Confidence Intervals for $\mu$

Case 1: normal population, known $\sigma$, any $n$

The 100(1 – $\alpha$)% CI for $\mu$ based on a random sample from a normal population with known $\sigma$ is

$$\left( \bar{x} - \frac{z_{\alpha/2}}{\sqrt{n}} \sigma, \bar{x} + \frac{z_{\alpha/2}}{\sqrt{n}} \sigma \right) \text{ or, } \bar{x} \pm \frac{z_{\alpha/2}}{\sqrt{n}} \sigma$$

Remarks:

- when $X \sim N(\mu, \sigma)$, $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = N(0,1)$

Example: The yield-point of a steel-reinforcing bar is known to be normally distributed with $\sigma = 73$ lbs. A random sample of 19 bars is selected, producing a mean yield-point of 8237 lbs. Find a 95% CI for $\mu$.

```r
alpha = 0.05
n = 19
xbar = 8237
sigma = 73
qnorm(1-alpha/2)
## [1] 1.959964
me = qnorm(1-alpha/2)*sigma/sqrt(n)
(CI = c(xbar-me,xbar+me))
## [1] 8204.176 8269.824
```

Example: The displacement of particle under diffusive and convective forces after time $t = 1$ s is normally distributed with $\sigma = 2$ m. A random sample of 14 particles are tracked at the mean displacement is $\bar{x} = 3$ m. Determine a 99% lower confidence bound for $\mu$.

```r
```
Case 2: any population, (typically) unknown $\sigma$, large $n$

The *approximate* $100(1 - \alpha)$% CI for $\mu$ from a large sample ($n$ sufficiently large) is

$$ \left( \bar{x} - \frac{s}{\sqrt{n}} z_{\alpha/2}, \bar{x} + \frac{s}{\sqrt{n}} z_{\alpha/2} \right) \quad \text{or,} \quad \bar{x} \pm \frac{s}{\sqrt{n}} z_{\alpha/2} $$

Remarks:

- $n > 40$ is typically *sufficiently large* (30 for the CLT and 10 more for $\sigma$)
- when $X \sim N(\mu, \sigma)$, $\frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \approx \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

**Example:** The fluid ounces in a random sample of $n = 100$ milk jugs is measured. From the sample, we find $\bar{x} = 127.3$ and $s = 3.2$. Find the large-sample 90% CI for $\mu$.

```r
alpha = 0.1
n = 100
xbar = 127.3
s = 3.2
```

**Example:** The tensile strength (in ksi) is measured for a sample of $n = 42$ metallic components for aerospace vehicles producing a sample mean of 135.9 ksi and standard deviation of 4.59 ksi. Find a 99% lower confidence bound for the true mean tensile strength.

**Example:** Find the 99% CI for the mean waiting time between Old Faithful eruptions based on the sample data `faithful$waiting`. A “common misconception” ([link](http://yellowstone.net/geysers/old-faithful/)) is that Old Faithful erupts “on the hour, every hour.” What does the CI suggest about this claim?

```r
alpha = 0.01
(n = length(faithful$waiting))
## [1] 272
(xbar = mean(faithful$waiting))
## [1] 70.89706
(s = sd(faithful$waiting))
## [1] 13.59497
```
Exercises

1. Find $P(T > 2.1)$ if $T$ has a $t$ distribution with 19 degrees of freedom.

2. Find $P(T < -1.3)$ if $T$ has a $t$ distribution with 10 degrees of freedom.

3. Find $P(-2 < T < 2)$ if $T$ has a $t$ distribution with 12 degrees of freedom. Then, compare this number to the empirical rule for the normal distribution.

4. Determine $t_{0.05,20}$ and compare to $z_{0.05}$.

5. Determine $t_{0.025,10}$ and compare to $z_{0.025}$.

6. Determine $t_{0.025,10}$ and compare to $z_{0.025}$.

Some R Code: `pt(t,nu)` and `qt(1-alpha,nu)`

We consider $T$ that has a $t$ distribution with 11 degrees of freedom.

\begin{verbatim}
pt(1,11)
  ## [1] 0.8305997
1-pt(1,11)
  ## [1] 0.1694003
qt(0.05,11)
  ## [1] -1.795885
qt(0.975,11)
  ## [1] 2.200985
\end{verbatim}
Case 3: normal population, unknown $\sigma$, any $n$

The $100(1 - \alpha)\%$ CI for $\mu$ of a normally distributed population with unknown $\sigma$ is

$$
\left( \bar{x} - t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}, \ \bar{x} + t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}} \right)
$$

or,

$$
\bar{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}
$$

Remark:

- when $X \sim N(\mu, \sigma)$, $\frac{\bar{X} - \mu}{s/\sqrt{n}}$ has a Student’s $t$-distribution with $n - 1$ degrees of freedom

For more details about the $t$ distribution, see
http://web02.gonzaga.edu/faculty/burchn/R_files/Probability/t_distribution.html

**Example:** A sample of 14 beams provided a sample mean limit stress of 8.48 MPa and a sample standard deviation of 0.79 MPa. Assume the limit stress is normally distributed. Calculate a 95% lower confidence bound for the true mean limit stress.

```r
alpha = 0.05
n = 14
xbar = 8.48
s = 0.79
t_crit = qt(1-alpha,n-1)
(lcb = xbar - t_crit*s/sqrt(n))
## [1] 8.106092
```

**Example:** Breakdown voltage for a particular insulator is normally distributed. A sample of $n = 17$ insulators is taken and gives $s^2 = 137.3$ and $\bar{x} = 583.7$. Determine the 95% confidence interval for $\mu$.

Example: The indentation caused from projectiles being fired into ceramic body armor is measured for 41 experimental runs, resulting in $\bar{x} = 33.31$ mm and $s = 5.27$ mm. Assume indentation depth is normally distributed. Find the 90% upper confidence bound for the true mean indentation depth.
Exercises

1. Find $P(\chi^2 > 1.2)$ for $\nu = 7$

2. Find $P(\chi^2 < 11.2)$ for $\nu = 7$

3. Find $P(1.2 < \chi^2 < 11.2)$ for $\nu = 7$

4. Determine $\chi^2_{0.05,15}$ and $\chi^2_{0.95,15}$.

5. Determine $\chi^2_{0.025,31}$ and $\chi^2_{0.975,31}$.

Some R Code: pchisq(x,nu) and qchisq(p,nu)

```
pchisq(7,10)
## [1] 0.274555
1-pchisq(7,10)
## [1] 0.725445
qchisq(0.025,10)
## [1] 3.246973
qchisq(0.975,10)
## [1] 20.48318
```
Confidence Intervals for $\sigma^2$ (and $\sigma$)

The $100(1 - \alpha)\%$ CI for $\sigma^2$ is

$$\left( \frac{(n - 1)s^2}{\chi^2_{\alpha/2, n-1}}, \frac{(n - 1)s^2}{\chi^2_{1 - \alpha/2, n-1}} \right)$$

Remark:

- when $X \sim N(\mu, \sigma)$, $\frac{(n - 1)S^2}{\sigma^2}$ has a chi-squared ($\chi^2$) distribution with $n - 1$ df

The $\chi^2$ refers to a so-called chi-squared distribution with parameter $\nu > 0$; for more details, see http://web02.gonzaga.edu/faculty/burchn/R_files/Probability/Chi2_distribution.html

The $100(1 - \alpha)\%$ CI for $\sigma$ is

$$\left( \sqrt{\frac{(n - 1)s^2}{\chi^2_{\alpha/2, n-1}}}, \sqrt{\frac{(n - 1)s^2}{\chi^2_{1 - \alpha/2, n-1}}} \right)$$

**Example:** A sample of 14 beams provided a sample mean limit stress of 8.48 MPa and a sample standard deviation of 0.79 MPa. Assume the limit stress is normally distributed. Calculate a 99% confidence interval for the true variance $\sigma^2$ of limit stress.

**Example:** The level of Lake Huron (in feet) is assumed to have a normal distribution. A total of 98 lake level measurements have been made and are stored in the vector `LakeHuron`. Now, construct a 90% CI for $\sigma^2$ based off of the sample.

```r
alpha = 0.1
(n = length(LakeHuron))
## [1] 98
(s = sd(LakeHuron))
## [1] 1.318299
```

**Example:** Suppose breakdown voltage is normally distributed. We sample $n = 17$ breakdown voltages and $s^2 = 137,324.3$. Find the 95% confidence interval for $\sigma^2$. 