Chapter 7, #P-1

April 7, 2013

Problem P-1
Using the language for the Poincaré disk model, translate the following theorems in hyperbolic geometry to theorems in Euclidean geometry:

• a. If two triangles are similar, then they are congruent.
• b. If two lines are divergently parallel, then they have a common perpendicular and the latter is unique.
• c. The fourth angle of a Lambert quadrilateral is acute.

Solution
The phrasing is by no means unique for these. Below is one option. For all of the following let $\gamma$ be a fixed circle.

a. If $A, B, C, A', B', C'$ are points such that the arcs of circles $\hat{AB}, \hat{BC}, \hat{AC}, \hat{A'B'}, \hat{B'C'}, \hat{A'C'}$ all meet $\gamma$ perpendicularly, and such that the angles made by the tangents to these circles are respectively congruent (e.g. if angle $\angle A$ is the angle made by the tangents to $\hat{AB}$ and $\hat{AC}$, and $\angle A'$ is the angle made by the tangents to $\hat{A'B'}$ and $\hat{A'C'}$, then $\angle A \cong \angle A'$, etc.). Then the corresponding cross-ratios of the arcs with the intersections of their respective circles with $\gamma$ are equal. That is, for $P_{XY}$ the intersection of $\hat{XY}$ with $\gamma$ such that $P_{XY} \neq X \neq Y$:

\[
(AB, P_{AB}P_{BA}) = (A'B', P_{A'B'}P_{B'A'})
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b. If arcs $\hat{AB}$ and $\hat{CD}$ intersect $\gamma$ perpendicularly at $A, B$ and $C, D$ resp., and do not intersect each other, then there is a unique arc $\hat{EF}$ which intersects $\gamma$ perpendicularly at $E$ and $F$, and intersects both $\hat{AB}$ and $\hat{CD}$ perpendicularly, at points inside $\gamma$. 
c. Given four points $A, B, C, D$ in $\gamma$, such that the arcs of circles perpendicular to $\gamma$ and passing through the pairs $(A, B), (B, C), (C, D),$ and $(D, A)$ meet each other perpendicularly at $A$, $B$, and $C$. Then the remaining angle formed by the tangents to $DA$ and $CD$ at $D$ form an acute angle.