Designing a Suspension System

The suspension system on a vehicle (springs and shock absorbers), essentially turns the vehicle into a harmonic oscillator (see, e.g., https://www.youtube.com/watch?v=oaID-ujTOM8). The shock absorbers keep the vehicle from continuing to bounce up and down after every bump. If the damping is too small, the vehicle would allow several oscillations after each bounce. On the other hand, if the damping is too large, the vehicle would provide a stiff and uncomfortable ride. To get a sense of what a shock absorber should (and should not) do, take a look at https://www.youtube.com/watch?v=RAVXZcQHOT4.

We model this situation with the equation of motion

\[ my'' + cy' + ky = F_0 \cos(\omega t). \]

The solution has the form \( y = y_p + y_c \), where \( y_p \) is a particular solution and \( y_c \) is the solution to

\[ my'' + cy' + ky = 0. \]

For simplicity, we'll study the specific example (with \( m = 1 \), \( c = 2 \), \( k = 5 \), and \( F_0 = 3 \))

\[ y'' + 2y' + 5y = 3 \cos(\omega t) \]

From an engineering perspective, we'd like to investigate the behavior for different values of \( \omega \).

1. Find \( y_c \) and show that \( \lim_{t \to \infty} y_c = 0 \). For this reason, \( y_c \) is called the transient component of the solution.

\[ 1^2 + 2 \cdot 1 + 5 = 0 \]
\[ \lambda = \frac{-2 \pm \sqrt{4 - 20}}{2} \]
\[ \lambda = -1 \pm 2i \]

\[ y_c = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t)) \]

Since \( C_1 \cos(2t) + C_2 \sin(2t) \) is bounded and \( e^{-t} \to 0 \) as \( t \to \infty \),

\[ y_c \to 0 \] as \( t \to \infty \).

2. Determine \( y_p \). Since \( y = y_c + y_p \) and \( y_c \to 0 \) as \( t \to \infty \), \( y \approx y_p \) for large values of \( t \). Consequently, \( y_p \) is called the steady-state component of the solution. We can thus largely ignore \( y_c \) and focus on \( y_p \).

\[ y_p = A \cos(t) + B \sin(t) \]

\[ -A \cos(t) + B \sin(t) - 2A \sin(t) + 2B \cos(t) + 5A \cos(t) + 5B \sin(t) = 3 \cos(t) \]

\[ \begin{align*}
4A + 2B &= 3 \\
-2A + 4B &= 0
\end{align*} \]

\[ \begin{align*}
10B &= 3 \\
B &= \frac{3}{10}
\end{align*} \]

\[ A = \frac{6}{10} \]

\[ y_p = \frac{6}{10} \cos(t) + \frac{3}{10} \sin(t) \]
3. Recalling that \( \omega_0^2 = k/m \), we can rewrite \( y_p \) in phase-amplitude form

\[
y_p = R \cos(\omega t - \phi), \quad \text{where} \quad R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}.
\]

Determine the value of \( \omega \) that maximizes \( R \), denoted with \( \omega_{\max} \). 

**Hint:** in order to maximize \( R \), we can equivalently minimize \( m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2 \)

\[
\omega_0^2 = 5/1 = 5
\]

\[
R = \frac{3}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}
\]

\[
\psi' = -2\omega \cdot 2(5-\omega^2) + 8\omega = 4\omega(\omega^2 - 3)
\]

Critical values:

\[
\psi'' = 12\omega^2 - 12 = 12(\omega^2 - 1)
\]

So, \( \omega_{\max} = \sqrt{3} \) by 2nd derivative test.

4. Using \( \omega_{\max} \) above, determine the maximum amplitude, denoted with \( R_{\max} \).

\[
R_{\max} = \frac{3}{\sqrt{(5-(\sqrt{3})^2)^2 + 4(\sqrt{3})^2}} = \frac{3}{\sqrt{4 + 12}} = \frac{3}{4}
\]

5. Considering \( R \) as a function of \( \omega \), determine conditions on \( \omega \) that guarantee \( R < 0.5 \).

\[
R(\omega) = \frac{3}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}
\]

To guarantee \( R < 0.5 \),

\[
\frac{3}{\sqrt{(5-\omega^2)^2 + 4\omega^2}} = 0.5
\]

\[
36 = (5-\omega^2)^2 + 4\omega^2
\]

\[
36 = \omega^4 - 6\omega^2 + 25
\]

\[
0 = \omega^4 - 6\omega^2 - 11
\]

\[
12\omega^2 = 6 + \sqrt{36 + 44}
\]

\[
\omega^2 = 3 + \sqrt{20}
\]

\[
\omega = \sqrt{3 + \sqrt{20}}
\]