \in 2^\omega$. Consider the set

$$U_n := \left\{g \in 3^\omega : \mu\left(A_{g,n}\right) > 2^{-k}\right\}.$$

As in the proof of lemma 4.1.4, if $g \in U_n$, then $[g \restriction 2^{n+1}] \subseteq U_n$. It follows that U_n is clopen and it's P-measure is uniformly computable in $\mu \oplus P$. In addition, $U_1 \supseteq U_2 \supseteq \dots$ and $\bigcap_{n\in\omega} U_n = \{g \in 3^\omega : \mu\left(Z(g)\right) \geq 2^{-k}\}$. Thus $\lim_{n\to\infty} \mu(U_n) = \mu\left(\{g \in 3^\omega : \mu\left(Z(g)\right) \geq 2^{-k}\}\right) = 0$.

Let $V_i = U_n$ where n is the least number such that $P\left(U_n\right) \leq 2^{-i}$. Then $\{V_i\}_{i\in\omega}$ is a P-ML test over oracle μ and

$$\bigcap_{i\in\omega} V_i = \bigcap_{n\in\omega} U_n = \left\{g \in 3^\omega : \mu\left(Z(g)\right) > 2^{-k}\right\}.$$

Suppose that $F \in \mathcal{F}$ has $\mu(F) > 2^{-k}$. Then $Z^{-1}(F) \in \bigcap_i V_i$. By definition

$Z^{-1}(F)$ is non P-ML random over oracle μ. Therefore by corollary 4.1.8 F is not $\mathbf{P}_z$-ML random over oracle μ.

This holds for any $k \in \omega$ and therefore if F is $\mathbf{P}_Z$-ML random over oracle μ, then $\mu(F) = 0$. □

Note that the condition that $P\left(\{g \in 3^\omega : f \in Z(g)\}\right) = 0$ is simply a way of writing $\mathbf{P}_Z\left(\mathcal{F}_{\{f\}}\right) = 0$. This is not a strong condition. For example, any Bernoulli measure (discussed in section 5.2) on 3^ω satisfies this condition.

4.1.1 Fractal properties

We can also investigate the fractal properties of random closed sets by translating Z into an example of the construction described in section 2.5. The following proposition then is a direct result of theorem 2.5.3. It agrees exactly with a result in Barmpalias et al. [1] but also differs in some respects. That result proves that a BBCDW-random closed set F must have box dimension $2 - \log_2(3)$. Our result proves only that $\mathbf{P}_Z$-almost every $F \in \mathcal{F}$ has Hausdorff and box dimension $2 - \log_2(3)$.

The work in this section has been (independently) noted and extended by Mauldin and McLinden [4].

First we must specify a metric for the space 2^ω. This metric is compatible with the standard topology on 2^ω in the sense that defining open sets from open balls under the metric gives rise to the standard topology.

Definition 4.1.10. Define a metric d on 2^ω as follows. For $f \neq g \in 2^\omega$ let $n \in \omega$ be the least number such that $f(n) \neq g(n)$. Then

$$d(f, g) = 2^{-n}.$$

Proposition 4.1.11. *A closed set* $F \in \mathcal{F}(3^\omega)$ $\mathbf{P}_Z$*-almost certainly has Hausdorff and box dimension* $2 - \log_2(3)$.

Proof. The goal here is simply to show that the map $Z : 3^\omega \to \mathcal{F}(2^\omega)$ fits the framework for a random recursive construction. In this proof we use the approximating sets $A_{g,n}$ of the proof of proposition 4.1.9. These sets play the role of E_n in section 2.5.

Take $\mathbb{E} = 2^\omega$ with the usual metric d. Take $V = \mathbb{E}$ and $m = 2$. We will define $V_0, V_1 \subseteq 2^\omega$. The set V_0 is either $[0]$ or $\emptyset$ as determined by a random ratio of similarity s_0, and V_1 is either $[1]$ or $\emptyset$ as determined by a random ratio of similarity s_1. The similarities s_0 and s_1 are two (dependent) random variables taking values in $\{0, \frac{1}{2}\}$

We define $s_0 : 3^\omega \to [0, 1]$ and $s_1 : 3^\omega \to [0, 1]$ using the following. Let $f \in 3^\omega$ and let $T_f \subseteq 2^{<\omega}$ be the extensible tree coded by f (so $Z(f) = [T_f]$). Let $\sigma_0, \sigma_1, \ldots$ be the nodes of T_f listed by length and then lexicographically within lengths. Thus $f(n)$ determines the branching at node σ_n as in the definition 4.1.1.

We can now define s_0 and s_1:

$$s_0(f) = \begin{cases} 0 & \text{if } f(0) = 1 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$s_1(f) = \begin{cases} 0 & \text{if } f(0) = 0 \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Note that this gives the following probabilities (which entirely determine the distributions of s_0 and s_1):

1. $P(s_i = 0 | s_{i-1} = 0) = 0$ for $i = 0, 1$.

2. $P(s_i = 0 | s_{i-1} = \frac{1}{2}) = \frac{1}{2}$ for $i = 0, 1$.

3. $P(s_i = 0) = \frac{1}{3}$ for $i = 0, 1$.

Now for $i \in \{0, 1\}$, V_i is given by

$$V_i = \begin{cases} \emptyset & \text{if } s_i = 0 \\ [i] & \text{if } s_i = \frac{1}{2}. \end{cases}$$

We have done this so that $V_0 = \emptyset$ just in the case that $Z(f) \subseteq [1]$. Otherwise $V_0 = [0]$. Similarly, $V_1 = \emptyset$ just in the case that $Z(f) \subseteq [0]$ and $V_1 = [1]$ otherwise. Then $A_{f,1} = V_0 \cup V_1$.

Now suppose that $A_{f,k} = \bigcup_{\sigma \in 2^k} V_\sigma$ is defined. Suppose also that if V_σ is nonempty then $V_\sigma = [\sigma]$ and that this happens if and only if $\sigma \in T_f$.

Define $s_{\sigma^\frown j}$ for $j \in \{0, 1\}$ as follows:

$$s_{\sigma^\frown 0}(f) = \begin{cases} \frac{1}{2} & \text{if } \sigma = \sigma_n \in T_f \text{ and } f(n) \neq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\sigma^\frown 1}(f) = \begin{cases} \frac{1}{2} & \text{if } \sigma = \sigma_n \in T_f \text{ and } f(n) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $s_{\sigma^\frown j}$ is a is random variable with the same distribution as s_j for $j \in \{0, 1\}$ and that it is independent everywhere possible.

Then define

$$V_{\sigma^\frown j} \begin{cases} \emptyset & \text{if } s_{\sigma^\frown i} = 0 \\ [\sigma^\frown i] & \text{otherwise.} \end{cases}$$

Thus $V_{\sigma^\frown j}$ is nonempty if and only if $\sigma^\frown j \in T_f$. Now define $A_{f,k+1} = \bigcup_{\sigma \in 2^{k+1}} V_\sigma$.

Clearly $Z(f) = \bigcap_{k\in\omega} A_{f,k}$. Thus the map $Z : 3^\omega \to \mathcal{F}(2^\omega)$ is a random self-similar construction of the kind described in section 2.5. We now apply theorem 2.5.3 to this construction. That theorem says in in part that we need only to consider the random variables s_0 and s_1. Let N be the number of s_0 and s_1 that are non-zero. In this case we have 3 possibilities which occur with the following probabilities:

1. $P(N = 0) = P(s_0 = 0 \;\&\; s_1 = 0) = 0$;

2. $P(N = 1) = P(s_0 = 1 \;\&\; s_1 = 0) + P(s_0 = 0 \;\&\; s_1 = 1) = \frac{2}{3}$;

3. $P(N = 2) = P(s_0 = 1 \;\&\; s_1 = 1) = \frac{1}{3}$.

Then by theorem 2.5.3 $P(Z = \emptyset)$ is the least non-negative solution to

$$\frac{2}{3}t + \frac{1}{3}t^2 = t.$$

Clearly 0 is a solution and therefore $P(Z = \emptyset) = 0$.

Theorem 2.5.3 also states that with probability one Z has Hausdorff and box dimension d where d solves $E\left(s_0^d + s_1^d\right) = 1$ (here E is the expected value as reviewed in section 2.2). By definition

$$\begin{aligned} E\left(s_0^d + s_1^d\right) &= \frac{1}{3}\left(\frac{1}{2}\right)^d + \frac{1}{3}\left(\frac{1}{2}\right)^d + \frac{1}{3}\left(\left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d\right) \\ &= \frac{1}{3}2^{2-d}. \end{aligned}$$

We solve the equation $\frac{1}{3}2^{2-d} = 1$ for d to find that $d = 2 - \log_2(3)$. Therefore Z almost certainly has Hausdorff and box dimension $2 - \log_2(3)$.

□

4.2 Galton-Watson random closed sets

This section explores the approach to algorithmically random closed sets taken by Kjos-Hanssen and Diamondstone [14]. Our goal once again is to show that this approach is compatible with the framework developed in chapter 3. As in the previous section 4.1, the correspondence between binary-branching trees and closed sets of 2^ω gives a RACS. This time, however, the trees are coded in 2^ω rather than 3^ω and Kjos-Hanssen and Diamondstone allow for non-extensible trees.

The idea is to construct a tree by extending each node in the tree according to the outcome of a series of coin flips. This is formalized as a map $X : 2^\omega \to \mathcal{F}(2^\omega)$ defined by constructing a tree, $T(f)$, according to $f \in 2^\omega$. As in the case of the decoding map of section 4.1 the tree is constructed recursively. At stage s determines which strings of length s are members of $T(f)$. After each stage s we will have a finite binary tree $T(f)[s]$. The tree $T(f)$ will then be defined to be $\bigcup_{s\in\omega} T(f)[s]$.

We begin by setting $T(f)[0] = \{\lambda\}$. At stage 1 we determine which of the strings 0 and 1 are in $T(f)[1]$. We have $0 \in T(f)[1]$ if $f(0) = 1$ and $1 \in T(f)[1]$ if $f(1) = 1$. To reiterate, for $i \in \{0, 1\}$ the string $i \in T(f)[1] \iff f(i) = 1$.

For later stages we simply continue this process. Let $T(f)[s] \subseteq 2^{<\omega}$ be the tree after stage s of the construction. At this point $T(f)[s]$ contains $n(s)$ strings, added according to the first $n(s) - 1$ bits of f (recall that the empty string is always added). The tree $T(f)[s]$ also contains some number of strings of length s: $\sigma_0, \ldots, \sigma_k$ (ordered alphabetically). We extend each of these strings as follows:

$$\sigma_j{}^\frown i \in T(f)[(s+1)] \iff f\left(n(s) + 2j + i\right) = 1$$

for $j \in \{0, 1, \dots k\}$ and $i \in \{0, 1\}$.

Definition 4.2.1. The map $X : 2^\omega \to \mathcal{F}(2^\omega)$ is given by $X(f) = [T(f)]$.

This is very similar to the canonical coding of Barmpalias et al. [1]. In this case, however, not every tree is extensible. Classically this does not make much of a difference but it does add considerable complications from the computability perspective. Diamondstone and Kjos-Hanssen [14] have resolved the complications to prove that a version of the classical results holds effectively (their result is stated as theorem 4.2.7 below). We will prove that the map X defined above is a RACS and use this translate the results of Kjos-Hanssen and Diamondstone [14] into the context of Martin-Löf random closed sets. We also prove a result, lemma 4.2.13, that Kjos-Hanssen and Diamondstone cite in the proof of theorem 4.2.7.

Throughout the following we wish to consider a more general measure on 2^ω than the fair coin flip measure. We will generalize to Bernoulli measures, i.e. measures generated by flipping a biased coin.

Definition 4.2.2. Let $p \in (0, 1)$. The *Bernoulli measure with parameter* p is the Borel measure P on 2^ω such that:

1. $\forall \sigma \in 2^{<\omega}\ P([\sigma^\frown 1] = pP([\sigma])$ and

2. $P(2^\omega) = 1$.

Definition 4.2.3. Let $p \in (0, 1)$ and let P be the Bernoulli measure on 2^ω with parameter p. Then we say that $f \in 2^\omega$ is *p-Martin-Löf random* if it is P-Martin-Löf random.

Trees produced from $(2^\omega, P)$, where P is the Bernoulli measure with parameter p, and according to the algorithm described above are called (binary branching)

Galton-Watson trees with survival probability p. Such trees were originally studied in the 19th century (by Sir Francis Galton) in the context of the extinction of noble surnames. A basic result in the theory of these trees is the following classical proposition about the probability of the existence of a path through a Galton-Watson tree. The existence of such a path means that a surname does not become extinct. We note that this result also follows from theorem 2.5.3, though this connection will not matter at all in the following. The result that we will actually use is the easy corollary 4.2.5.

Proposition 4.2.4 (Galton and Watson)**.** *The probability that a (binary branching) Galton-Watson tree with survival probability $p \in (0,1)$ has no (infinite) paths is the least positive solution to the equation*

$$x = p^2x^2 + 2p(1-p)x + (1-p)^2. \tag{4.1}$$

Proof idea.

$$\begin{aligned} P(\text{no path}) = P(&\text{no path} \mid 0 \text{ and } 1 \text{ both survive})P(0 \text{ and } 1 \text{ both survive}) \\ &+ P(\text{no path} \mid \text{exactly one survives})P(\text{exactly one survives}) \\ &+ P(\text{neither } 0 \text{ or } 1 \text{ survives}) \end{aligned}$$

□

Corollary 4.2.5. *If T is a (binary branching) Galton-Watson tree with survival probability $p \in (0,1)$, then the probability that T has no infinite paths is 1 if and only if $p \le \frac{1}{2}$. Otherwise the probability that T has no infinite paths is*

$$\left(\frac{1-p}{p}\right)^2.$$

Proof. We first find solutions to equation 4.1.

$$\begin{aligned} x = p^2x^2 + 2p(1-p)x + (1-p)^2 &\iff p^2x^2 + (2p - 2p^2 - 1)x + (1-p)^2 = 0 \\ &\iff x^2 - \left(1 + \frac{1 - 2p + p^2}{p^2}\right)x + (1-p)^2 = 0 \\ &\iff (x-1)\left(x - \left(\frac{1-p}{p}\right)^2\right) = 0 \end{aligned}$$

Therefore 1 and $\left(\frac{1-p}{p}\right)^2$ are the solutions to this equation. We now note that

$$\left(\frac{1-p}{p}\right)^2 < 1 \iff p > \frac{1}{2}.$$

Therefore by the preceding proposition T has no infinite paths with probability 1 if and only if $p \leq \frac{1}{2}$. It also follows that the probability that T has no infinite paths is $\left(\frac{1-p}{p}\right)^2$ otherwise. □

We are now ready to define the Galton-Watson random closed sets of Kjos-Hanssen and Diamondstone [14] and to state their result.

Definition 4.2.6 (Kjos-Hanssen and Diamondstone [14])**.** Let $p \in (0, 1)$. A closed set $F \subseteq 2^\omega$ is *p-Galton-Watson random* (p-GW random) if there is a p-ML random $f \in 2^\omega$ such that $X(f) = F$.

Theorem 4.2.7 (Kjos-Hanssen and Diamondstone [14])**.** *Closed set $F \subseteq 2^\omega$ is BBCDW-random if and only if F is $\frac{2}{3}$-Galton-Watson random and $F \neq \emptyset$.*

We wish to bring this result into the context of Martin-Löf random closed sets. The first step is to apply corollary 4.1.4. This gives the following.

Corollary 4.2.8. *Let $Z : 3^\omega \to \mathcal{F}(2^\omega)$ be the canonical decoding map of section 4.1. $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Z$-ML random if and only if F is $\frac{2}{3}$-GW random and $F \neq \emptyset$.*

Proof. This is a direct consequence of theorems 4.2.7 and corollary 4.1.4 which showed that a closed set $F \subseteq 2^\omega$ is BBCDW-random if and only if F is $\mathbf{P}_Z$-ML random. □

The next step is to prove that the map X is measurable.

Proposition 4.2.9. *The map X is a RACS.*

Proof. Consider $X^{-1}\left(\mathcal{F}^{[\sigma]}\right)$. We take advantage of compactness. In this case we have

$$[T(f)] \cap [\sigma] = \emptyset \iff (\exists n \in \omega)\,(\forall \tau \in 2^n)\,(\tau \succeq \sigma \implies \tau \notin T(f))\,.$$

Thus

$$X^{-1}\left(\mathcal{F}^{[\sigma]}\right) = \{f \in 2^\omega \;:\; (\exists n \in \omega)\,(\forall \tau \in 2^n)\,(\tau \succeq \sigma \implies \tau \notin T(f))\}\,. \qquad (4.2)$$

There are $2^{n+1} - 1$ strings of length at most n. Thus, if $f \in 2^\omega$ and $n \in \omega$ are such that $(\forall \tau \in 2^n)\,[\tau \succeq \sigma \implies \tau \notin T(f)]$, then

$$\left[f \restriction 2^{n+1}\right] \subseteq X^{-1}\left(\mathcal{F}^{[\sigma]}\right).$$

Hence the set $X^{-1}\left(\mathcal{F}^{[\sigma]}\right)$ is Σ^0_1 and thus measurable. Because sets of the form $\mathcal{F}^{[\sigma]}$ generate the Borel σ-algebra on $\mathcal{F}$ this suffices to prove the proposition. □

Let $\mathbf{P}_X$ be the measure induced on $\mathcal{F}(2^\omega)$ by X. Ideally the p-Galton-Watson random closed sets would be exactly the $\mathbf{P}_X$-Martin-Löf random closed sets. Unfortunately we have only been able to show containment in one direction: every p-GW random closed set is $\mathbf{P}_X$-ML random. Before we prove this we must determine the computational power of $\mathbf{P}_X$.

Lemma 4.2.10. *Let P the Bernoulli measure with parameter $p \in \left(\frac{1}{2}, 1\right)$ on 2^ω. Then $\mathbf{P}_X \equiv_T P \equiv_T p$.*

Proof. It is clear from the definition of Bernoulli measure that $P \equiv_T p$.

Now let $c = \mathbf{P}_X\left(\mathcal{F}^{2^\omega}\right)$. Then c is the measure of a basic open set of $\mathcal{F}(2^\omega)$ and hence computable from $\mathbf{P}_X$. But $\mathcal{F}^{2^\omega} = \emptyset$ and so by corollary 4.2.5 $c = \left(\frac{1-p}{p}\right)^2$. Hence $1 - 2p + (1-c)p^2 = 0$. Thus p can be computed from c using the quadratic equation. Therefore $p \leq_T \mathbf{P}_X$.

Now we wish to show that $\mathbf{P}_X \leq_T p$. We claim that it suffices to show that using p as an oracle we can compute $\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}\right)$ for any $\sigma_1, \ldots, \sigma_n \in 2^{<\omega}$. To prove this claim we appeal to lemma 3.0.11 which gives sufficient conditions for computing $\mathbf{P}_T$ from the capacity T_X. The catch is that lemma 3.0.11 only applies when the basis for 2^ω is closed under finite unions. This means that we must pass to the basis on 2^ω of all clopen sets, that is, the closure of the standard basis under finite unions. Let c_0 be the code for $\mathbf{P}_X$ with respect to the standard basis of cylinders. Let c_1 be the code for $\mathbf{P}_X$ with respect to the basis of clopen sets. If we code the clopen sets in a reasonable way it is clear that $c_0 \leq_T c_1$ since every cylinder is clopen.

By lemma 3.0.11 $c_1 \leq_T T_X$, where T_X is coded with respect to the basis of clopen sets. Coding T_X with respect to the basis of clopen sets by definition means that for each $\sigma_1, \ldots, \sigma_n \in 2^{<\omega}$ we know $\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]\cup\cdots\cup[\sigma_n]}\right)$. Then by inclusion-exclusion

$$\begin{aligned}\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]\cup\dots\cup[\sigma_n]}\right) &= \mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]}\cup\dots\cup\mathcal{F}_{[\sigma_n]}\right)\\ &= \sum_{1\le i\le n}\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]}\right)\\ &\quad - \sum_{1\le i<j\le n}\mathbf{P}_X\left(\mathcal{F}_{[\sigma_i],[\sigma_j]}\right)+\dots\\ &\quad + (-1)^{n-1}\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],\dots,[\sigma_n]}\right).\end{aligned}$$

Therefore $\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1]\cup\dots\cup[\sigma_n]}\right)$ can be computed if we know $\mathbf{P}_X\left(\mathcal{F}_{[\sigma'_1],\dots,[\sigma'_k]}\right)$ for every $\sigma'_1,\dots,\sigma'_k \in 2^{<\omega}$. This proves the claim since Turing reducibility is transitive.

We now find a formula for calculating $\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],\dots,[\sigma_n]}\right)$. If there are $i \neq j \in \{1,\dots,n\}$ such that $[\sigma_i]\cap[\sigma_j] \neq \emptyset$, then we know that $[\sigma_i]\cap[\sigma_j] = [\sigma_i]$ (by swapping indices if necessary). Consequently $\mathcal{F}_{[\sigma_1],\dots,[\sigma_n]} = \mathcal{F}_{[\sigma_1],\dots,[\sigma_{j-1}],[\sigma_{j+1}],\dots,[\sigma_n]}$ and so we can drop $[\sigma_j]$ from the expression. We can thus assume, without loss of generality, that $[\sigma_i]\cap[\sigma_j] = \emptyset$ for every $i \neq j \in \{1,\dots,n\}$.

Let $S = \{\tau \in 2^{<\omega} : \exists i \in \{1,\dots,n\}\ \tau \preceq \sigma_i\}$, i.e. the finite tree of all predecessors of $\sigma_1,\dots,\sigma_n$. Let $N = |S|-1$. We subtract 1 because by construction all trees include the empty string λ, and so we do not use any information from $f \in 2^\omega$ to determine if λ is in $T(f)$. In particular, if $S \subseteq T(f)$, then the N bits of f coding for the non-empty strings of S must be 1. Hence $P\left(\{f \in 2^\omega : S \subseteq T(f)\}\right) = p^N$.

Our construction is self-similar in the sense that if we know $\sigma \in 2^{<\omega}$ is in our tree, then the probability that the sub-tree of extensions of σ has an infinite path is exactly $1-\left(\frac{1-p}{p}\right)^2$. In addition, the sub-trees of extensions of $\sigma_1,\dots,\sigma_n$ are (statistically) independent. Hence

$$\begin{aligned}\mathbf{P}_X\left(\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}\right) &= P\left(X^{-1}\left(\mathcal{F}_{[\sigma_1],\ldots,[\sigma_n]}\right)\right)\\ &= P\left(\{f \in 2^\omega : S \subseteq T(f) \;\&\; (\forall i \in \{1,\ldots,n\})[T(f)] \cap [\sigma_i] \neq \emptyset\}\right)\\ &= p^N\left(1-\left(\frac{1-p}{p}\right)^2\right)^n.\end{aligned} \tag{4.3}$$

Therefore $\mathbf{P}_X \leq_T p$. □

We are now ready to prove that the $\mathbf{P}_X$-ML random closed sets include the p-GW random closed sets. We actually prove the equivalent (by definition) result that if $f \in 2^\omega$ is p-ML random then $X(f)$ is $\mathbf{P}_X$-ML random.

Lemma 4.2.11. *Let $p \in (0,1)$. If $f \in 2^\omega$ is p-Martin-Löf random, then $X(f)$ is $\mathbf{P}_X$-Martin-Löf random. (That is, if $F \in \mathcal{F}(2^\omega)$ is p-GW random, then F is $\mathbf{P}_X$-ML random).*

Proof. We have two cases to consider: $p \leq \frac{1}{2}$ and $p > \frac{1}{2}$.

Case 1: $p \leq \frac{1}{2}$. In this case $\mathbf{P}_X(\{\emptyset\}) = 1$ by corollary 4.2.5. This means that $\mathbf{P}_X\left(\mathcal{F}_{2^\omega}\right) = 0$. Every non-empty closed subset of 2^ω is in $\mathcal{F}_{2^\omega}$ and therefore $\emptyset$ is the only $\mathbf{P}_X$-ML random closed set. Thus we need to prove that if $f \in 2^\omega$ is p-ML random, then $X(f) = \emptyset$.

Applying equation 4.2 to $\{\emptyset\} = \mathcal{F}^{2^\omega}$ gives

$$X^{-1}\left(\{\emptyset\}\right) = \{f \in 2^\omega : (\exists n \in \omega)(\forall \tau \in 2^n)[\tau \notin T(f)]\}$$

and shows that this set is Σ^0_1. We also know that $P\left(X^{-1}\left(\{\emptyset\}\right)\right) = 1$. A Σ^0_1 set of measure 1 must contain every p-ML random $f \in 2^\omega$. Therefore if $f \in 2^\omega$ is p-ML random, then $X(f) = \emptyset$. This finishes case 1.

Case 2: $p > \frac{1}{2}$. In this case $0 < \mathbf{P}_X(\{\emptyset\}) = \left(\frac{1-p}{p}\right)^2 < 1$ by corollary 4.2.5. We prove the contrapositive: if $F \in \mathcal{F}(2^\omega)$ is not $\mathbf{P}_X$-ML random and $X(f) = F$, then f is not p-ML random. We would like to apply lemma 3.0.12 but this is not possible: $X^{-1}(\mathcal{F}_{[\sigma]})$ is Π^0_1 and not Σ^0_1 as required in the hypotheses of the lemma. The technique of this proof, however, is similar to the technique of the proof of lemma 3.0.12. In particular, the goal is to take a $\mathbf{P}_X$-ML test in $\mathcal{F}(2^\omega)$ and pull it pack via X to a p-ML test in 2^ω.

In order to proceed we approximate the Π^0_1 sets $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ by clopen sets:

$$A_{\sigma,s} = \left\{f \in 2^\omega : \left(\exists \tau \in 2^{|\sigma|+s}\right)[\tau \succeq \sigma \ \& \ \tau \in T(f)]\right\}.$$

That $A_{\sigma,s}$ is a clopen set for each $\sigma \in 2^{<\omega}$ and each $s \in \omega$ is clear from the definition. We can also see that $A_{\sigma,0} \supseteq A_{\sigma,1} \supseteq \dots$ for each $\sigma \in 2^{<\omega}$. Furthermore

$$\bigcap_{s \in \omega} A_{\sigma,s} = X^{-1}\left(\mathcal{F}_{[\sigma]}\right).$$

To prove this, suppose that $f \in \bigcap_{s\in\omega} A_{\sigma,s}$. Then

$$(\forall s \in \omega)\left(\exists \tau \in 2^{|\sigma|+s}\right)(\tau \succeq \sigma \ \& \ \tau \in T(f)).$$

Thus $[\sigma] \cap [T(f)] \neq \emptyset$. By definition $X(f) = [T(f)]$ and so $X(f) \in \mathcal{F}_{[\sigma]}$.

Now we wish to calculate $P\left(X^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right)$. We can apply equation 4.3 from the proof of lemma 4.2.10. This gives

$$P\left(X^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right) = p^{|\sigma|}\left(1 - \left(\frac{1-p}{p}\right)^2\right).$$

We will use this to calculate exactly how close an approximating set $A_{\sigma,s}$ is to $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$. Because $A_{\sigma,s}$ is clopen for every $s \in \omega$ we can calculate $P\left(A_{\sigma,s}\right)$. Then the exact error of the approximation is

$$P\left(A_{\sigma,s}\right) - p^{|\sigma|}\left(1 - \left(\frac{1-p}{p}\right)^2\right).$$

Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a $\mathbf{P}_X$-ML test. We produce a p-ML test $\{V_i\}_{i\in\omega}$. To define V_k we consider $\mathcal{U}_{k+1}$. If $\mathcal{F}^C_{[\sigma_1],\ldots,[\sigma_n]}$ is the s^{th} basic open set enumerated into $\mathcal{U}_{k+1}$ then we find $t \in \omega$ large enough so that

$$\sum_{j=1}^{n}\left[P\left(A_{\sigma_j,t}\right) - \left(1 - p^{|\sigma_j|}\left(\frac{1-p}{p}\right)^2\right)\right] \leq 2^{-(k+1+s)}. \tag{4.4}$$

This ensures that the total error of approximating $X^{-1}\left(\mathcal{F}^C_{[\sigma_1],\ldots,[\sigma_n]}\right)$ by the Σ^0_1 set $X^{-1}\left(\mathcal{F}^C\right) \cap A_{\sigma_1,t} \cap \cdots \cap A_{\sigma_n,t}$ is small.

We then add the Σ^0_1 set $X^{-1}\left(\mathcal{F}^C\right) \cap A_{\sigma_1,t} \cap \cdots \cap A_{\sigma_n,t}$ to V_k. By equation 4.4

$$P(V_k) \leq \mathbf{P}_X(\mathcal{U}_{k+1}) + \sum_{s=1}^{\infty} 2^{-(k+1+s)} \leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

Furthermore, $\{V_i\}_{i\in\omega}$ is a uniformly $\Sigma^{0,\mathbf{P}_X}_1$ sequence. By lemma 4.2.10 $\mathbf{P}_X \equiv_T P$ and thus $\{V_i\}_{i\in\omega}$ is a uniformly $\Sigma^{0,P}_1$ sequence. Therefore $\{V_i\}_{i\in\omega}$ is a P-ML test.

Suppose that $F \in \bigcap_{i\in\omega} \mathcal{U}_i$. Then $X^{-1}(\{F\}) \subseteq \bigcap_{i\in\omega} V_i$ by our definition of V_i. This holds for every $\mathbf{P}_X$-ML test $\{\mathcal{U}_i\}_{i\in\omega}$. Therefore if $F \in \mathcal{F}(2^\omega)$ is not $\mathbf{P}_X$-ML random and $X(f) = F$, then f is not p-ML random. This completes case 2 and the proof. □

Because $\emptyset$ is an atom of the measure $\mathbf{P}_X$ it must be $\mathbf{P}_X$-ML random. Note that $X^{-1}(\{\emptyset\})$ is Σ^0_1 and has positive measure and so must contain both random and non-random elements of 2^ω. This is worth emphasizing: in this case non-p-ML random elements of 2^ω map to a $\mathbf{P}_X$-ML random element of $\mathcal{F}(2^\omega)$. Of course, this $\mathbf{P}_X$-ML random closed set is $\emptyset$, which is unusual in that it is isolated in $\mathcal{F}(2^\omega)$ and and atom of the measure $\mathbf{P}_X$. The question of whether other $\mathbf{P}_X$-ML random closed sets are also the image of non-p-ML random elements of 2^ω remains open.

Kjos-Hanssen and Diamondstone [14] cite the following lemma 4.2.13 in their proof of theorem 4.2.7. This lemma connects the map X and the canonical decoding $Z : 3^\omega \to \mathcal{F}(2^\omega)$ of section 4.1. As in section 4.1, let $\mathbf{P}_Z$ be the measure induced on $\mathcal{F}(2^\omega)$ by Z (and the "fair coin" measure on 3^ω). We establish this connection by considering the space $(\mathcal{F}(2^\omega), \mathbf{P}_Z)$ and a measure, ν_Z, on 2^ω such that $\mathbf{P}_Z(\mathcal{H}) = \nu_Z\left(X^{-1}(\mathcal{H})\right)$ for measurable $\mathcal{H} \subseteq \mathcal{F}(2^\omega)$.

Definition 4.2.12 (Kjos-Hanssen and Diamondstone [14])**.** Define a Borel probability measure ν_Z on 2^ω as follows:

1. $\nu_Z\left(2^\omega\right) = 1$

2. If σ has even length, then

$$\nu_Z\left([\sigma^\frown 01]\right) = \nu_Z\left([\sigma^\frown 10]\right) = \nu_Z\left([\sigma^\frown 11]\right) = \frac{1}{3}\nu_Z\left([\sigma]\right)$$

   and

$$\nu_Z\left([\sigma^\frown 00]\right) = 0.$$

Note that this defines ν_Z on the entire Borel σ-algebra since the measure of the cylinders determined by strings with odd length is implicitly set. For example, $\nu_Z([0]) = \nu_Z([00]) + \nu_Z([01]) = \frac{1}{3}$ and $\nu_Z([1]) = \nu_Z([10]) + \nu_Z([11]) = \frac{2}{3}$.

Lemma 4.2.13. *The following are equivalent:*

1. *A closed set* $F \in \mathcal{F}(2^\omega)$ *is* $\mathbf{P}_Z$*-Martin-Löf random;*

2. *There is some* ν_Z*-Martin-Löf random* $f \in 2^\omega$ *such that* $X(f) = F$*;*

3. *Every* $f \in 2^\omega$ *such that* $X(f) = F$ *is* ν_Z*-Martin-Löf random.*

Proof. We first note that $\mathbf{P}_Z \equiv_T \nu_Z \equiv_T \emptyset$. We proved that $\mathbf{P}_Z \equiv_T \emptyset$ in theorem 4.1.4. That $\nu_Z \equiv_T \emptyset$ is clear from the definition of ν_Z.

We wish to show that the ν_Z-ML random reals correspond via the map X with the $\mathbf{P}_Z$-ML random closed sets. First we show that if $X(f) \in \mathcal{F}(2^\omega)$ is not $\mathbf{P}_Z$-ML random, then f is not ν_Z-ML random.

As in the proof of lemma 4.2.11 we must work around the fact that $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ is Π^0_1. As before we approximate this set by clopen sets. This time, however, the approximation is much easier. Define

$$A_\sigma := \{f \in 2^\omega : \sigma \in T(f)\}.$$

Then $A_\sigma \supseteq X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ and A_σ is clopen. Moreover

$$\nu_Z(A_\sigma) = \nu_Z\left(X^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right). \tag{4.5}$$

These measures are equal because under the measure ν_Z almost every tree is extensible, i.e. $\nu_Z(\{f \in 2^\omega : T(f) \text{ is extensible}\}) = 1$.

Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a $\mathbf{P}_Z$-ML test in $\mathcal{F}(2^\omega)$. Because $\mathbf{P}_Z$ and ν_Z are computable we do not need to worry about the oracles $\mathbf{P}_Z$ or ν_Z. We construct a ν_Z-ML test $\{V_i\}_{i\in\omega}$ as follows. If $\mathcal{F}^C_{[\sigma_1],\ldots,[\sigma_n]}$ is enumerated into $\mathcal{U}_i$, then we enumerate

$$A_{\sigma_1} \cap \cdots \cap A_{\sigma_n} \cap X^{-1}\left(\mathcal{F}^C\right)$$

into V_i. It is clear that $\{V_i\}_{i\in\omega}$ is uniformly Σ^0_1. By equation 4.5

$$\begin{aligned}\nu_Z(V_i) &= \nu_Z\left(X^{-1}(\mathcal{U}_i)\right)\\ &= \mathbf{P}_Z(\mathcal{U}_i)\\ &\le 2^{-i}.\end{aligned}$$

Therefore $\{V_i\}_{i\in\omega}$ is a ν_Z-ML test.

By construction, if $F \in \bigcap_{i\in\omega}\mathcal{U}_i$ and $X(f) = F$, then $f \in \bigcap_{i\in\omega} V_i$. This holds for every $\mathbf{P}_Z$-ML test $\{\mathcal{U}_i\}_{i\in\omega}$. Therefore, if $X(f)$ is not $\mathbf{P}_Z$-ML random, then f is not ν_Z-ML random.

Now we prove the converse: if $f \in 2^\omega$ is not ν_Z-ML random, then $X(f)$ is not $\mathbf{P}_Z$-ML random.

Each $\sigma \in 2^{<\omega}$ determines a finite binary tree $T(\sigma)$ as described in definition 4.2.1. This tree has a finite number of nodes whose extensions have not yet been completely determined. Let these nodes be $\tau_1, \ldots, \tau_n$. Define $C_\sigma = ([\tau_1] \cup \ldots [\tau_n])^{\complement}$. The set C_σ is the largest set such that for any $f \succ \sigma$, $X(f) \cap C_\sigma = \emptyset$. This means that $X([\sigma]) \subseteq \mathcal{F}^{C_\sigma}$. C_σ is also clopen and so $\mathcal{F}^{C_\sigma}$ is a basic open set of $\mathcal{F}(2^\omega)$.

Let $f \in 2^\omega$ and let $\{U_i\}_{i\in\omega}$ be a ν_Z-ML test such that $f \in \bigcap_{i\in\omega} U_i$. Construct a $\mathbf{P}_Z$-ML test $\{\mathcal{V}_i\}_{i\in\omega}$ as follows. If $[\sigma]$ is enumerated into U_i, then $\mathcal{F}^{C_\sigma}$ is enumerated into $\mathcal{V}_i$. Clearly $\{\mathcal{V}_i\}_{i\in\omega}$ is uniformly Σ^0_1. Additionally, $X^{-1}(\mathcal{V}_i) = U_i \cup E$ where $E \subseteq \{f \in 2^\omega : T(f) \text{ is not extensible}\}$. Non-extensible trees occur with probability 0 under ν_Z and hence $\nu_Z(E) = 0$. Consequently $\mathbf{P}_Z(\mathcal{V}_i) = \nu_Z(U_i \cup E) = \nu_Z(U_i) \le 2^{-i}$. Therefore $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}_Z$-ML test.

Now because $f \in \bigcap_{i\in\omega} U_i$ if follows that $X(f) \in \bigcap_{i\in\omega} \mathcal{V}_i$. Therefore if $f \in 2^\omega$ is not ν_Z-ML random, then $X(f)$ is not a $\mathbf{P}_Z$-ML random closed set.

Thus far we have shown that if $X(f) = F$, then F is $\mathbf{P}_Z$-ML random if and only if f is ν_Z-ML random. That is, statements 1 and 3 of the lemma are equivalent. Finally, X is a surjection and hence we conclude that F is a $\mathbf{P}_Z$-ML random closed set if and only if there is a ν_Z-ML random $f \in 2^\omega$ such that $X(f) = F$.

□

4.3 Generalized random fractal constructions

We follow Mauldin and McLinden's [4] generalization of random fractal constructions in 2^ω. Mauldin and McLinden have proved a number of results about the Hausdorff dimension and Hausdorff measure of such constructions.

Let $\mathcal{T}$ be the set of all finite binary trees (so $\mathcal{T}$ is a countable subset of the power set of $2^{<\omega}$). For convenience we require that every tree contain at least the empty string λ. Let P be a probability measure on $\mathcal{T}$. One can then construct an (potentially infinite) binary tree randomly according to P as follows. Let $T[0]$ be a tree randomly chosen from $\mathcal{T}$. Given $T[s]$ we extend the tree to $T[s+1]$ by affixing a finite tree chosen randomly to each live terminal node of $T[s]$. We say that terminal node is dead if we have extended it by the tree $\{\lambda\}$ (where λ is the empty string). The non-dead terminal nodes are live. Then $T = \bigcup_{s \in \omega} T[s]$ is binary tree which may have infinite paths. Binary trees correspond to closed subsets of 2^ω (as in proposition 2.3.8) and so we can think of $[T]$ as a randomly generated closed set. In other words, we are using P to produce a RACS.

This RACS can be formalized as a map $X : \mathcal{T}^\omega \to \mathcal{F}(2^\omega)$. This map is defined in almost exactly the same way as the RACS in section 4.2. We begin with $f \in \mathcal{T}^\omega$ and construct $T(f) = \bigcup_{s \in \omega} T(f)[s]$ in stages. Define $T(f)[0] = f(0)$ (recall that $f(0)$ is a finite binary tree). Then $T(f)[0]$ has finitely many live terminal nodes $\tau_1, \ldots, \tau_n$. Extend to $T(f)[1]$ by adding $\tau_i^\frown f(i)$ to $T(f)[0]$ for each $i \in \{1, \ldots, n\}$ (where $\tau_i^\frown f(i)$ is defined to be the set $\{\tau_i^\frown \sigma : \sigma \in f(i)\}$). Iterating this process gives $T(f)[s]$ for every $s \in \omega$. We then define $T(f) = \bigcup_{s\in\omega} T(f)[s]$ and $X(f) = [T(f)]$.

We note that every live terminal node in $T(f)[s]$ is a string (i.e. an element of $2^{<\omega}$) of length at least $s+1$. At stage 0 we have $T(f)[0] = f(0)$ and any live terminal nodes of $f(0)$ have length at least 1. Then at each stage $s \geq 1$ each live terminal node is either extended by one or more strings of length at least 1, or extended by only the empty string λ. Extension by λ means that the node is no longer alive and thus will not be extended at any future stage. This means that for each $\sigma \in 2^{<\omega}$ other than the empty string,

$$\sigma \in T(f) \iff \sigma \in T(f)\left[|\sigma| - 1\right].$$

We now prove that X is a RACS. To do this we must first prove that $\mathcal{T}^\omega$ is a probability space. We actually prove the stronger result that $\mathcal{T}^\omega$ is a Martin-Löf probability space. We recall that Martin-Löf probability spaces were defined in 3.0.5.

Lemma 4.3.1. *The space $\mathcal{T}^\omega$ is a Martin-Löf probability space.*

Proof. First we note that topologically $\mathcal{T}^\omega$ is a countable product of countable discrete spaces. In particular it has a basis consisting of sets of the form

$$[\sigma] = \{f \in \mathcal{T}^\omega : \sigma \prec f\}$$

for $n \in \omega$ and $\sigma \in \mathcal{T}^{<\omega}$. This basis is certainly countable. By adding $\emptyset$ we get a basis closed under finite intersection. We note that for $\sigma_1 \not\preceq \sigma_2 \in \mathcal{T}^{<\omega}$

$$[\sigma_1] \cap [\sigma_2] = \begin{cases} [\sigma_1] & \text{if } \sigma_2 \prec \sigma_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

The set $\mathcal{T}^{<\omega}$ can be enumerated as $\sigma_1, \sigma_2, \ldots$ in such a way that determining if σ_i is a predecessor of σ_j is computable. In fact we can ensure that determining which tree is represented by $\sigma_i(j)$ is uniformly computable over $i, j \in \omega$. We will use the fact in our analysis, though it is not important in proving this result. Now let $B_0 = \emptyset$ and $B_i = [\sigma_i]$ for $i \geq 1$. Clearly $B_0, B_1, \ldots$ is a basis for $\mathcal{T}^\omega$ such that the intersection function $\mathfrak{g} : \omega^2 \to \omega \iff B_i \cap B_j = B_{\mathfrak{g}(i,j)}$ is computable.

The set $\mathcal{T}$ is equipped with a probability measure P. Hence $\mathcal{T}^\omega$ is a probability space under the infinite product measure, P^ω defined by

$$P^\omega([\sigma]) = \prod_{i<|\sigma|} P(\sigma(i))$$

for $\sigma \in \mathcal{T}^{<\omega}$ Therefore P^ω is a Martin-Löf probability space. □

Proposition 4.3.2. *The map $X : \mathcal{T}^\omega \to \mathcal{F}(2^\omega)$ is a RACS.*

Proof. To prove that X is measurable it suffices to prove that $X^{-1}\left(\mathcal{F}^{[\sigma]}\right)$ is measurable for each $\sigma \in 2^{<\omega}$. By the compactness of 2^ω, $[T(f)] \cap [\sigma] = \emptyset$ if and only if there is $n \in \omega$ such that if $\tau \in 2^n$ and $\tau \succeq \sigma$, then $\tau \notin T(f)$. But if $\tau \in 2^n$, then the membership of τ in $T(f)$ is determined by the membership of τ in $T(f)[n]$. Because $\mathcal{T}$ consists only of finite trees we know that $T(f)[n]$ is finite for every $n \in \omega$. Hence deciding membership in $T(f)[n]$ takes only finitely much information from f. Thus if $f \in X^{-1}\left(\mathcal{F}^{[\sigma]}\right)$, then there exists $m \in \omega$ such that

$$[f \upharpoonright m] \subseteq X^{-1}\left(\mathcal{F}^{[\sigma]}\right).$$

This means that $X^{-1}\left(\mathcal{F}^{[\sigma]}\right)$ is open. Therefore X is a RACS. □

Mauldin and McLinden [4] broke their analysis of these random fractals into two cases: the case when the distribution on $\mathcal{T}$ is such that the tree $\{\lambda\}$ occurs with positive probability and the case in which $\{\lambda\}$ occurs with probability 0. The distinction between these two cases is whether or not we must worry about the possibility of non-extensible trees. Whether or not $\{\lambda\}$ occurs with probability 0 makes a difference for our analysis as well.

The more important distinction for us, however, is whether or not the measure P^ω is carried on a compact subset of $\mathcal{T}^\omega$. When P^ω is carried on a compact subset we find that the situation becomes much more tractable. This is because closed subsets of compact spaces can be approximated by finite unions of basic open sets. This kind of approximation has been key to our work so far (cf. lemmas 4.2.11 and 4.2.13).

The easiest case for us is when $\{\lambda\}$ occurs with probability 0 and P^ω is carried on a compact subset of $\mathcal{T}^\omega$. This happens if and only if the distribution P on $\mathcal{T}$ is carried on a finite set of trees $\{\Gamma_1, \ldots, \Gamma_n\}$ and $\Gamma_i \neq \{\lambda\}$ for each $i \in \{1, \ldots, n\}$. In this case the situation is very much analogous to that in lemma 4.2.13. In that example we proved that X mapped the Martin-Löf random elements of its domain to Martin-Löf random elements of its range. We now prove the corresponding result for this example.

Proposition 4.3.3. *Suppose that P is carried on the finite set $\{\Gamma_1, \ldots, \Gamma_n\} \subseteq \mathcal{T}$*

and $\Gamma_i \neq \{\lambda\}$ *for each* $i \in \{1, \ldots, n\}$*. If* $f \in \mathcal{T}^\omega$ *is* P^ω*-Martin-Löf random, then* $X(f)$ *is a* $\mathbf{P}_X$*-Martin-Löf random closed set.*

Proof. As usual in this kind of situation we will prove the contrapositive. That is, if $F \in \mathcal{F}(2^\omega)$ is not $\mathbf{P}_X$-ML random and $X(f) = F$, then f is not P^ω-ML random. Our proof is very similar to the proof of lemma 4.2.13. As in that case $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ is Π^0_1 and so we cannot apply lemma 3.0.12 directly. Instead we approximate these sets by Σ^0_1 sets with the same measure.

First we note that the measure P^ω is carried on $\{\Gamma_1, \ldots, \Gamma_n\}^\omega$. Moreover, $P^\omega \backslash \{\Gamma_1, \ldots, \Gamma_n\}^\omega$ is an Σ^0_1 set. We also note that for any $\tau \in \mathcal{T}^{<\omega}$ the complement of $[\tau]$ can be expressed as

$$[\tau]^{\complement} = \bigcup \left\{[\sigma] : \sigma \in \mathcal{T}^{|\tau|} \ \& \ \sigma \neq \tau\right\}.$$

Thus $[\tau]^{\complement}$ open and therefore $[\tau]$ is closed. Of course $[\tau]$ is a basic open set for $\mathcal{T}^\omega$ and thus $[\tau]$ is clopen.

Now let $t \in \omega$ be such that for every $i \in \{1, \ldots, n\}$, Γ_i has fewer than t terminal nodes. Let $\sigma \in 2^{<\omega}$. Then if $f \in X^{-1}\left(\mathcal{F}^{[\sigma]}\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega$ we claim that

$$\left[f \restriction t^{|\sigma|+1}\right] \subseteq X^{-1}\left(\mathcal{F}^{[\sigma]}\right).$$

To prove this claim we rely on the fact that $\Gamma_i \neq \{\lambda\}$ for each $i \in \{1, \ldots, n\}$. Consequently, if $f \in \{\Gamma_1, \ldots, \Gamma_n\}^\omega$, then $T(f)$ is extensible. This means that if $f \in X^{-1}\left(\mathcal{F}^{[\sigma]}\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega$, then $\sigma \notin T(f)$.

Now recall that the membership of $\sigma \in 2^{<\omega}$ with length greater than 0 in $T(f)$ is determined by its membership in $T(f)[|\sigma| - 1]$. But if $f \in \{\Gamma_1, \ldots, \Gamma_n\}^\omega$, then constructing $T(f)[|\sigma| - 1]$ uses no more than $t^{|\sigma|+1}$ bits of f. This can be proved inductively by proving that for $f \in \{\Gamma_1, \ldots, \Gamma_n\}^\omega$, $T(f)[s]$ has at most t^{s+1} terminal nodes for each $s \in \omega$. We leave this to the reader. We recall that if $f \in X^{-1}\left(\mathcal{F}^{[\sigma]}\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega$, then $\sigma \notin T(f)$. Hence if $g \in \left[f \upharpoonright t^{|\sigma|+1}\right]$, then $\sigma \notin T(g)$. Therefore

$$\left[f \upharpoonright t^{|\sigma|+1}\right] \subseteq X^{-1}\left(\mathcal{F}^{[\sigma]}\right).$$

This proves the claim.

By the claim there are $\tau_1, \ldots, \tau_k \in \{\Gamma_1, \ldots, \Gamma_n\}^{|\sigma|+1}$ such that

$$X^{-1}\left(\mathcal{F}^{[\sigma]}\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega = ([\tau_1] \cup \cdots \cup [\tau_k]) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega.$$

By taking complements we see that there are $\tau_1', \ldots, \tau_l' \in \{\Gamma_1, \ldots, \Gamma_n\}^{|\sigma|+1}$ such that

$$X^{-1}\left(\mathcal{F}_{[\sigma]}\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega = ([\tau_1'] \cup \cdots \cup [\tau_l']) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega.$$

Furthermore, both $\tau_1, \ldots, \tau_k$, and $\tau_1', \ldots, \tau_k'$ can be computed uniformly from $\sigma \in 2^{<\omega}$. Consequently $X^{-1}\left(\mathcal{F}^{[\sigma]}\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega$ and $X^{-1}\left(\mathcal{F}_{[\sigma]}\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega$ are both clopen sets that can be computed uniformly from $\sigma \in 2^{<\omega}$.

The sets $\mathcal{F}^{[\sigma]}$ and $\mathcal{F}_{[\sigma]}$ for $\sigma \in 2^{<\omega}$ form a sub-basis for $\mathcal{F}(2^\omega)$ and thus for $\mathcal{B} \subseteq \mathcal{F}(2^\omega)$ a basic open set, $X^{-1}(\mathcal{B}) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega$ is clopen set and can be computed uniformly. Recall that $\mathbf{P}^\omega$ is carried on $\{\Gamma_1, \ldots, \Gamma_n\}^\omega$ and that $\mathbf{P}_X(\mathcal{B}) = P^\omega(X^{-1}(\mathcal{B}))$ by definition. Therefore

$$\mathbf{P}_X(\mathcal{B}) = P^\omega\left(X^{-1}(\mathcal{B}) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega\right).$$

The right hand side of this equation is uniformly computable from P^ω because $X^{-1}(\mathcal{B}) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega$ is clopen and uniformly computable. Therefore $\mathbf{P}_X \leq_T$

P^ω.

Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a $\mathbf{P}_X$-ML test. Define

$$V_i = \left(X^{-1}\left(\mathcal{U}_i\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega\right) \cup \left(\mathcal{T}^\omega \setminus \{\Gamma_1, \ldots, \Gamma_n\}^\omega\right).$$

For each $i \in \omega$ V_i is the union of a uniformly computable clopen set and a Σ_1^0 set that does not depend on i. Hence $\{V_i\}_{i\in\omega}$ is uniformly $\Sigma_1^{0,\mathbf{P}_X}$. As we have seen $\mathbf{P}_X \leq_T P^\omega$ and hence $\{V_i\}_{i\in\omega}$ is uniformly Σ_1^{0,P^ω}. Furthermore, P^ω is carried on $\{\Gamma_1, \ldots, \Gamma_n\}^\omega$ and thus

$$\begin{aligned} P^\omega\left(V_i\right) &= P^\omega\left(X^{-1}\left(\mathcal{U}_i\right) \cap \{\Gamma_1, \ldots, \Gamma_n\}^\omega\right) \\ &= P^\omega\left(X^{-1}\left(\mathcal{U}_i\right)\right) \\ &= \mathbf{P}_X\left(\mathcal{U}_i\right) \\ &\leq 2^{-i}. \end{aligned}$$

Therefore $\{V_i\}_{i\in\omega}$ is a P^ω-ML test.

Now suppose that $F \in \mathcal{F}(2^\omega)$ is such that $F \in \bigcap_{i\in\omega}\mathcal{U}_i$. Then $X^{-1}(\{F\}) \subseteq \bigcap_{i\in\omega} V_i$. Therefore if $X(f) = F$, then f is not P^ω-ML random. This holds for every $\mathbf{P}_T$-ML test $\{\mathcal{U}_i\}_{i\in\omega}$ and so we have completed the proof. □

When P is carried on a finite set of trees that includes $\{\lambda\}$ the situation is analogous to that in theorem 4.2.11. The measure P^ω is still carried on a compact subset of $\mathcal{T}^\omega$ in this case. As in theorem 4.2.11 it should be possible to prove that X maps P^ω-ML random sequences from $\mathcal{T}^\omega$ to $\mathbf{P}_X$-ML random closed sets. This proof will require a different approximation of the Π_1^0 sets $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ than that given in the preceding theorem. The details remain to be worked out.

The final case is when the distribution P is such that infinitely many trees occur with positive probability. This means that P^ω is not carried on a compact subset of $\mathcal{T}^\omega$. This case seems to be much more complicated and little is known.

4.4 Random intervals

This section deals with another canonical example from the probability theory of random closed sets: random intervals. Usually the context is $\mathcal{F}([0,1])$ in which case we define a RACS by $x \mapsto [0,x]$. Note that in this case the capacity of the RACS is $K \mapsto 1 - \inf(K)$. In the case of the RACS $x \mapsto [x,1]$ the capacity is $K \mapsto \sup(K)$. Generalizations of these kinds of examples are examined in section 4.5.

For the time being, however, we wish to illustrate the range of possible measures on $\mathcal{F}(2^\omega)$ and so it is more instructive to examine the analogous situation for this space. In sections 4.1 and 4.2 we have already seen examples of RACS in this context. In both of those examples the RACS came from some sort of coding of a tree as an element of a Cantor space. In this example we have a very different RACS coming from the lexicographical ordering on 2^ω that has little to do with trees.

Definition 4.4.1. For $f, g \in 2^\omega$ with $f \neq g$ let $n \in \omega$ be the least number such that $f(n) \neq g(n)$. Then we define $f < g$ if $f(n) = 0$.

For $\sigma, \tau \in 2^{<\omega}$ we will say that $\sigma < \tau$ if $\sigma \prec \tau$ or if $\sigma(n) = 0$ where n is the least number such that $\sigma(n) \neq \tau(n)$.

Definition 4.4.2. A map $X : 2^\omega \to \mathcal{F}(2^\omega)$ is then defined by $X(f) = \{g \in 2^\omega : g \leq f\}$.

We recall that our basis for $\mathcal{F}(2^\omega)$ consists of sets of the form

$$\mathcal{F}^{[\sigma_{n+1}]\cup\cdots\cup[\sigma_k]}_{[\sigma_1],\ldots,[\sigma_n]}$$

for $\sigma_1, \ldots, \sigma_k \in 2^{<\omega}$.

Proposition 4.4.3. *The map X is a RACS.*

Proof. To prove that the map X is measurable we consider $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ for $\sigma \in 2^{<\omega}$:

$$\begin{aligned} X^{-1}\left(\mathcal{F}_{[\sigma]}\right) &= \left\{f \in 2^\omega : f \geq \sigma^\frown 0^\omega\right\} \\ &= \bigcup \left\{[\tau] : \tau \in 2^{|\sigma|} \;\&\; \sigma \leq \tau\right\} \end{aligned}.$$

Therefore $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ is clopen (and hence measurable). □

This, it turns out, allows us to apply lemma 3.0.12. We will also be able to apply lemma 3.0.13. A little extra work then gives the following result. As usual we let $\mathbf{P}_X$ be the measure on $\mathcal{F}(2^\omega)$ induced by X.

Theorem 4.4.4. *Closed set $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_X$-ML random if and only if $F \in X(2^\omega)$ and if $X(f) = F$, then f is ML random.*

Proof. ($\Leftarrow$) We first prove that $\mathbf{P}_X$ is computable. Let T_X be the capacity functional of the RACS X. Then by definition

$$\begin{aligned} T_X([\sigma]) &= \mathbf{P}_X\left(\mathcal{F}_{[\sigma]}\right) \\ &= m\left(X^{-1}\left(\mathcal{F}_{[\sigma]}\right)\right) \\ &= m\left(\left\{f \in 2^\omega : f \geq \sigma^\frown 0^\omega\right\}\right) \\ &= 1 - \sum_{i=0}^{|\sigma|-1} \sigma(i) 2^{-i}. \end{aligned}$$

The sum in the last line is just the rational number with binary expansion σ. This is clearly computable. It follows easily that $T_X(C)$ is computable for any clopen set $C \subseteq 2^\omega$. Hence by lemma 3.0.11 $\mathbf{P}_X \equiv_T T_X \equiv_T \emptyset$.

By the preceding discussion we know that $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ is clopen for each $\sigma \in 2^{<\omega}$. This correspondence is also computable and so we can apply lemma 3.0.12. Therefore if $f \in 2^\omega$ is a ML random real, then $X(f)$ is a $\mathbf{P}_X$-ML random closed set.

($\Rightarrow$) We wish to use lemma 3.0.13. To do so we need to find (uniformly) for each $\sigma \in 2^{<\omega}$ a Σ^0_1 set $\mathcal{H} \subseteq \mathcal{F}$ such that $X^{-1}(\mathcal{H}) = [\sigma]$. By our calculations so far $[\sigma] \subseteq X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$. Unfortunately $X^{-1}\left(\mathcal{F}_{[\sigma]}\right)$ also contains the set $\{f \in 2^\omega : \sigma^\frown 1^\omega < f\}$. Let $R_\sigma = \{f \in 2^\omega : \sigma^\frown 1^\omega < f\}$. We note that R_σ is actually a clopen subset of $2^{<\omega}$ by expressing it as follows:

$$R_\sigma = \bigcup \left\{[\tau] : \tau \in 2^{|\sigma|} \;\&\; \sigma < \tau\right\}.$$

Consequently the set $\mathcal{F}^{R_\sigma}_{[\sigma]}$ is a basic open set in $\mathcal{F}$. Furthermore $X^{-1}\left(\mathcal{F}^{R_\sigma}_{[\sigma]}\right) = [\sigma]$. This allows us to apply lemma 3.0.13. Therefore if $F \in \mathcal{F}$ is $\mathbf{P}_X$-ML random and $f \in 2^\omega$ is such that $X(f) = F$, then f is ML random.

It remains to show, however, that if $F \in \mathcal{F}$ is $\mathbf{P}_X$-ML random, then there is some $f \in 2^\omega$ such that $X(f) = F$. We will prove the contrapositive. Suppose that $F \notin X(2^\omega)$. Then there is some $f \in 2^\omega \setminus F$ and some $g \in F$ such that $f < g$. F is compact and 2^ω is Hausdorff and so there must be $\sigma \in 2^{<\omega}$ such that $\sigma \prec f$ and $[\sigma] \cap F = \emptyset$. Let $\tau \in 2^{<\omega}$ such that $\tau \prec g$. Note that for every $h_1 \in [\sigma]$ and every $h_2 \in [\tau]$ $h_1 < h_2$.

We now consider the set $\mathcal{F}^{[\sigma]}_{[\tau]}$. This is a basic open set in $\mathcal{F}$ and $F \in \mathcal{F}^{[\sigma]}_{[\tau]}$. Additionally $X^{-1}\left(\mathcal{F}^{[\sigma]}_{[\tau]}\right) = \emptyset$. This means that $\mathbf{P}_X\left(\mathcal{F}^{[\sigma]}_{[\tau]}\right) = 0$. Therefore F is not $\mathbf{P}_X$-ML random. $\square$

So far in all of our examples with RACS $X : \Omega \to \mathcal{F}(\mathbb{E})$ we have found that if $F \in \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_X$-ML random, then $F \in X(\Omega)$. Is this always the case? The answer is no and we are able to construct an example in which this is not the case by modifying the random interval map we have just discussed.

Fix $r \in 2^\omega$ such that r is ML random. Define $Y : 2^\omega \to \mathcal{F}(2^\omega)$ by

$$Y(f) = \begin{cases} \{g \in 2^\omega : g \leq f\} & \text{if } f \neq r \\ \emptyset & \text{if } f = r \end{cases}.$$

This map is very closed related to X:

$$Y^{-1}\left(\mathcal{F}_{[\sigma]}\right) = X^{-1}\left(\mathcal{F}_{[\sigma]}\right) \setminus \{r\}.$$

This equation proves that Y is measurable and, moreover, $\mathbf{P}_X = \mathbf{P}_Y$. Therefore $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_Y$-ML random if and only in F is $\mathbf{P}_X$-ML random. But there is one $\mathbf{P}_X$-ML random closed set that is not in the image of Y, namely $X(r) = \{g \in 2^\omega : g \leq r\}$.

We can also apply Robbins' theorem to the RACS X to find the expected value of the measure of X, $E(m \circ X)$. In applying Robbins' theorem we will use the identification of 2^ω with $[0, 1]$ as discussed in section 2.4.1. For $f \in 2^\omega$ let $r(f)$

be the real number with binary expansion given by f.

$$\begin{aligned} E(m \circ X) &= \int_{2^\omega} P(f \in X) dm(f) \\ &= \int_{2^\omega} r(f) dm(f) \\ &= \int_{[0,1]} x dx \\ &= \frac{1}{2} \end{aligned}$$

This example can be generalized to deal with intervals with two random endpoints (in which case the analogue of theorem 4.4.4 holds) or to higher dimensions. The generalization to intervals with two random endpoints that we considered is again a map $Z : 2^\omega \to \mathcal{F}(2^\omega)$. To do this we split an element $f \in 2^\omega$ into its even and odd bits:

$$\begin{aligned} f_0(n) &= f(2n) \\ f_1(n) &= f(2n+1). \end{aligned}$$

The map Z is then given by

$$Z(f) = \{g \in 2^\omega : f_0 \leq g \leq f_1\} \cup \{g \in 2^\omega : f_1 \leq g \leq f_0\}.$$

In this case Robbins' theorem can be used to show that $E(m \circ Z) = \frac{1}{3}$.

4.5 Maxitive capacities

This section, despite being located in the chapter on Martin-Löf random closed subsets of 2^ω, deals with more general spaces. The capacities we examine here generalize the RACS of the preceding section 4.4. For the following let $\mathbb{E}$ be a locally compact, Hausdorff, second countable (LCHS) space. We will assume that Martin-Löf random subsets of $\mathbb{E}$ are generated with respect to a canonical basis for the space $\mathcal{F}(\mathbb{E})$ as in proposition 3.0.9 and definition 3.0.10. We recall that this means that we have a fixed locally compact basis for $\mathbb{E}$ such that the sub-basis for $\mathcal{F}(\mathbb{E})$ consists of all sets of the form $\mathcal{F}_{B_1}$ and $\mathcal{F}^{\overline{B}_2}$, where B_1, B_2 are basic open sets of $\mathbb{E}$.

Recall from section 2.4.3 that for $\varphi : \mathbb{E} \to [0,1]$ upper semi-continuous the functional $T(K) = \sup_{x \in K} \varphi(x)$ is a Choquet Capacity. Such capacities are called *maxitive* because $T(K_1 \cup K_2) = \max\{T(K_1), T(K_2)\}$ for any compact $K_1, K_2 \subseteq \mathbb{E}$. As usual let $\mathbf{P}_T$ be the measure induced by this capacity.

Proposition 4.5.1. *If $F \in \mathcal{F}(\mathbb{E})$ is $\mathbf{P}_T$-ML random and $\varphi : \mathbb{E} \to [0,1]$ is upper semi-continuous with global maximum $m \in [0,1]$, then one of the following must hold:*

1. $F = \emptyset$;
2. $F \subseteq \{x \in \mathbb{E} : \varphi(x) = m\}$;
3. $\{x \in \mathbb{E} : \varphi(x) = m\} \subseteq F$.

Furthermore if $m = 1$, then only the last two are possible, i.e. $F \neq \emptyset$.

Proof. Fix any countable basis for $\mathbb{E}$ satisfying the conditions of lemma 2.4.7. In particular we have that the closure of any basic open is compact. By proposition 3.0.9 $\mathcal{F}(\mathbb{E})$ is a Martin-Löf space under any probability measure. We work with the canonical basis for $\mathcal{F}(\mathbb{E})$ which is generated from the sub-basis of all sets of the form $\mathcal{F}_{B_1}$ and $\mathcal{F}^{\overline{B}_2}$, where B_1, B_2 are basic open sets of $\mathbb{E}$.

Suppose that $m \in [0,1]$ is a maximum for φ and that there are points $x_0 \in F \setminus \{x \in \mathbb{E} : \varphi(x) = m\}$ and $x_1 \in \{x \in \mathbb{E} : \varphi(x) = m\} \setminus F$. The space $\mathbb{E}$ is Hausdorff and so there are disjoint basic open sets B_0 and B_1 such that $x_0 \in B_0$ and $x_1 \in B_1$. In fact we can even ensure that $F \cap \overline{B_1} = \emptyset$. Consider the basic open set $\mathcal{F}_{B_0}^{\overline{B_1}}$. We will use the fact that φ attains its maximum on B_1 and hence $T(A) \leq T(B_1)$ for any $A \subseteq \mathbb{E}$. Thus

$$\begin{aligned}\mathbf{P}_T\left(\mathcal{F}_{B_0}^{\overline{B_1}}\right) &= \Delta_{B_0} T\left(\overline{B_1}\right) \\ &= T\left(B_0 \cup \overline{B_1}\right) - T\left(B_1\right) \\ &= 0.\end{aligned}$$

But $F \in \mathcal{F}_{B_0}^{\overline{B_1}}$ and thus F is not $\mathbf{P}_T$-ML random. Therefore if $F \neq \emptyset$ is $\mathbf{P}_T$-ML random then either $F \subseteq \{x \in \mathbb{E} : \varphi(x) = m\}$ or $\{x \in \mathbb{E} : \varphi(x) = m\} \subseteq F$.

Suppose now that $m = 1$. We wish to show that $\emptyset$ is not $\mathbf{P}_T$-ML random. Suppose that $x \in \mathbb{E}$ such that $\varphi(x) = 1$. Let B be a basic open neighborhood of x. Then

$$\mathbf{P}_T\left(\mathcal{F}^{\overline{B}}\right) = 1 - \mathbf{P}_T\left(\mathcal{F}_{\overline{B}}\right) = 1 - T(\overline{B}) = 0.$$

Furthermore, $\emptyset \in \mathcal{F}^{\overline{B}}$ and $\mathcal{F}^{\overline{B}}$ is a basic open in the Fell topology. Therefore $\emptyset$ is not $\mathbf{P}_T$-ML random. □

4.6 Random singletons

For this example we deal once again with a map $X : 2^\omega \to \mathcal{F}(2^\omega)$. This time X is given by $X(f) = \{f\}$, that is, f maps to the singleton set $\{f\}$. We have already shown in proposition 2.4.22 that this map is measurable. In this example we can again prove that the random closed sets are exactly the images of random elements of 2^ω. As in previous examples we apply lemmas 3.0.13 and 3.0.12 and then prove that all ML-random closed sets are in the image.

Proposition 4.6.1. *Closed set $F \in \mathcal{F}(2^\omega)$ is $\mathbf{P}_X$-Martin-Löf random if and only if $F = \{f\}$ for some Martin-Löf random $f \in 2^\omega$.*

Proof. We first note that $\mathbf{P}_X$ and the fair coin measure on 2^ω are both computable.

We now apply lemma 3.0.12. In this case X^{-1} is particularly easy to describe. Let $C \subseteq 2^\omega$ be clopen and $\sigma_1, \ldots, \sigma_n \in 2^{<\omega}$. Then

$$X^{-1}(\mathcal{F}^C_{[\sigma_1],\ldots,[\sigma_n]}) = \left(\bigcap_{i=1}^{n} [\sigma_i] \right) \setminus C. \tag{4.6}$$

The latter is a clopen subset of 2^ω. Therefore by lemma 3.0.12 if $f \in 2^\omega$ is ML random, then $X(f) = \{f\}$ is $\mathbf{P}_X$-ML random.

Note that $X^{-1}(\mathcal{F}_{[\sigma]}) = [\sigma]$ for any $\sigma \in 2^{<\omega}$. We can thus apply lemma 3.0.13. Therefore if F is $\mathbf{P}_X$-ML random, then $X^{-1}(\{F\})$ is either $\emptyset$ or consists entirely of ML random elements of 2^ω. Of course X is an injection, and so this means that if $F \in X(2^\omega)$ is $\mathbf{P}_X$-ML random, then $X^{-1}(F)$ is a ML random element in 2^ω.

It remains to be shown that if $F \in \mathcal{F}$ is $\mathbf{P}_X$-ML random, then $F \in X(2^\omega)$. We prove the contrapositive. Suppose that $F \notin X(2^\omega)$. Then there are $f, g \in F$ such that $f \neq g$. Hence there are $\sigma, \tau \in 2^{<\omega}$ such that $[\sigma] \cap [\tau] = \emptyset$ and $f \in [\sigma]$ and $g \in [\tau]$. This means that $F \in \mathcal{F}_{[\sigma],[\tau]}$, a basic open set of the Fell topology. But $X^{-1}\left(\mathcal{F}_{[\sigma],[\tau]}\right) = \emptyset$. Therefore $\mathbf{P}_X\left(\mathcal{F}_{[\sigma],[\tau]}\right) = 0$. This means that F is an element of a basic open set with $\mathbf{P}_X$-measure 0. Consequently F is not $\mathbf{P}_X$-random. This completes the proof. □

CHAPTER 5

MARTIN-LÖF RANDOM CLOSED SUBSETS OF THE REAL NUMBERS UNDER GENERALIZED POISSON PROCESSES

In this chapter we address the Martin-Löf random closed subsets of $\mathbb{R}$ under the measure on $\mathcal{F}(\mathbb{R})$ induced by a capacity of the form $T(K) = 1 - e^{-\lambda m(K)}$, where λ is a positive real constant and m is the Lebesgue measure on $\mathbb{R}$. Such a functional is a Choquet capacity by proposition 2.4.23 of section 2.4.3. The RACS induced by these capacities are known as generalized Poisson processes. We begin this chapter by working with a specific generalized Poisson process, namely $T(K) = 1-2^{-m(K)}$. We are able to determine many of the properties of the Martin-Löf random closed sets for this RACS. This includes a new characterization of the Martin-Löf random reals as exactly those reals contained in some Martin-Löf random closed set. We then generalize and extend these results in the second section.

5.1 A generalized Poisson process

This section examines a well known classical example from the theory of random sets. We work in the space of closed subsets of $\mathbb{R}$, $\mathcal{F}(\mathbb{R})$. By proposition 3.0.9 $\mathcal{F}(\mathbb{R})$ is a Martin-Löf space. We must specify exactly which basis we are working with, both for $\mathbb{R}$ and $\mathcal{F}(\mathbb{R})$. The space $\mathbb{R}$ has a basis consisting of all intervals $I = (q_1, q_2)$ where $q_1 < q_2 \in \mathbb{Q}$. Let $I_0, I_1, \ldots$ be an enumeration of this basis such that the function $i \mapsto \langle q_1, q_2 \rangle$ is computable. We will refer to this as the *standard basis* for $\mathbb{R}$. This also gives us a canonical enumeration, $\mathcal{B}_0, \mathcal{B}_1 \ldots$ of the basis for $\mathcal{F}(\mathbb{R})$ (as in proposition 3.0.9 and definition 3.0.10). Note that a basic open set $\mathcal{B} \subseteq \mathcal{F}(\mathbb{R})$ is of the form

$$\mathcal{B} = \mathcal{F}^{\overline{I}_{i_{n+1}} \cup \cdots \cup \overline{I}_{i_k}}_{I_{i_1}, \ldots, I_{i_n}}.$$

Definition 5.1.1. Let m be Lebesgue measure on $\mathbb{R}$ and let $\lambda \in (0, \infty)$. Define $T : \mathcal{K}(\mathbb{R}) \to [0, 1]$ by

$$T(K) = 1 - e^{-\lambda m(K)}.$$

By proposition 2.4.23 T is a Choquet capacity and hence gives rise to a Borel probability measure $\mathbf{P}_T$ on $\mathcal{F}(\mathbb{R})$. Closed sets chosen randomly according to this measure are *generalized Poisson processes*. We will address the reason behind this terminology later. Because $m(I) = m(\overline{I})$ for any basic open $I \subseteq \mathbb{R}$ we will be able to apply lemma 3.0.11 to find that $\mathbf{P}_T \equiv_T T \equiv_T \lambda$.

Proposition 5.1.2. *When coded with respect to the standard bases for* $\mathbb{R}$ *and* $\mathcal{F}(\mathbb{R})$, $\mathbf{P}_T \equiv_T T \equiv_T \lambda$.

Proof. We cannot apply lemma 3.0.11 directly because the standard basis for $\mathbb{R}$ is not closed under finite unions. We pass to the basis of finite unions from the standard basis for $\mathbb{R}$. Let $B_0, B_1, \ldots$ be an enumeration of all the finite unions of open rational intervals (i.e. the closure of our basis $I_0, I_1, \ldots$ under finite union) such that the correspondence $B_j = I_{i_1} \cup \cdots \cup I_{i_n}$ is computable. This means that there is an obvious computable union function. Clearly, for each $j \in \omega$, $\overline{B}_j$ is compact. In addition, for each $j \in \omega$, $m(B_j) = m\left(\overline{B}_j\right)$. Hence for $j, k \in \omega$

$$T\left(\overline{B}_j \cup \overline{B}_k\right) = T\left(\overline{B}_j \cup B_k\right).$$

Therefore we can apply lemma 3.0.11 to the basis $B_0, B_1, \ldots$.

The catch is that in changing our basis for $\mathbb{R}$ we have changed the canonical enumeration of our basis for $\mathcal{F}(\mathbb{R})$. Let $\mathcal{B}'_0, \mathcal{B}'_1, \ldots$ be this new basis. A basic open set $\mathcal{B}'$ is of the form

$$\mathcal{B}'_i = \mathcal{F}^{\overline{B}_{i_{n+1}}}_{B_{i_1},\ldots B_{i_n}}.$$

The conclusion of lemma 3.0.11 is that if T is coded by $t \in 2^\omega$ such that

$$t^{[j]} = T\left(\overline{B}_j\right)$$

and if $\mathbf{P}_T$ is coded by $p \in 2^\omega$ such that

$$p^{[i]} = \mathbf{P}_T\left(\mathcal{B}'_i\right),$$

then $t \equiv_T p$. We recall that the coding of measures is defined in 3.0.4.

Let s be the coding of T with respect to our standard basis for $\mathbb{R}$, $I_0, I_1, \ldots$. For each $i \in \omega$ we can uniformly computably find $j \in \omega$ such that $I_i = B_j$. Hence $s \leq_T t$. We also wish to show that $t \leq_T s$. Consider

$$T\left(\overline{B}_j\right) = 1 - 2^{-\lambda m\left(\overline{B}_j\right)} = 1 - 2^{-\lambda m\left(\overline{I}_{i_1} \cup \cdots \cup \overline{I}_{i_n}\right)}.$$

where $B_j = I_{i_1} \cup \cdots \cup I_{i_n}$. The correspondence $B_j = I_{i_1} \cup \cdots \cup I_{i_n}$ is computable and m is computable and consequently $t \leq \lambda$. But $\lambda \leq_T s$ via

$$\lambda = -\ln\left(1 - T\left([0,1]\right)\right).$$

Therefore $t \equiv_T s \equiv_T \lambda$.

Now let q be the coding of $\mathbf{P}_T$ with respect to our standard basis for $\mathcal{F}(\mathbb{R})$, $\mathcal{B}_0, \mathcal{B}_1, \ldots$. For each $i \in \omega$ we can uniformly computably find $j \in \omega$ such that $\mathcal{B}_i = \mathcal{B}'_j$. Hence $q \leq_T p$. Codes for capacities are always computable from codes for the corresponding measure when they are coded with respect to the same basis. Hence $s \leq_T q$. Therefore $q \leq_T p \equiv_T t \equiv_T \lambda \equiv_T s \leq_T q$. In other words, $q \equiv_T p \equiv_T \lambda$. □

For our analysis it is convenient to work with $\lambda = \ln 2$ so that $T(K) = 1 - 2^{-m(K)}$ for compact $K \subseteq \mathbb{R}$. This assignment of λ means that $T \equiv_T \mathbf{P}_T \equiv_T m \equiv_T \emptyset$ and hence computability issues are minimized. This has the added advantage of preventing confusion with our usual assignment of λ to represent the empty string. Changing the parameter λ to a different computable real will not significantly change any of what follows. We will address the case when λ is not computable more thoroughly in section 5.2.

We know that $\mathbf{P}_T$ will inherit some of the properties of the Lebesgue measure. One example is translation invariance (which is called *stability* for a RACS).

Definition 5.1.3. Let $\mathcal{A} \subseteq \mathcal{F}(\mathbb{R})$, $F \in \mathcal{F}(\mathbb{R})$, and $c \in \mathbb{R}$.

1. $F + c := \{x + c : x \in F\}$.
2. $\mathcal{A} + c := \{E + c : E \in \mathcal{A}\}$.

A RACS $X : \Omega \to \mathcal{F}(\mathbb{R})$ is *stable* if for every $c \in \mathbb{R}$ the RACS X and $X + c$ are identically distributed.

A generalized Poisson process is the canonical example of a stable RACS. In this case stability means that $\mathbf{P}_T(\mathcal{A}) = \mathbf{P}_T(\mathcal{A}+c)$ for every measurable $\mathcal{A} \subseteq \mathcal{F}(\mathbb{R})$ and every $c \in \mathbb{R}$. We now prove an effective version of stability for our RACS.

Lemma 5.1.4. *If $F \in \mathcal{F}(\mathbb{R})$ is $\mathbf{P}_T$-ML random and $c \in \mathbb{Q}$ then $F + c$ is also $\mathbf{P}_T$-ML random.*

Proof. We prove the contrapositive. Suppose that $F + c$ is not $\mathbf{P}_T$-ML random and that $\{\mathcal{U}_i\}_{i\in\omega}$ is a $\mathbf{P}_T$-ML test such that $F + c \in \bigcap_{i\in\omega} \mathcal{U}_i$. Define $\mathcal{V}_i = \mathcal{U}_i - c$ for each $i \in \omega$. Observe that for $r_1, \dots, r_k, q_1, \dots, q_n \in \mathbb{Q}$,

$$\mathcal{F}^{[r_1,r_2]\cup\cdots\cup[r_{k-1},r_k]}_{(q_1,q_2),\dots,(q_{n-1},q_n)} \subseteq \mathcal{U}_k \iff \mathcal{F}^{[r_1-c,r_2-c]\cup\cdots\cup[r_{k-1}-c,r_k-c]}_{(q_1-c,q_2-c),\dots,(q_{n-1}-c,q_n-c)} \subseteq \mathcal{V}_k.$$

Hence $\{\mathcal{V}_i\}_{i\in\omega}$ is a uniformly Σ^0_1 sequence and $\mathbf{P}_T(\mathcal{V}_i) = \mathbf{P}_T(\mathcal{U}_i) \leq 2^{-i}$. Thus $\{\mathcal{V}_i\}_{i\in\omega}$ is a $\mathbf{P}_T$-ML test and $F \in \bigcap_{i\in\omega} \mathcal{V}_i$. Therefore F is not $\mathbf{P}_T$-ML random. □

We are interested in the relationship between $\mathbf{P}_T$-Martin-Löf random closed subsets of $\mathbb{R}$ and the m-Martin-Löf random elements of $\mathbb{R}$. The following theorem is a characterization of the m-Martin-Löf random reals. It should be contrasted with the result of Barmpalias et al. [1] that each BBCDW-random closed set contains a non-random element of 2^ω.

Theorem 5.1.5. *A real number $r \in \mathbb{R}$ is m-Martin-Löf random if and only if there is some $\mathbf{P}_T$-Martin-Löf random closed set F such that $r \in F$.*

Proof. First we prove that if F contains a non m-ML random real r, then F is not $\mathbf{P}_T$-ML random. Let $r \in F$ and suppose that $\{U_i\}_{i\in\omega}$ is a ML test such that $r \in \bigcap_{i\in\omega} U_i$. Let $I_0, I_1, \dots$ be our standard enumeration of all open rational intervals of $\mathbb{R}$ and let $f \in 2^\omega$ be c. e. such that $U_i = \bigcup_{f(\langle i,j\rangle)=1} I_j$. Consider the sequence of subsets of $\mathcal{F}$, $\{\mathcal{F}_{U_i}\}_{i\in\omega}$. This is a uniformly Σ^0_1 sequence because

$$\mathcal{F}_{U_i} = \bigcup_{f(\langle i,j\rangle)=1} \mathcal{F}_{I_j}.$$

By definition $F \in \bigcap_{i\in\omega} \mathcal{F}_{U_i}$. Furthermore,

$$\mathbf{P}_T\left(\mathcal{F}_{U_i}\right) = 1 - 2^{-m(U_i)} \leq 1 - 2^{-2^{-i}}.$$

Consequently, if $n \geq \log_2 \log_2 (1 - 2^{-i})$, then $\mathbf{P}_T\left(\mathcal{F}_{U_n}\right) \leq 2^{-i}$. Let

$$n = \left\lceil \log_2 \log_2 \left(1 - 2^{-i}\right)\right\rceil$$

and let

$$\mathcal{V}_i = \mathcal{F}_{U_n}.$$

Then $\{\mathcal{V}_i\}_{i\in\omega}$ a $\mathbf{P}_T$-ML test and $F \in \bigcap_{i\in\omega} \mathcal{V}_i$. Therefore F is not $\mathbf{P}_T$-ML random.

To prove the converse suppose that $r \in \mathbb{R}$ is such that there is no $\mathbf{P}_T$-ML random closed set containing r. Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a universal $\mathbf{P}_T$-ML test (the existence of which is proved in lemma 3.0.8). No closed set containing r is random and thus for all $i \in \omega$, $\mathcal{F}_{\{r\}} \subseteq \mathcal{U}_i$. The first step is to show that open covers of $\mathcal{F}_{\{r\}}$ must in fact cover more than just $\mathcal{F}_{\{r\}}$. The second step is to use this information to construct an m-ML test that catches r.

Step 1: Prove that for each $i \in \omega$ there is some closed rational interval J_i such that $\mathcal{F}_{J_i} \subseteq \mathcal{U}_i$ and $r \in J_i$.

The set $\{r\}$ is compact in $\mathbb{R}$ and hence $\mathcal{F}_{\{r\}}$ is closed in $\mathcal{F}(\mathbb{R})$. The space $\mathcal{F}(\mathbb{R})$ is compact and so $\mathcal{F}_{\{r\}}$ is compact. Let $f \in 2^\omega$ be c.e. such that

$$\mathcal{U}_i = \bigcup_{f(\langle i,j\rangle)=1} \mathcal{B}_j$$

(where $\mathcal{B}_0, \mathcal{B}_1, \ldots$ is the standard basis for $\mathcal{F}$). By compactness, there must be some $N \in \omega$ such that

$$\mathcal{F}_{\{r\}} \subseteq \bigcup_{j \leq N \ \& \ f(\langle i,j \rangle)=1} \mathcal{B}_j.$$

By definition $\mathcal{B}_j = \mathcal{F}^{K_j}_{I_{k_{j,1}},\ldots,I_{k_{j,n_j}}}$ where K_j is some finite union of closed rational intervals. We know that the closed set $\{r\}$ is a member of $\mathcal{F}_{\{r\}}$. Hence there is $j_0 \leq N$ such that $f(\langle i, j_0 \rangle) = 1$ and $\{r\} \in \mathcal{B}_{j_0}$. Because $\{r\} \in \mathcal{B}_{j_0}$ we know that $r \in I_{k_{j_0,m}}$ for each $m \leq n_{j_0}$. We define

$$I := \bigcap \left\{ I_{k_{j,m}} : j \leq N \ \& \ f(\langle i,j \rangle) = 1 \ \& \ m \leq n_j \ \& \ r \in I_{k_{j,m}} \right\}.$$

Then I is a finite intersection of open rational intervals, each of which contains r. Therefore I is a nonempty open rational interval.

We now claim that $\mathcal{F}_I \subseteq \mathcal{U}_i$. Suppose that $F \in \mathcal{F}_I$. Note that $F \cup \{r\} \in \bigcup_{j \leq N \ \& \ f(\langle i,j \rangle)=1} \mathcal{B}_j$. Hence there is $j \leq N$ such that $f(\langle i, j \rangle) = 1$ and

$$F \cup \{r\} \in \mathcal{B}_j = \mathcal{F}^{K_j}_{I_{k_{j,1}},\ldots,I_{k_{j,n_j}}}.$$

Fix this j. By definition $(F \cup \{r\}) \cap K_j = \emptyset$ and $(F \cup \{r\}) \cap I_{k_{j,m}} \neq \emptyset$ for each $m \in \{1, \ldots, n_j\}$. Clearly $F \cap K_j = \emptyset$ and hence $F \in \mathcal{F}^{K_j}$. Now, if $r \notin I_{k_{j,m}}$ then it must be the case that $F \cap I_{k_{j,m}} \neq \emptyset$. On the other hand, if $r \in I_{k_{j,m}}$ then $I \subseteq I_{k_{j,m}}$ by the definition of I. But $F \in \mathcal{F}_I$ and so by definition $F \cap I_{k_{j,m}} \neq \emptyset$. Thus $F \cap I_{k_{j,m}} \neq \emptyset$ for each $m \in \{1, \ldots, n_j\}$. Consequently

$$F \in \mathcal{F}^{K_j}_{I_{k_{j,1}},\ldots,I_{k_{j,n_j}}} = \mathcal{B}_j.$$

Therefore

$$F \in \bigcup_{j \leq N \ \& \ f(\langle i,j \rangle)=1} \mathcal{B}_j$$

and we have proved the claim.

So far we have proved the existence of an open rational interval I such that $\mathcal{F}_I \subseteq \mathcal{U}_i$ and $r \in I$. By taking a closed rational interval $J_i \subseteq I$ such that $r \in J_i$ we find the desired closed rational interval J_i such that $\mathcal{F}_{J_i} \subseteq \mathcal{U}_i$ and $r \in J_i$. This completes the first step.

Step 2: Construct an m-ML test $\{V_i\}_{i \in \omega}$ in $\mathbb{R}$ such that $r \in \bigcap_{i \in \omega} V_i$.

To build V_k we use $\mathcal{U}_n$ where

$$n = \left\lceil -\log_2 \left(1 - 2^{-2^{-(k+1)}}\right) \right\rceil .$$

We need this because it guarantees that

$$2^{-n} \leq 1 - 2^{-2^{-(k+1)}}. \tag{5.1}$$

Recall that we have a c.e. $f \in 2^\omega$ such that $\mathcal{U}_n = \bigcup_{f(\langle n,j \rangle)=1} \mathcal{B}_j$. Let $\mathcal{U}_{n,s}$ be the stage s approximation to $\mathcal{U}_n$:

$$\mathcal{U}_{n,s} := \bigcup_{f(\langle n,j \rangle)=1 \ \& \ j \leq s} \mathcal{B}_j .$$

Note that for each $i \in \omega$, $\mathcal{F}_{\overline{I}_i}$ is compact in $\mathcal{F}(\mathbb{R})$. Because $\mathcal{F}_{\overline{I}_i}$ is compact it follows that $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_n$ if and only if there is a stage s such that $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,s}$.

We want to be able to identify these stages in a computable manner. This is possible by the following calculations. First,

$$\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,s} \iff \mathcal{F}_{\overline{I}_i} \setminus \mathcal{U}_{n,s} = \emptyset.$$

The set $\mathcal{F}_{\overline{I}_i} \setminus \mathcal{U}_{n,s}$ is a closed set of the form

$$\mathcal{F}^{O_1}_{K^1_1,K^1_2,\ldots K^1_{k_1}} \cup \cdots \cup \mathcal{F}^{O_m}_{K^m_1,K^m_2,\ldots K^m_{k_m}} \tag{5.2}$$

where K^j_l is a finite union of closed rational intervals and O_l is a finite union of open rational intervals. Finding such an expression for this set is also uniformly computable. Consequently $\mathcal{F}_{\overline{I}_i} \setminus \mathcal{U}_{n,s} = \emptyset$ if and only if each component of the formula 5.2 is empty. We know that

$$\mathcal{F}^{O_j}_{K^j_1,K^j_2,\ldots K^j_{k_j}} = \emptyset \iff (\exists l \leq k_j)\, K^j_l \subseteq O_j.$$

Each K^j_l is a finite union of closed rational intervals and O_j is a finite union of open rational intervals and all of these sets were obtained uniformly computably. We can thus determine uniformly computably if $K^j_l \subseteq O_j$. Therefore we can computably determine if $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,s}$.

To construct our m-ML test we watch for closed rational intervals J such that $\mathcal{F}_J \subseteq \mathcal{U}_{n,s}$ and use them to build V_k. At the same time we construct a set $C_k \subseteq V_k$ that we will use later to calculate the measure of V_k. We build V_k and C_k in stages. At stage 0 we set $V_{k,0} = C_{k,0} = \emptyset$.

Suppose that we have enumerated j intervals into $V_{k,s}$. Wait for the next stage t such that there is $i \leq t$ with $\overline{I}_i \nsubseteq V_{i,s}$ and $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,t}$. Let i be the least index such that I_i satisfies these conditions. We then take an open rational interval I such that $\overline{I}_i \subseteq I$ and $m(I \setminus \overline{I}_i) \leq 2^{-(2+k+j)}$ and set $V_{k,t} = V_{k,s} \cup I$. We also set

$C_{k,t} = C_{k,s} \cup \overline{I}_i$.

Let $V_k = \bigcup_{s\in\omega} V_{k,s}$ and $C_k = \bigcup_{s\in\omega} C_{k,s}$. By construction V_k is Σ_1^0 and we note that this is uniform over $k \in \omega$. We have also constructed a set $C_k \subseteq V_k$ such that

$$\begin{aligned} m(V_k) &= m(C_k) + m(V_k \setminus C_k) \\ &\leq m(C_k) + \sum_{j=0}^{\infty} 2^{-(2+k+j)} \\ &= m(C_k) + 2^{-(k+1)}. \end{aligned}$$

We also know that $\mathcal{F}_{C_k} \subseteq \mathcal{U}_n$ and hence $\mathbf{P}_T(\mathcal{F}_{C_k}) \leq 2^{-n}$. By definition $\mathbf{P}_T(\mathcal{F}_{C_k}) = 1 - 2^{-m(C_k)}$ and so we find that $m(C_k) \leq -\log_2(1-2^{-n})$. But n was chosen to satisfy equation 5.1 and therefore

$$\begin{aligned} m(C_k) &\leq -\log_2\left(1-2^{-n}\right) \\ &\leq -\log_2\left(1-\left(1-2^{-2^{-(k+1)}}\right)\right). \\ &= 2^{-(k+1)}. \end{aligned}$$

Therefore

$$m(V_k) \leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

This means that $\{V_i\}_{i\in\omega}$ is an m-Martin-Löf test.

We also know that for every $n \in \omega$ there is an closed rational interval J_n such that $r \in J_n$ and $\mathcal{F}_{J_n} \subseteq \mathcal{U}_n$. We claim that $J_n \subseteq V_k$. Because $\mathcal{F}_{J_n} \subseteq \mathcal{U}_n$ there is a stage s such that $\mathcal{F}_{J_n} \subseteq \mathcal{U}_{n,s}$. There is also some index i_n such that $J_n = \overline{I}_{i_n}$. If $J_n \nsubseteq V_{k,s}$ then at some stage $t \geq \max\{s, i_n\}$ i_n will be the least index i such that $\overline{I}_i \nsubseteq V_{k,t}$ and $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,t}$. At this stage we will set $C_{k,t} = C_{k,t-1} \cup \overline{I}_{i_n}$. As always $C_{k,t} \subseteq V_{k,t}$ and consequently $J_n = \overline{I}_{i_n} \subseteq V_k$. This proves the claim.

A clear consequence of the claim is that for every $i \in \omega$, $r \in V_i$. In other words, $r \in \bigcap_{i\in\omega} V_i$. We have also shown that $\{V_i\}_{i\in\omega}$ is an m-ML test. Therefore r is not m-ML random. $\square$

We are interested in understanding the $\mathbf{P}_T$-Martin-Löf random closed sets. The following theorems begin that process.

Proposition 5.1.6. *If $F \in \mathcal{F}(\mathbb{R})$ is $\mathbf{P}_T$-ML random, then for all $n \in \omega$ $F \nsubseteq (-n, n)$ (i.e. F is unbounded).*

Proof. We prove the contrapositive. Suppose that there is some $n \in \omega$ such that $F \subseteq (-n, n)$. Define $\mathcal{U}_i = \mathcal{F}^{[-(n+i+1),-n]\cup[n,n+i+1]}$. Then $\{\mathcal{U}_i\}_{i\in\omega}$ is a uniformly Σ^0_1 sequence of subsets of $\mathcal{F}(\mathbb{R})$. Furthermore, $\mathbf{P}_T(\mathcal{U}_i) = 2^{-2(i+1)} \leq 2^{-i}$. Therefore $\{\mathcal{U}_i\}_{i\in\omega}$ is a $\mathbf{P}_T$-ML test and $F \in \bigcap_{i\in\omega} \mathcal{U}_i$. Hence F is not $\mathbf{P}_T$-ML random. $\square$

We will now look at some topological properties of $\mathbf{P}_T$-ML random closed sets. We will be looking at properties that are defined locally and so we will restrict ourselves to $\mathcal{F}_{[0,1]}$. These properties will be invariant under translation and so by lemma 5.1.4 these properties will hold of any $\mathbf{P}_T$-ML random closed set. Working in the subspace $\mathcal{F}_{[0,1]}$ is convenient because it is similar to a space that we understand fairly well, $\mathcal{F}(2^\omega)$. The analogy should not be taken too far, however, as there are some important differences between the spaces. To help make use of the analogy between $\mathcal{F}_{[0,1]}$ and $\mathcal{F}(2^\omega)$ we introduce the following notation.

Definition 5.1.7. Let $I_\lambda = (0, 1)$ (for the empty string λ). If for $\sigma \in 2^{<\omega}$ I_σ is defined, then $I_{\sigma^\frown 0}$ is the open left half interval of I_σ and $I_{\sigma^\frown 1}$ is the open right half interval of I_σ.

For example $I_0 = \left(0, \frac{1}{2}\right)$, $I_{01} = \left(\frac{1}{4}, \frac{1}{2}\right)$, and so on.

Definition 5.1.8. For $x \in [0,1]$ and $n \geq 1$ let $x \upharpoonright n \in \mathbb{Q}$ be the rational represented by first n digits of the binary expansion of x. That is, if $x = 0.d_1 d_2 \ldots$ is the binary expansion of x, then $x \upharpoonright n = d_1 {}^\frown \ldots {}^\frown d_n$. For convenience define $x \upharpoonright 0$ to be the empty string λ. Where binary expansions are not unique we arbitrarily chose one to use; which we chose is unimportant so long as we are consistent.

We now show that the sequence of sets $\{I_{x \upharpoonright n}\}_{n \in \omega}$ behaves as we would expect.

Proposition 5.1.9. *For each $x \in [0,1]$ and each $n \in \omega$ $I_{x\upharpoonright n} \supseteq I_{x \upharpoonright n+1}$ and $\{x\} = \bigcap_{n \in \omega} \overline{I}_{x\upharpoonright n}$. Furthermore, if $x \notin \mathbb{Q}$, then $\bigcap_{n\in\omega} I_{x\upharpoonright n} = \{x\}$.*

Proof. That $I_{x\upharpoonright n} \supseteq I_{x\upharpoonright n+1}$ is clear from the definitions of $x \upharpoonright n$ and I_σ. Furthermore, $x \in \overline{I}_{x\upharpoonright n}$ for every $n \in \omega$ since $\overline{I}_{x\upharpoonright n}$ contains all reals whose binary expansions agree with x up to the n^{th} digit. We note that this is not true of the open interval $I_{x\upharpoonright n}$. Clearly the diameter of $\overline{I}_{x\upharpoonright n} \to 0$ as $n \to \infty$ and so $\{x\} = \bigcap_{n\in\omega} \overline{I}_{x\upharpoonright n}$.

Now if $x \notin \bigcap_{n\in\omega} I_{x\upharpoonright n}$, then there is some least $n \in \omega$ such that $x \in \overline{I}_{x\upharpoonright n} \setminus I_{x\upharpoonright n}$. This means that x is one of the end points of $I_{x\upharpoonright n}$, and hence $x \in \mathbb{Q}$. □

We now use the preceding to prove an effective version of the fact that $\mathbf{P}_T$-almost every closed set is discrete.

Proposition 5.1.10. *If $F \in \mathcal{F}_{[0,1]}$ is $\mathbf{P}_T$-Martin-Löf random and $x \in F \cap [0,1]$ then there is an open set $G \subseteq \mathbb{R}$ such that $x \in G$ and $G \cap F = \{x\}$ (i.e. x is isolated in F).*

Proof. Let $F \in \mathcal{F}_{[0,1]}$ be $\mathbf{P}_T$-ML random. By lemma 5.1.5 every element of F is m-ML random. In particular $F \cap \mathbb{Q} = \emptyset$. This means, by the preceding lemma (5.1.9), that for all $\sigma \in 2^{<\omega}$

$$F \cap I_\sigma = F \cap \overline{I}_\sigma.$$

We now claim that

$$x \in F \cap [0,1] \iff x \in [0,1] \ \& \ (\forall n \in \omega \ I_{x\restriction n} \cap F \neq \emptyset) .$$

To prove $\Rightarrow$ suppose that $x \in F$. Then by lemma 5.1.9 $x \notin \mathbb{Q}$ and moreover, for each $n \in \omega$, $x \in I_{x\restriction n}$. Thus $\forall n \in \omega \ I_{x\restriction n} \cap F \supseteq \{x\} \neq \emptyset$.

To prove $\Leftarrow$ suppose that $\forall n \in \omega \ I_{x\restriction n} \cap F \neq \emptyset$. We know that F is closed and hence that $[0,1] \cap F$ is compact. If $x \notin F$, then

$$F \cap \left(\bigcap_{n \in \omega} I_{x\restriction n} \right) = \emptyset.$$

We have already seen, however, that $F \cap I_{x\restriction n} = F \cap \overline{I}_{x\restriction n}$ for every $n \in \omega$. Consequently

$$[0,1] \cap F \subseteq \bigcup_{n \in \omega} \left(\overline{I}_{x\restriction n} \right)^{\complement}.$$

By compactness this cover must have a finite sub-cover, i. e. there is $N \in \omega$ such that

$$[0,1] \cap F \subseteq \bigcup_{n \leq N} \left(\overline{I}_{x\restriction n} \right)^{\complement}.$$

But this means that

$$([0,1] \cap F) \cap \overline{I}_{x\restriction N+1} = \emptyset.$$

Of course $\overline{I}_{x\restriction N+1} \subseteq [0,1]$ and so we have shown that $F \cap \overline{I}_{x\restriction N+1} = \emptyset$. As before $F \cap \overline{I}_{x\restriction N+1} = F \cap I_{x\restriction N+1}$. This is a contradiction of our assumption that $F \cap I_{x\restriction n} \neq \emptyset$ for every $n \in \omega$. Therefore $x \in F$. This completes the proof of the claim.

We now define

$$\mathcal{U}_i = \left\{ E \in \mathcal{F}_{[0,1]} : (\exists n \geq i)(\exists \sigma \in 2^n) I_{\sigma^\frown 0} \cap E \neq \emptyset \ \& \ I_{\sigma^\frown 1} \cap E \neq \emptyset \right\}.$$

We wish to show that the sequence $\{\mathcal{U}_i\}_{i\in\omega}$ is uniformly Σ^0_1. This is clear if we recall that for $\sigma \in 2^{<\omega}$ the set $\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}}$ is by defined by

$$\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}} = \{E \in \mathcal{F}(\mathbb{R}) : I_{\sigma^\frown 0} \cap E \neq \emptyset \ \& \ I_{\sigma^\frown 1} \cap E \neq \emptyset\}.$$

Hence

$$\mathcal{U}_i = \bigcup \left\{\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}} : \sigma \in 2^n,\ n \geq i\right\}.$$

This is clearly Σ^0_1.

Furthermore,

$$\mathbf{P}_T(\mathcal{U}_i) \leq \sum_{n\geq i} \left[\sum_{\sigma\in 2^n} \mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}}\right)\right] = \sum_{n\geq i} 2^n \left(1 - 2^{-2^{-(n+1)}}\right)^2.$$

By the familiar ratio test of calculus this series converges. In fact, the ratio rest shows that there is $N \in \omega$ and $q \in (0,1) \cap \mathbb{Q}$ such that for $n \geq N$

$$2^n \left(1 - 2^{-2^{-(n+1)}}\right)^2 < q^n.$$

Using this bound we can then pick out a subsequence $\{\mathcal{U}_{i_j}\}_{j\in\omega}$ such that $\mathbf{P}_T(\mathcal{U}_{i_j}) \leq 2^{-j}$. Therefore we have a $\mathbf{P}_T$-ML test.

A number $x \in [0,1] \cap F$ is not isolated in F if and only if

$$(\forall n \in \omega)(\exists m \geq n)\left(I_{(x\restriction m)^\frown 0} \cap F \neq \emptyset \ \& \ I_{(x\restriction m)^\frown 1} \cap F \neq \emptyset\right).$$

Thus, if there is $x \in [0,1] \cap F$ that is not isolated in F, then $F \in \bigcap_{i\in\omega} \mathcal{U}_i$ and so F is not $\mathbf{P}_T$-ML random. Therefore if F is $\mathbf{P}_T$-ML-random and $x \in [0,1] \cap F$, then x is isolated in F. $\square$

This proposition has some nice corollaries.

Corollary 5.1.11. *If $F \in \mathcal{F}(\mathbb{R})$ is $\mathbf{P}_T$-Martin-Löf random, then every $x \in F$ is isolated in F.*

Proof. Suppose F is $\mathbf{P}_T$-ML random and $x \in F$. There is $c \in \mathbb{Q}$ such that $x + c \in [0, 1]$. By lemma 5.1.4 $F + c$ is $\mathbf{P}_T$-ML random. Hence $x + c$ is isolated in $F + c$. Therefore x is isolated in F. □

Corollary 5.1.12. *If $F \in \mathcal{F}(\mathbb{R})$ is $\mathbf{P}_T$-Martin-Löf random, then F is countable.*

Proof. Let $F \in \mathcal{F}(\mathbb{R})$ be $\mathbf{P}_T$-ML random. By corollary 5.1.11 for every $n \in \mathbb{Z}$ each point of $F \cap [n, n+1]$ is isolated in F. For each $x \in F \cap [n, n+1]$ let $N_{n,x} \subseteq \mathbb{R}$ be an open neighborhood of x such that $N_{n,x} \cap F = \{x\}$. Then $\{N_{n,x} : x \in F \cap [n, n+1]\} \cup \{F^{\complement}\}$ is an open cover of $[n, n+1]$. Because $[n, n+1]$ is compact there is a finite sub-cover. This means that $F^{\complement}$ contains all but finitely many points of $F \cap [n, n+1]$. In other words, $F \cap [n, n+1]$ is finite. This is true for each $n \in \omega$ and therefore F is countable. □

It is easy to calculate using Robbins' theorem that $E(m(F)) = 0$. An effective version of this is also easy to prove using the previous corollary.

Corollary 5.1.13. *If $F \in \mathcal{F}(\mathbb{R})$ is $\mathbf{P}_T$-ML random, then $m(F) = 0$.*

Proof. The set F is countable and hence $m(F) = 0$. □

We now discuss why our random closed sets are called generalized Poisson processes. A Poisson process with rate λ is a family of random variables $N(t)$ such that for $t > 0$ $N(t)$ is the greatest $n \in \omega$ such that $X_0 + X_1 + \cdots + X_n \leq t$ (where $X_1, X_2, \ldots$ is an i.i.d. sequence of exponential random variables with mean λ). One of the canonical examples of a Poisson process is the number of particles emitted from a radioactive source by time t (with the caveat that the Poisson process is only a good model over periods of time too short for the radioactive source to stabilize). In this example the times between the emission of particles are the exponential random variables. The probability that a particle was emitted in the interval $(0, t)$, for example is $1 - e^{-\lambda t}$. The collection of emission times for the radioactive source is also an unbounded set of isolated points. This collection is something very much like one of our random closed sets.

More precisely, if we define $N(t) : \mathcal{F}(\mathbb{R}) \to [0, 1]$ by $N(t)(F) = |F \cap [0, t]|$, then $N(t)$ is a Poisson process (under the measure $\mathbf{P}_T$). Similarly $N'(t) : \mathcal{F}(\mathbb{R}) \to [0, 1]$ defined by $N'(t)(F) = |F \cap [-t, 0]|$, is also a Poisson process. This connection is one reason for calling our RACS a generalized Poisson process. The reason the word "generalized" has been added is that our RACS works equally well in $\mathbb{R}^n$, where it is not quite so clear what the traditional version of a Poisson process should be. Many of the preceding results describing the $\mathbf{P}_T$-Martin-Löf random closed sets are effective versions of classical descriptions of Poisson processes.

The following discussion may be helpful in understanding the nature of $\mathbf{P}_T$. It may safely be skipped, however. The goal here is to understand the likely behavior of sets in increasingly small intervals. As before we may restrict our attention to $\mathcal{F}_{[0,1]}$.

Definition 5.1.14. Let $\mathcal{H}_0, \mathcal{H}_1 \subseteq \mathcal{F}$ be measurable sets such that $\mathbf{P}_T(\mathcal{H}_1) > 0$. The conditional probability $\mathbf{P}_T(\mathcal{H}_0|\mathcal{H}_1)$ is defined to be

$$\frac{\mathbf{P}_T(\mathcal{H}_0 \cap \mathcal{H}_1)}{\mathbf{P}_T(\mathcal{H}_1)}.$$

We will continue to work with the intervals I_σ for $\sigma \in 2^{<\omega}$. Let $n \geq 1$ and let $\sigma \in 2^{n-1}$. Consider the conditional probability $\mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown i}}|\mathcal{F}_{I_\sigma}\right)$ for $i = 0, 1$. This is the probability of a closed set containing a point of $I_{\sigma^\frown i}$ given that it contains a point of I_σ. Because $I_{\sigma^\frown i} \subseteq I_\sigma$ it follows that

$$\begin{aligned}\mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown i}, I_\sigma}\right) &= \mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown i}}\right) \\ &= 1 - 2^{-2^{-n}}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown i}} | \mathcal{F}_{I_\sigma}\right) &= \frac{1 - 2^{-2^{-n}}}{1 - 2^{-2^{-(n-1)}}} \\ &= \frac{1 - 2^{-2^{-n}}}{\left(1 - 2^{-2^{-n}}\right)\left(1 + 2^{-2^{-n}}\right)} \\ &= \frac{1}{1 + 2^{-2^{-n}}}.\end{aligned}$$

Note that as $|\sigma| = n$ goes to infinity this probability increases to $\frac{1}{2}$. This means that the probability of a set containing a point in $I_{\sigma^\frown i}$ given that it contains a point of I_σ is usually fairly high.

Now consider $\mathbf{P}_T(\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}} | \mathcal{F}_{I_\sigma})$, the probability that a set contains a point in both halves of I_σ given that it contains a point of I_σ:

$$\begin{aligned}\mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}} \mid \mathcal{F}_{I_\sigma}\right) &= \frac{\mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}, I_\sigma}\right)}{\mathbf{P}_T\left(\mathcal{F}_{I_\sigma}\right)} \\ &= \frac{\mathbf{P}_T\left(\mathcal{F}_{I_{\sigma^\frown 0}, I_{\sigma^\frown 1}}\right)}{\mathbf{P}_T\left(\mathcal{F}_{I_\sigma}\right)} \\ &= \frac{\left(1 - 2^{-2^{-n}}\right)^2}{1 - 2^{-2^{-(n-1)}}} \\ &= \frac{1 - 2^{-2^{-n}}}{1 + 2^{-2^{-n}}}.\end{aligned}$$

As $|\sigma| = n$ goes to infinity this probability decreases to 0. Thus the probability of splittings decreases to 0 as the size of the intervals I_σ decreases. Note that we have already made use of this fact in proposition 5.1.10.

5.2 More on generalized Poisson processes

As in the last section we wish to consider a capacity $T : \mathcal{K}(\mathbb{R}) \to [0, 1]$ given by $T(K) = 1 - e^{-\lambda m(k)}$ where m is the Lebesgue measure on $\mathbb{R}$ and $\lambda \in (0, \infty)$. In this section, however, we will focus on the situation when λ is not computable. This produces new concerns for Martin-Löf randomness and brings up the possibility of different behavior for Hippocrates randomness (see definition 3.0.7).

We have already seen in proposition 5.1.2 that $\mathbf{P}_T \equiv_T T \equiv_T \lambda$. This allows us to state the following relativized version of theorem 5.1.5.

Theorem 5.2.1. *Let T be the capacity given by $T(K) = 1 - e^{-\lambda m(K)}$. A real number $x \in \mathbb{R}$ is m-Martin-Löf random relative to oracle λ if and only if there is a $\mathbf{P}_T$-Martin-Löf random closed set F such that $x \in F$.*

Proof. ($\Leftarrow$) The proof that if F is $\mathbf{P}_T$-Martin-Löf random, then every element of F is m-Martin-Löf random relativizes directly by including the oracle $\mathbf{P}_T$ everywhere. For non-computable $\mathbf{P}_T$ the statement becomes: F is $\mathbf{P}_T$-Martin-Löf random, then every element of F is Martin-Löf random relative to $\mathbf{P}_T$. By proposition 5.1.2 $\mathbf{P}_T \equiv_T \lambda$ and so we have the desired statement.

($\Rightarrow$) The other direction also relativizes directly. Key here is the existence of a universal Martin-Löf test relative to $\mathbf{P}_T$ (see lemma 3.0.8). The relativized proof shows that if $r \in \mathbb{R}$ is not contained in and $\mathbf{P}_T$-Martin-Löf test, then r is not m-Martin-Löf random relative to $\mathbf{P}_T$. Again by proposition 5.1.2 $\mathbf{P}_T \equiv_T \lambda$ and so we have the desired statement. □

Note that the other results describing the topological properties $\mathbf{P}_T$-Martin-Löf random closed sets also either relativize or simply apply directly in our current situation. We will make use of the relativized versions of some of them in the following. In particular we have the following.

Proposition 5.2.2. *If $F \in \mathcal{F}(\mathbb{R})$ is $\mathbf{P}_T$-Martin-Löf random, then F is countable, discrete, and unbounded.*

We might also ask to what extent theorem 5.2.1 holds if we change the measure but do not add extra oracular information about the measure, i.e. if we switch to Hippocrates randomness. Hippocrates randomness is defined in 3.0.7. We can prove that Hippocrates randomness is strong enough to ensure that members of $\mathbf{P}_T$-Hippocrates random closed sets are themselves m-random. Recall that every $\mathbf{P}_T$-Martin-Löf random closed set is also $\mathbf{P}_T$-Hippocrates random but that the converse is not necessarily true.

Proposition 5.2.3. *If $F \in \mathcal{F}(\mathbb{R})$ is $\mathbf{P}_T$-Hippocrates random, then every element of F is m-Martin-Löf random.*

Proof. We follow the proof of theorem 5.1.5. Suppose that $F \in \mathcal{F}(\mathbb{R})$ and $r \in F$. Suppose also that there is an m-ML test $\{U_i\}_{i\in\omega}$ such that $r \in \bigcap_{i\in\omega} U_i$. Then $\{\mathcal{F}_{U_i}\}_{i\in\omega}$ is a uniformly Σ^0_1 sequence of subsets of $\mathcal{F}(\mathbb{R})$. We cannot compute $\mathbf{P}_T\left(\mathcal{F}_{U_i}\right)$ from $m(U_i)$ as we did before because this time λ is not computable. We can, however, find compute upper bounds for the measures of these sets. Let $c \in \mathbb{Q}$ be such that $\ln c \geq \lambda$. Then

$$\mathbf{P}_T\left(\mathcal{F}_{U_i}\right) = 1 - e^{-\lambda m(U_i)} \leq 1 - c^{-m(U_i)}.$$

Consequently, if $i \geq \log_2 \log_c \left(1 - 2^{-j}\right)$, then $\mathbf{P}_T\left(\mathcal{F}_{U_i}\right) \leq 2^{-j}$. Therefore $\{\mathcal{F}_{U_i}\}_{i\in\omega}$ gives a $\mathbf{P}_T$-Hippocrates test and $F \in \bigcap_{i\in\omega} \mathcal{F}_{U_i}$. Hence F is not $\mathbf{P}_T$-Hippocrates random. Since this holds for every $r \in F$ and every m-ML test $\{U_i\}_{i\in\omega}$ the result follows. $\square$

This begs the question of what happens with the other direction of theorem 5.2.1 in the case of Hippocrates randomness. That is, are there m-Martin-Löf random reals that avoid $\mathbf{P}_T$-Hippocrates random closed sets for non-computable λ? This question remains open.

One might also look at random reals under different measures on $\mathbb{R}$ and ask if theorem 5.2.1 generalizes. Specifically, given a Borel measure μ on $\mathbb{R}$, is there a measure $\mathbf{P}$ on $\mathcal{F}(\mathbb{R})$ such that $r \in \mathbb{R}$ is μ-Martin-Löf random if and only if there is a $\mathbf{P}$-Martin-Löf random closed set F with $r \in F$? If μ is regular, $\mu(K) < \infty$ for every compact $K \subseteq \mathbb{R}$, and $\mu(\overline{I}) = \mu(I)$ for every open rational interval I, then we can answer this question in the affirmative. In this situation the functional $K \mapsto 1 - e^{-\mu(K)}$ is a Choquet capacity by proposition 2.4.23 (this is where we use the condition that $\mu(K) < \infty$). Let $\mathbf{P}$ be the measure on $\mathcal{F}$ generated by this capacity. Because $\mu(\overline{I}) = \mu(I)$ for any open rational interval I, the proof of lemma 5.1.2 can be adapted to show that $\mu \equiv_T \mathbf{P}$. The proof of theorem 5.1.5 can then be relativized to the measure μ to show that $r \in \mathbb{R}$ is μ-Martin-Löf random if and only if there is a $\mathbf{P}$-Martin-Löf random closed set F with $r \in F$. We note that the regularity of μ is important in working out the details of this proof.

We now turn our attention to selection functions, an important part of the theory of RACS.

Definition 5.2.4. Let Ω be a probability space, $\mathbb{E}$ an LCHS space, and $X : \Omega \to \mathcal{F}(\mathbb{E})$ a random closed set. A measurable function $s : \Omega \to \mathbb{E}$ is a *selection* for X if $s(\alpha) \in X(\alpha)$ for almost every $\alpha \in \omega$.

Selection functions are used extensively in the study of RACS. They can be used to define a kind of expected value for a RACS, called the selection expectation (see chapter 2 of Molchanov [18]). This is beyond the scope of our current work, however. For our purposes it is most important to note that a selection function induces an image measure, μ, on $\mathbb{E}$ (called the distribution of s). We recall that μ is defined by $\mu(A) = P(s^{-1}(A))$ for measurable $A \subseteq \mathbb{E}$. If s is a selection for a RACS X, then we have the following for any compact $K \subseteq \mathbb{E}$:

$$\begin{aligned} \mu(K) &= P\left(s^{-1}(K)\right) \\ &= P\left(\{\alpha \in \Omega : s(\alpha) \in K\}\right) \\ &\leq P\left(\{\alpha \in \Omega : X(\alpha) \cap K \neq \emptyset\}\right) \\ &= P\left(X^{-1}\left(\mathcal{F}_K\right)\right) \\ &= T_X\left(K\right) \end{aligned}$$

where T_X is the capacity induced by X. We have shown that if μ is the distribution of a selection for X, then $\mu(K) \leq T(K)$ for any compact $K \subseteq \mathbb{E}$.

The following theorem proves that the converse also holds. That is, if μ is a measure and T is a Choquet capacity such that $\mu(K) \leq T(K)$ for compact $K \subseteq \mathbb{E}$, then there is a RACS with capacity T a selection function for the RACS with distribution μ. A proof can be found on page 35 of Molchanov [18].

Theorem 5.2.5 (Artstein's Selectionability Theorem)**.** *Let $\mathbb{E}$ be a Polish space, μ be a probability measure on $\mathbb{E}$, and T be a Choquet capacity. Then there is a random closed set X and a selection s for X such that T is the capacity for X and μ is the distribution of s if and only if*

$$(\forall K \in \mathcal{K}) \;\; T(K) \geq \mu(K).$$

We now return to the situation of the measure $\mathbf{P}_T$ induced by the capacity $T(K) = 1 - e^{-\lambda m(K)}$. We have seen that the elements of any $\mathbf{P}_T$-Martin-Löf random set are themselves m-Martin-Löf random relative to λ. We can use the ideas of selection functions to characterize the elements of $\mathbf{P}_T$-Martin-Löf random sets in a new way. We note that the identity function $F \mapsto F$ is a RACS under the measure $\mathbf{P}_T$. We follow the custom and abuse language slightly by calling this RACS $\mathbf{P}_T$. We prove in theorem 5.2.6 that the elements of a $\mathbf{P}_T$-Martin-Löf random closed set are exactly the reals that are Hippocrates random under the distribution of some selection function of the RACS $\mathbf{P}_T$.

We note that theorem 5.2.6 is not a generalization of theorems 5.1.5 and 5.2.1. In the case of theorems 5.1.5 and 5.2.1 we are able to determine that reals are m-Martin-Löf random if and only if they are members of $\mathbf{P}_T$-Martin-Löf random closed sets. In this theorem we will find that reals are Hippocrates random with respect to *some* measure μ with certain properties if and only they are members of a $\mathbf{P}_T$-Martin-Löf random closed set. In this case one of the properties of the measure μ is that $\mu(K) \leq T(K)$ for every compact $K \subseteq \mathbb{R}$. Every measure μ on $\mathbb{R}$ such that $\mu(K) \leq T(K)$ must satisfy the inequality $\mu(\mathbb{R}) \leq 1$. This means that the following theorem is only applies probability measures on $\mathbb{R}$. The Lebesgue measure m is not a probability measure and so theorem 5.2.6 is addressing a very different situation than theorems 5.1.5 and 5.2.1.

The following theorem is the result of joint work with Bjørn Kjos-Hanssen.

Theorem 5.2.6. *Let $T : \mathcal{K}(\mathbb{R}) \to [0,1]$ be a generalized Poisson process given by $T(K) = 1 - e^{-\lambda m(K)}$ with $\lambda \in (0,\infty)$. Let $x \in \mathbb{R}$. Then there is a $\mathbf{P}_T$-Martin-Löf random closed set F such that $x \in F$ if and only if there is a regular Borel measure μ on $\mathbb{R}$ such that $(\forall K \in \mathcal{K})\ \mu(K) \leq T(K)$ and x is μ-Hippocrates random relative* *to oracle* λ.

Proof. First we note that by lemma 5.1.2 $\mathbf{P}_T \equiv_T T \equiv_T \lambda$.

($\Rightarrow$) Let F be a $\mathbf{P}_T$-ML random closed set and $x \in F$. We produce a probability measure μ such that x is μ-H random relative to λ and $\mu(K) \leq T(K)$ for each compact set K. We do this by finding a selection function s for the RACS $\mathbf{P}_T$, such that $s(F) = x$. It is not actually important for our proof that the function s is a selection for $\mathbf{P}_T$, it suffices that s be measurable. It seems to be worth noting, however, that the function we define is a selection. Our measure is then the distribution of the selection.

Because F is $\mathbf{P}_T$-ML random we know that each element of F is m-ML random and that F is discrete (by theorem 5.2.1 and proposition 5.2.2). Hence here is $q \in \mathbb{Q}$ such that x is the least member of F larger than q. We note that $q \notin F$.

Define the map $s : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ by

$$s(E) = \begin{cases} \min\left(E \cap [q, \infty)\right) & \text{if } E \cap [q,\infty) \neq \emptyset \\ q & \text{otherwise.} \end{cases}$$

We claim that s is a selection function for the RACS $\mathbf{P}_T$. To prove the claim we must show that s is measurable and that $s(E) \in E$ for $\mathbf{P}_T$-almost every $E \in \mathcal{F}$. To prove that s is measurable is suffices to show that $s^{-1}\left((a,b)\right)$ is measurable for any $a < b \in \mathbb{Q}$.

$$s^{-1}\left((a,b)\right) = \begin{cases} \emptyset & b \leq q \\ \mathcal{F}_{[q,b)} \cup \mathcal{F}^{[q,\infty)} & a < q < b \\ \mathcal{F}_{(a,b)} & q \leq a \end{cases} \tag{5.3}$$

The sets $\emptyset$ and $\mathcal{F}_{(a,b)}$ are clearly measurable but we must prove that $\mathcal{F}_{[q,b)} \cup \mathcal{F}^{[q,\infty)}$ is measurable.

We first claim that

$$\mathcal{F}_{[q,b)} = \bigcap_{i\in\omega} \mathcal{F}_{(q-2^{-i},b)}.$$

Clearly $\mathcal{F}_{[q,b)} \subseteq \bigcap_{i\in\omega} \mathcal{F}_{(q-2^{-i},b)}$. To show the reverse containment, suppose that $\mathcal{F}_{[q,b)} \subsetneq \bigcap_{i\in\omega} \mathcal{F}_{(q-2^{-i},b)}$ and let $E \in \left(\bigcap_{i\in\omega} \mathcal{F}_{(q-2^{-i},b)}\right) \setminus \mathcal{F}_{[q,b)}$. This means that for every $i \in \omega$, $E \cap (q-2^{-i}, b) \neq \emptyset$ but also that $E \cap [q,b) = \emptyset$. Hence for every $i \in \omega$, $E \cap (q-2^{-i}, q) \neq \emptyset$.

Define $\hat{E} = E \cap [q-1, q]$. Then $\hat{E}$ is closed and bounded and hence compact. In addition, $\hat{E} \cap [q,b) = \emptyset$ and hence $\hat{E} \subseteq \bigcup_{i\in\omega} (q-2, q-2^{-i})$. By compactness there is $N \in \omega$ such that $\hat{E} \subseteq \bigcup_{i\leq N} (q-2, q-2^{-i})$. Consequently $\hat{E} \cap \left(q-2^{-(N+1)}, q\right) = \emptyset$. But $\hat{E} \cap \left(q-2^{-(N+1)}, q\right) = E \cap \left(q-2^{-(N+1)}, q\right)$ and so this is a contradiction. Therefore $\mathcal{F}_{[q,b)} = \bigcap_{i\in\omega} \mathcal{F}_{(q-2^{-i},b)}$ and so $\mathcal{F}_{[q,b)}$ is measurable.

Now $\mathcal{F}^{[q,\infty)} = \bigcap_{i\geq 1} \mathcal{F}^{[q,q+i]}$. This means that $\mathcal{F}^{[q,\infty)}$ is measurable. Hence $\mathcal{F}_{[q,b)} \cup \mathcal{F}^{[q,\infty)}$ is a measurable set. Therefore s is a measurable function.

We now prove that $s(E) \in E$ for P_T-almost every $E \in \mathcal{F}$. As noted above, this is actually not an essential part of the proof. If desired the reader may skip to the paragraph beginning "We can now define our measure." By definition $s(E) \in E$ if and only if $E \cap [q,\infty) \neq \emptyset$. Thus we must actually prove that

$$\mathbf{P}_T\left(\{E \in \mathcal{F} : E \cap [q,\infty) = \emptyset\}\right) = 0.$$

By definition $\{E \in \mathcal{F} : E \cap [q,\infty) = \emptyset\} = \mathcal{F}^{[q,\infty)}$. By our preceding calculations

$\mathcal{F}^{[q,\infty)} = \bigcap_{i\geq 1} \mathcal{F}^{[q,q+i]}$. Furthermore $\mathcal{F}^{[q,q+1]} \supseteq \mathcal{F}^{[q,q+2]} \supseteq \ldots$. Thus

$$\begin{aligned}\mathbf{P}_T\left(\mathcal{F}^{[q,\infty)}\right) &= \mathbf{P}_T\left(\bigcap_{i\geq 1} \mathcal{F}^{[q,q+i]}\right) \\ &= \lim_{i\to\infty} \mathbf{P}_T\left(\mathcal{F}^{[q,q+i]}\right) \\ &= \lim_{i\to\infty} e^{-\lambda i} \\ &= 0.\end{aligned}$$

Therefore $s(E) \in E$ for P_T-almost every $E \in \mathcal{F}$. This completes the proof that s is a selection.

We can now define our measure:

$$\mu(K) = \mathbf{P}_T\left(s^{-1}(K)\right).$$

The measure μ extends to a regular Borel measure on $\mathbb{R}$ by Carathéodory's construction. It is clear from equation 5.3 that $s^{-1}(K) \subseteq \mathcal{F}_K \cup \mathcal{F}^{[q,\infty)}$. It follows that

$$\mu(K) = \mathbf{P}_T\left(s^{-1}(K)\right) \leq \mathbf{P}_T(\mathcal{F}_K) = T(K).$$

We have now produced a regular Borel measure μ such that $\mu(K) \leq T(K)$ for every compact $K \subseteq \mathbb{R}$. It remains to be shown that x is μ-H random relative to λ. We proceed by contradiction.

Suppose that $\{U_i\}_{i\in\omega}$ is a μ-H test relative to λ such that $x \in \bigcap_{i\in\omega} U_i$. We then define $\mathcal{V}_i = s^{-1}(U_i) \cap \mathcal{F}^{\{q\}}$. From equation 5.3 we know that for $a < b \in \mathbb{Q}$

$$s^{-1}((a,b)) \cap \mathcal{F}^{\{q\}} = \begin{cases} \emptyset & b \leq q \\ \mathcal{F}_{(q,b)} \cup \mathcal{F}^{[q,\infty)} & a < q < b \\ \mathcal{F}^{\{q\}}_{(a,b)} & q \leq a. \end{cases}$$

In the first and third cases $s^{-1}((a,b)) \cap \mathcal{F}^{\{q\}}$ is a basic open set in the Fell topology. For the middle case have shown that $\mathcal{F}^{[q,\infty)} = \bigcup_{i \geq 1} \mathcal{F}^{[q,q+i]}$. Hence $\mathcal{F}_{(q,b)} \cup \mathcal{F}^{[q,\infty)}$ is Σ^0_1. Thus $\{\mathcal{V}_i\}_{i \in \omega}$ is a uniformly $\Sigma^{0,\lambda}_1$ sequence. Moreover, $\mathcal{V}_i \subseteq s^{-1}(U_i)$ and therefore

$$\mathbf{P}_T(\mathcal{V}_i) \leq \mathbf{P}_T\left(s^{-1}(U_i)\right) = \mu(U_i) \leq 2^{-i}.$$

In other words, $\{\mathcal{V}_i\}_{i \in \omega}$ is a $\mathbf{P}_T$-ML test.

Recall that $q \notin F$. Moreover, for each $i \in \omega$, $x \in F \cap U_i$. Hence $F \in \mathcal{F}^{\{q\}}_{U_i} = \mathcal{V}_i$ for every $i \in \omega$, i.e. $F \in \bigcap_{i \in \omega} \mathcal{V}_i$. This contradicts the assumption that F is $\mathbf{P}_T$-ML random. Therefore x must be μ-H random relative to oracle λ.

($\Leftarrow$) This direction of the proof is implicit in Kjos-Hanssen [13]. For completeness we provide a (different) proof. This proof is essentially the same as the proof of the corresponding direction of theorem 5.1.5.

Let T be a generalized Poisson process with rate $\lambda \in (0,\infty)$ and let μ be a regular Borel measure on $\mathbb{R}$ such that $\mu(K) \leq T(K)$ for all compact $K \subseteq \mathbb{R}$. Suppose that $r \in \mathbb{R}$ is such that no closed set containing r is $\mathbf{P}_T$-ML random. We will show that r is not μ-H random relative to oracle λ.

Because no $\mathbf{P}_T$-ML random closed sets contain r it follows that $\mathcal{F}_{\{r\}}$ consists entirely of non-random sets. We have exactly the same topological situation as in the proof of theorem 5.1.5. Consequently if $\{\mathcal{U}_i\}_{i \in \omega}$ is a universal $\mathbf{P}_T$-ML test, then for each $i \in \omega$ there is a closed rational interval J_i such that $r \in J_i$ and $\mathcal{F}_{J_i} \subseteq \mathcal{U}_i$. As before we wish to use this knowledge that these intervals exist to construct a randomness test that catches r. This time, however, we are using the rate λ as an oracle. We do not use μ as an oracle, however, just the fact that $\mu(K) \leq T(K)$ for compact $K \subseteq \mathbb{R}$.

We wish to construct an H-test relative to oracle λ, $\{V_i\}_{i\in\omega}$, such that $r \in \bigcap_{i\in\omega} V_i$. Let $\{\mathcal{U}_i\}_{i\in\omega}$ be a universal $\mathbf{P}_T$-ML test. Note that this means that $\mathcal{U}_n$ is $\Sigma_1^{0,\lambda}$. Let $\mathcal{U}_{n,s}$ be the stage s approximation to $\mathcal{U}_n$. Let $I_0, I_1, \ldots$ be the standard enumeration of all open rational intervals of $\mathbb{R}$. Note that $\mathcal{F}_{\overline{I}_i}$ is compact for every $i \in \omega$. Because $\mathcal{F}_{\overline{I}_i}$ is compact it follows that $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_n$ if and only if there is a stage s such that $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n,s}$. For $n \geq 1$ we watch for intervals such that $\overline{I}_i \subseteq \mathcal{U}_{n,s}$ and use them to build V_{n-1}. It is uniformly computable in λ to decide if $\overline{I}_i \subseteq \mathcal{U}_{n,s}$ by the same arguments used in the proof of theorem 5.1.5. At the same time we will construct a set $C_{n-1} \subseteq V_{n-1}$ that we will use later to calculate the measure of V_{n-1}.

We build V_n and C_n in stages. At stage 0 we set $V_{n,0} = C_{n,0} = \emptyset$. We note that at each stage s of the construction $C_{n,s}$ is compact.

Suppose that we have enumerated j intervals into $V_{n,s}$ (with $s \geq 1$). We set $V_{n,t} = V_{n,s}$ until the next stage t such that there is $i \leq t$ with $\overline{I}_i \nsubseteq V_{n,s}$ and $\mathcal{F}_{\overline{I}_i} \subseteq \mathcal{U}_{n+1,t}$. Let i be the least index such that $\overline{I}_i$ satisfies these conditions. We then take an open rational interval I such that $\overline{I}_i \subseteq I$ and

$$T\left(\overline{\left(I \setminus \overline{I}_i\right)}\right) \leq 2^{-(2+n+j)}. \tag{5.4}$$

We then set $V_{n,t} = V_{n,t-1} \cup I$ and $C_{n,t} = C_{n,t-1} \cup \overline{I}_i$.

Let $V_n = \bigcup_{s\in\omega} V_{n,s}$ and $C_n = \bigcup_{s\in\omega} C_{n,s}$. Equation 5.4 ensures that V_n will not be much larger than C_n. The total error, $\mu(V_n \setminus C_n)$, is no more than the sum over $j \in \omega$ of the errors $\mu\left(I \setminus \overline{I}_i\right)$. But

$$\begin{aligned}\mu\left(I \setminus \overline{I}_i\right) &\leq \mu\left(\overline{\left(I \setminus \overline{I}_i\right)}\right) \\ &\leq T\left(\overline{\left(I \setminus \overline{I}_i\right)}\right) \\ &\leq 2^{-(2+n+j)}.\end{aligned}$$

Consequently

$$\mu(V_n \setminus C_n) \leq \sum_{j \in \omega} 2^{-(2+n+j)} = 2^{-(n+1)}.$$

By construction V_n is $\Sigma_1^{0,\lambda}$ and we note that this is uniform over $n \in \omega$. We have also constructed $C_n \subseteq V_n$ such that

$$\begin{aligned}\mu(V_n) &= \mu(C_n) + \mu(V_n \setminus C_n) \\ &\leq \mu(C_n) + 2^{-(n+1)}.\end{aligned} \tag{5.5}$$

We must now calculate an upper bound for $\mu(C_n)$.

By regularity $\mu(C_n) = \lim_{s\to\infty} \mu(C_{n,s})$. But $C_{n,s}$ is compact for every $s \in \omega$ and hence $\mu(C_{n,s}) \leq T(C_{n,s})$. Consequently $\mu(C_n) \leq \lim_{s\to\infty} T(C_{n,s})$. By definition

$$\lim_{s\to\infty} T(C_{n,s}) = \lim_{s\to\infty} \mathbf{P}_T\left(\mathcal{F}_{C_{n,s}}\right) = \mathbf{P}_T\left(\mathcal{F}_{C_n}\right).$$

By construction $\mathcal{F}_{C_n} \subseteq \mathcal{U}_{n+1}$ and hence $\mathbf{P}_T(\mathcal{F}_{C_n}) \leq 2^{-(n+1)}$. Therefore

$$\mu(C_n) \leq 2^{-(n+1)}.$$

Substituting this inequality back into equation 5.5 gives $\mu(V_n) \leq 2^{-n}$. Therefore $\{V_i\}_{i\in\omega}$ is a μ-H test relative to oracle λ. Exactly as in the proof of theorem

5.1.5 we know that $r \in V_i$ for each $i \in \omega$. Consequently r in not μ-H random relative to oracle λ. Therefore if $r \in \mathbb{R}$ is μ-Hippocrates random relative to λ, then there is a $\mathbf{P}_T$-Martin-Löf random $F \in \mathcal{F}(\mathbb{R})$ such that $r \in F$. □

REFERENCES

1. George Barmpalias, Paul Brodhead, Douglas Cenzer, Seyyed Dashti, and Rebecca Weber. Algorithmic randomness of closed sets. *J. Logic Comput.*, 17 (6):1041–1062, 2007. ISSN 0955-792X.

2. Paul Brodhead. Computable aspects of closed sets. Ph.D. Thesis, University of Florida, 2008.

3. Gustave Choquet. Theory of capacities. *Ann. Inst. Fourier, Grenoble*, 5: 131–295 (1955), 1953–1954. ISSN 0373-0956.

4. R. Daniel Mauldin and Alexander P. McLinden. Random closed sets viewed as random recursions. *Arch. Math. Logic*, 48(3-4):257–263, 2009. ISSN 0933-5846.

5. Rod Downey, Denis R. Hirschfeldt, André Nies, and Sebastiaan A. Terwijn. Calibrating randomness. *Bull. Symbolic Logic*, 12(3):411–491, 2006. ISSN 1079-8986.

6. Kenneth Falconer. *Fractal geometry.* John Wiley & Sons Inc., Hoboken, NJ, second edition, 2003. ISBN 0-470-84861-8. Mathematical foundations and applications.

7. Gerald B. Folland. *Real analysis.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1999. ISBN 0-471-31716-0. Modern techniques and their applications, A Wiley-Interscience Publication.

8. Willem Fouché. Arithmetical representations of Brownian motion. I. *J. Symbolic Logic*, 65(1):421–442, 2000. ISSN 0022-4812.

9. Willem L. Fouché. The descriptive complexity of Brownian motion. *Adv. Math.*, 155(2):317–343, 2000. ISSN 0001-8708.

10. Siegfried Graf, R. Daniel Mauldin, and S. C. Williams. The exact Hausdorff dimension in random recursive constructions. *Mem. Amer. Math. Soc.*, 71 (381):x+121, 1988. ISSN 0065-9266.

11. Geoffrey R. Grimmett and David R. Stirzaker. *Probability and random processes*. Oxford University Press, New York, third edition, 2001. ISBN 0-19-857223-9.

12. Peter Hertling and Klaus Weihrauch. Random elements in effective topological spaces with measure. *Inform. and Comput.*, 181(1):32–56, 2003. ISSN 0890-5401.

13. Bjørn Kjos-Hanssen. Infinite subsets of random sets of integers. *Math. Res. Lett.*, 16(1):103–110, 2009. ISSN 1073-2780.

14. Bjørn Kjos-Hanssen and David Diamondstone. Members of random closed sets. In *CiE 2009*, volume 5635 of *Lecture Notes in Computer Science*, pages 144–153. Springer, Berlin, 2009.

15. Bjørn Kjos-Hanssen and Anil Nerode. The law of the iterated logarithm for algorithmically random Brownian motion. In *Logical foundations of computer science*, volume 4514 of *Lecture Notes in Computer Science*, pages 310–317. Springer, Berlin, 2007.

16. G. Matheron. *Random sets and integral geometry*. John Wiley & Sons, New York-London-Sydney, 1975. With a foreword by Geoffrey S. Watson, Wiley Series in Probability and Mathematical Statistics.

17. R. Daniel Mauldin and S. C. Williams. Random recursive constructions: asymptotic geometric and topological properties. *Trans. Amer. Math. Soc.*, 295(1):325–346, 1986. ISSN 0002-9947.

18. Ilya Molchanov. *Theory of random sets*. Probability and its Applications (New York). Springer-Verlag London Ltd., London, 2005. ISBN 978-185223-892-3; 1-85233-892-X.

19. Hung T. Nguyen. *An introduction to random sets*. Chapman & Hall/CRC, Boca Raton, FL, 2006. ISBN 978-1-58488-519-1; 1-58488-519-X.

20. André Nies. *Computability and Randomness*. Oxford Logic Guides. Oxford University Press, New York, USA, first edition, 2009. ISBN 978-0199230761.

21. Jan Reimann. Randomness beyond Lebesgue measure. Invited article for Proceedings of Logic Colloquium 06, Nijmegen. To appear. Available at `http://math.berkeley.edu/~reimann/publications.html`.

22. Jan Reimann and Theodore A. Slaman. Measures and their random reals. Submitted for publication. Available at `http://math.berkeley.edu/~reimann/publications.html`, 2010.

23. Jan Reimann and Theodore A. Slaman. Randomness for continuous measures. In preparation. Available at `http://math.berkeley.edu/~reimann/publications.html`, 2010.

24. H. E. Robbins. On the measure of a random set. *Ann. Math. Statistics*, 15: 70–74, 1944. ISSN 0003-4851.