**Special Equations**

**Hermite's Equation** The differential equation

\[ y'' - 2xy' + 2y = 0 \]

is called Hermite's equation and appears in the study of quantum harmonic oscillators (the quantum analog of the classical mass-spring system).

- Which points are singular points? On what intervals is there a unique solution?

\[ \text{None} \quad (-\infty, \infty) \]

- Assume \( y = \sum_{n=0}^{\infty} a_n x^n \) and find the recursion formula for \( a_{n+2} \) in terms of \( a_n \) and \( a_{n+1} \).

\[
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^n + 2x \sum_{n=0}^{\infty} a_n x^n = 0
\]

\[
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2x a_n x^n = 0
\]

When \( n=0 \):

\[ 2a_2 + 2x a_0 = 0 \implies a_2 = -\alpha a_0 \]

When \( n \geq 1 \):

\[ a_{n+2} = \frac{2(n+1)-2\alpha}{(n+2)(n+1)} a_n \]

- What is the largest open interval that the above power series will converge?

- Suppose \( \alpha = 2, y(0) = 2, \) and \( y'(0) = -1 \). Determine the coefficients \( a_0, \ldots, a_6 \).

\[ y(0) = 2 \implies a_0 = 2 \]

\[ y'(0) = -1 \implies a_1 = -1 \]

\[ a_2 = -4 \]

\[ a_3 = \frac{1}{3} \]

\[ a_4 = 0 \]

\[ a_5 = 0.0333... \]

\[ a_6 = 0 \]

\[ a_7 = 0.0047619 \]

\[ a_8 = 0 \]
Chebyshev's Equation  The differential equation
\[(1-x^2)y'' - xy' + \alpha^2 y = 0\]
is called Chebyshev's equation, the solutions to which are useful in evaluating integrals numerically.

- Which points are singular points? On what intervals is there a unique solution?
\[x = \pm 1\]  
\[\((-\infty, -1) \cup (1, \infty)\)\]

- Assume \(y = \sum_{n=0}^{\infty} a_n x^n\) and find the recursion formula for \(a_{n+2}\) in terms of \(a_n\) and \(a_{n+1}\).
\[\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0\]
\[\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} x^n - \sum_{j=2}^{\infty} n(n-1) a_n x^{n-j} - \sum_{j=1}^{\infty} j a_n x^{n-j} + \sum_{j=0}^{\infty} j^2 a_n x^n = 0\]

\[n = 0: \quad 2a_2 + \alpha^2 a_0 = 0 \quad \Rightarrow \quad a_2 = -\frac{\alpha^2}{2} a_0\]
\[n \geq 2: \quad a_{n+2} = \frac{n^2 - \alpha^2}{(n+2)(n+1)} a_n\]

\[n = 1: \quad 6a_3 - a_1 + \alpha^2 a_1 = 0 \quad \Rightarrow \quad a_3 = \frac{1 - \alpha^2}{6} a_1\]

- What is the largest open interval that the above power series will converge?

- If \(\alpha = m\), where \(m\) is in \(\{1, 2, 3, \ldots\}\), there is a nonzero polynomial solution
\[P_m(t) = x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0,\]
called the Chebyshev polynomial of degree \(m\). Find the Chebyshev polynomial of degree \(m = 2\).  
*hint: set \(\alpha = 2\) and either use the above recurrence relation or guess a solution of the form \(y = x^2 + a_1 x + a_0\)*

\[(1-x^2)y'' - xy' + 4y = 0 \quad \Rightarrow \quad y = x^2 + a_1 x + a_0\]

\[2(1-x^2) - x(2x + a_1) + 4(x^2 + a_1 x + a_0) = 0\]

\[2 + 4a_0 = 0 \quad \Rightarrow \quad a_0 = 0\]
\[-a_1 + 4a_1 = 0 \quad \Rightarrow \quad a_1 = 0\]
\[a_0 = -\frac{1}{2}\]

\[P_0, \quad y = x^2 - \frac{1}{2}\]
Airy's Equation  The differential equation

\[ y'' - xy = 0 \]

is called Airy's equation and models the refraction of light.

- Which points are singular points? On what intervals is there a unique solution?

\[ n_{\infty} \quad (-\infty, \infty) \]

- Assume \( y = \sum_{n=0}^{\infty} a_n x^n \). First show that \( a_2 = 0 \). Then, find the recurrence relation for the coefficients.

\[
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0
\]

\[
\sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_n x^n = 0
\]

\[ n=0: \quad 2a_2 = 0 \quad \Rightarrow \quad a_2 = 0 \]

\[ n \geq 1: \quad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad \text{three-step recurrence} \]

- Suppose \( y(0) = 1 \) and \( y'(0) = 0 \). Show that the solution is

\[ y = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2 \cdot 3)(5 \cdot 6) \cdots ((3k-1) \cdot 3k)} \]

hint: compute \( a_0, a_1, a_2, \ldots \) until a pattern emerges

\[ a_0 = 1 \quad \text{from initial conditions} \]

\[ a_1 = 0 \]

\[ a_2 = 0 \quad \text{Thus, } \quad a_4 = a_6 = a_8 = a_{10} = \ldots = 0 \]

\[ a_3 = \frac{1}{3 \cdot 2} \]

\[ a_5 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} \]

\[ a_7 = \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} \]

\[ y = 1 + \frac{1}{3 \cdot 2} x^3 + \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \ldots \]

\[ y = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2 \cdot 3)(5 \cdot 6) \cdots ((3k-1) \cdot 3k)} \]
Legendre's Equation  The differential equation

\[(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0\]

is called Legendre's equation and is used for modeling spherically symmetric potentials in the theory of Newtonian gravitation and in electricity and magnetism.

- Which points are singular points? On what intervals is there a unique solution?

\[x = \pm 1\]

\[(-\infty, -1) \cup (-1, 1) \cup (1, \infty)\]

- Assume \( y = \sum_{n=0}^{\infty} a_n x^n \) and find the recursion formula for \( a_{n+2} \) in terms of \( a_n \) and \( a_{n+1} \).

\[
(1-x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha(\alpha+1) \sum_{n=0}^{\infty} a_n x^n = 0
\]

Again, our solution will be defined on \((-1, 1)\)

\[
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \alpha(\alpha+1) a_n x^n = 0
\]

\[
\begin{align*}
\text{\( n=0 \): } & \quad 2a_2 + \alpha(\alpha+1)a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{-\alpha(\alpha+1)}{2} a_0, \quad k \geq 2: \\
\text{\( n=1 \): } & \quad 6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0 \quad \Rightarrow \quad a_3 = \frac{2 - \alpha(\alpha+1)}{6} a_1, \quad a_{n+2} = \frac{n(n+1) - \alpha(\alpha+1)}{(n+2)(n+1)} a_n
\end{align*}
\]

- If \( \alpha = m \), where \( m \) is in \( \{1, 2, 3, \ldots\} \), there is a nonzero polynomial solution

\[P_m(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0,\]

called the Legendre polynomial of degree \( m \). Find the Legendre polynomial of degree \( m = 3 \). Hint: set \( \alpha = 3 \) and guess a solution of the form \( y = x^3 + a_2 x^2 + a_1 x + a_0 \)

\[
(1-x^2)y'' - 2xy' + 12y = 0
\]

\[
\begin{align*}
(1-x^2)\left(6x+2a_2\right) - 2x\left(3x^2 + 2a_2 x + a_1\right) + 12\left(x^3 + a_2 x^2 + a_1 x + a_0\right) &= 0 \\
2a_2 + 12a_0 &= 0 \quad \Rightarrow \quad a_0 = 0 \\
6 - 2a_1 + 12a_1 &= 0 \quad \Rightarrow \quad a_1 = -\frac{6}{10} \\
-2a_2 - 4a_2 + 12a_2 &= 0 \quad \Rightarrow \quad a_2 = \frac{6}{10}
\end{align*}
\]

\[
\begin{cases}
\gamma = \frac{3}{10} x \quad \text{if } x \neq 0 \\
\gamma = \frac{3}{10} \quad \text{if } x = 0
\end{cases}
\]