The Concordance Genus of 11-Crossing Knots

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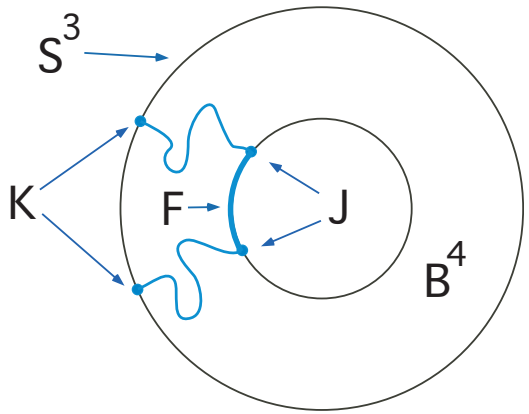
Recall: Two knots, K and J, are called concordant if $K \cup J$ is the boundary of a cylinder in $S^3 \times I$.

Under the equivalence relation of concordance, knots form a group called the concordance group, $\mathcal{C}$.

- The identity is the equivalence class of the unknot (slice knots).
- Addition in this group is the connect sum, $\#$.
- The inverse of a knot, K is $-K$.

We will use the following notation:

- $g_3(K) := \min\{g(F) \mid F \hookrightarrow S^3, \partial F = K\}$.
- $g_4(K) := \min\{g(F) \mid F \hookrightarrow B^4, K = \partial F \hookrightarrow S^3 = \partial B^4\}$.
- $g_c(K) := \min\{g_3(K') \mid K' \sim K\}$



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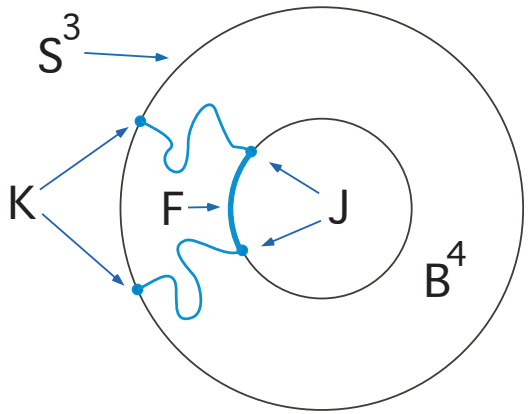

Fact $g_4(K) \leq g_c(K) \leq g_3(K)$. If K is slice, $g_c(K) = g_4(K) = 0$.

Goal: To compute the concordance genus of knots. In particular to illustrate the complexity of the invariant with a variety of examples.

We begin by considering a number of obstructions:

- $g_4(K) \leq g_c(K) \leq g_3(K)$. If K is slice, $g_c(K) = g_4(K) = 0$.
- $\frac{1}{2}|\sigma_\omega(K)| \leq g_4(K) \leq g_c(K)$ (recall, $\sigma_\omega(K) := \textit{signature}((1-\omega)V + (1-\bar{\omega})V^T)$, where V is a Seifert matrix for K).
- $\frac{1}{2}\deg(\Delta_K(t)) \leq g_3(K)$ ($\Delta_K(t)$ is the Alexander polynomial) .
- For slice K, $\Delta_K(t) = f(t)f(t^{-1})$ for some polynomial $f(t)$.
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Here are some examples:

	Alexander Polynomial	Signature	g_3	g_4	g_c
unknot	1	0	0	0	0
3_1	$1-t+t^2$	-2	1	1	1
4_1	$1-3t+t^2$	0	1	1	1
5_1	$1-t+t^2-t^3+t^4$	-4	2	2	2
5_2	$2-3t+2t^2$	-2	1	1	1

Many 11-crossing prime knots these obstructions give an immediate result, as with lower crossing number knots:

$11a_1$: $\Delta_{11a_1}(t) = 2 - 12t + 30t^2 - 39t^3 + 30t^4 - 12t^5 + 2t^6$, which is irreducible over $\mathbb{Z}[t, t^{-1}]$. Thus

$$3 = \frac{1}{2} deg(\Delta_{11a_1}(t)) \leq g_c(11a_1) \leq g_3(11a_1) = 3.$$

$11a_{51}$:

$$\begin{aligned}\Delta_{11a_{51}}(t) &= 1 - 9t + 28t^2 - 39t^3 + 28t^4 - 9t^5 + t^6 \\ &= (1 - 3t + t^2)(1 - 3t + 5t^2 - 3t^3 + t^4)\end{aligned}$$

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Sometimes the Alexander polynomial is not enough. For $11a_{43}$, the Alexander polynomial is

$$\begin{aligned}\Delta_{11a_{43}}(t) &= 4 - 14t + 30t^2 - 37t^3 + 30t^4 - 15t^5 + 4t^6 \\ &= (1 - t + t^2)^2(4 - 7t + 4t^2),\end{aligned}$$

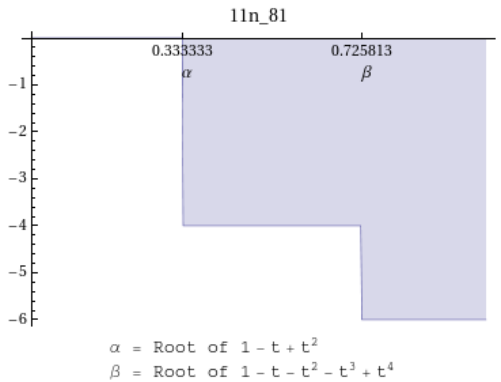
On the other hand, the signature is -6 so we have

$$3 = \frac{1}{2}|\sigma(11a_{43})| \leq g_4(11a_{43}) \leq g_c(11a_{43}) \leq g_3(11a_{43}) = 3.$$

On the other hand, for $11n_{81}$,

$$\begin{aligned}\Delta_{11n_{81}}(t) &= 1 - 3t + 4t^2 - 4t^3 + 3t^4 - 4t^5 + 4t^6 - 3t^7 + t^8 \\ &= (1 - t + t^2)^2(1 - t + t^2 - t^3 + t^4),\end{aligned}$$

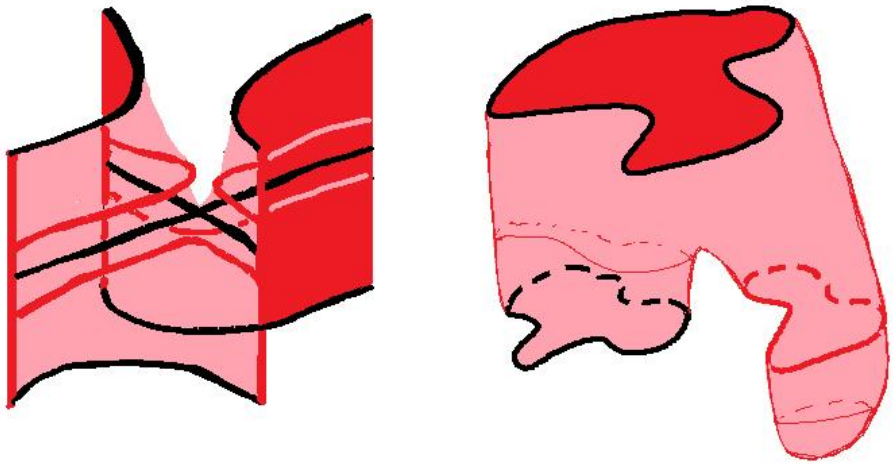
We also know $g_3(11n_{81}) = 4$, and $\sigma(11n_{81}) = -6$, and $g_4(11n_{81}) = 3$



Then for any knot, K', concordant to $11n_{81}$,
$4 \leq \frac{1}{2}deg(\Delta_{K'}(t)) \leq g_3(K')$. So we may conclude

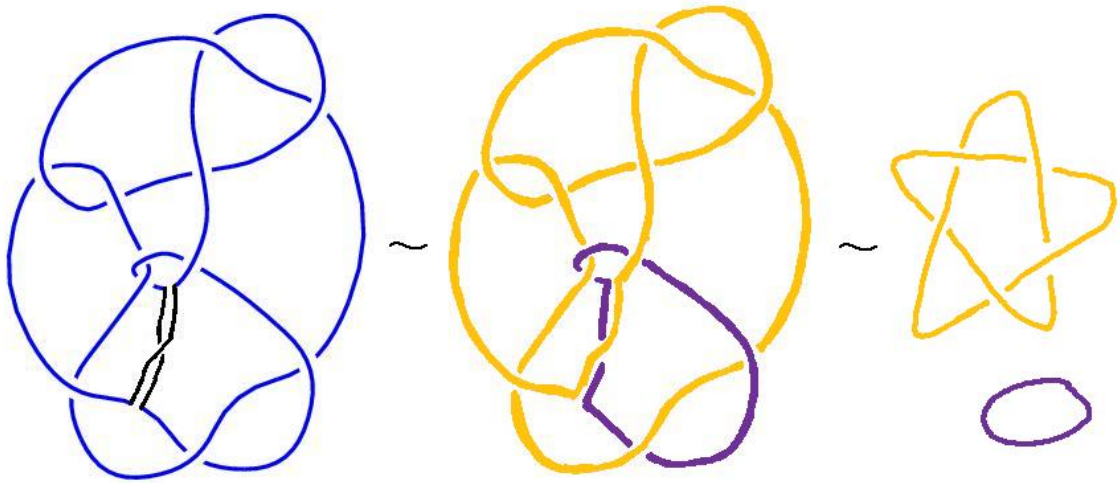
$$4 \leq g_c(11n_{81}) \leq g_3(11n_{81}) = 4.$$

In some cases, to find the concordance genus, you must find a concordance to a simpler knot.



You can construct a concordance using band moves.

For example, in the case of 11_{n69}, the $g_3(11_{n69}) = 3$, but $\sigma(11_{n69}) = -4$, $\Delta_{11_{n69}}(t) = (-2+t)(-1+2t)(1-t+t^2-t^3+t^4)$, and $g_4(11_{n69}) = 2$, so bounds don't give us enough information. However, the Alexander polynomial indicates a possible concordance to 5_1.



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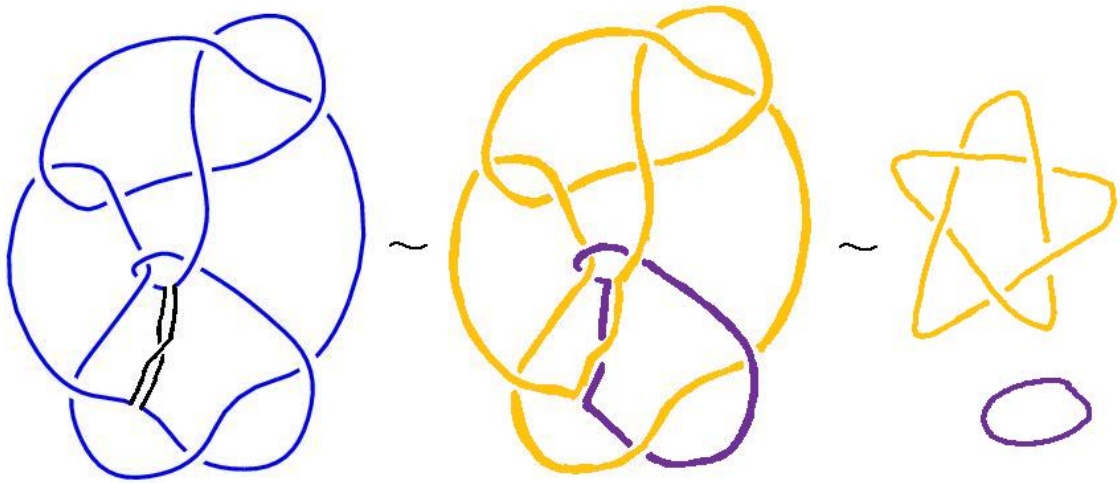

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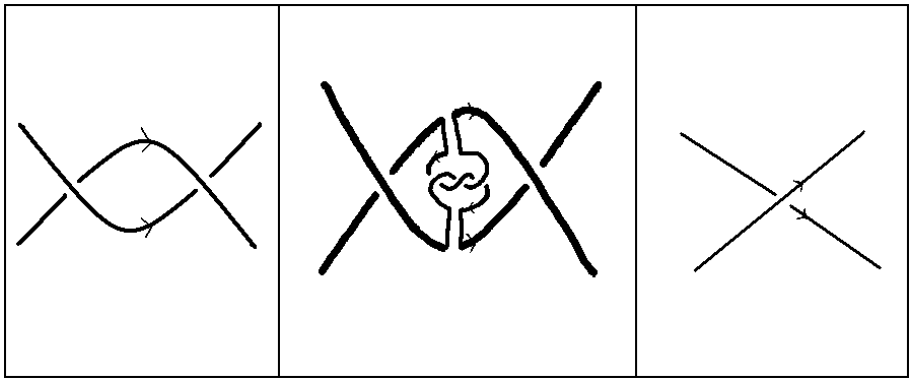
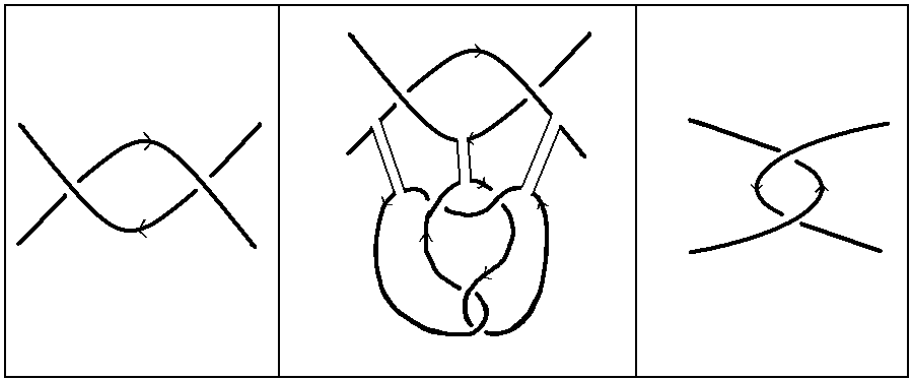
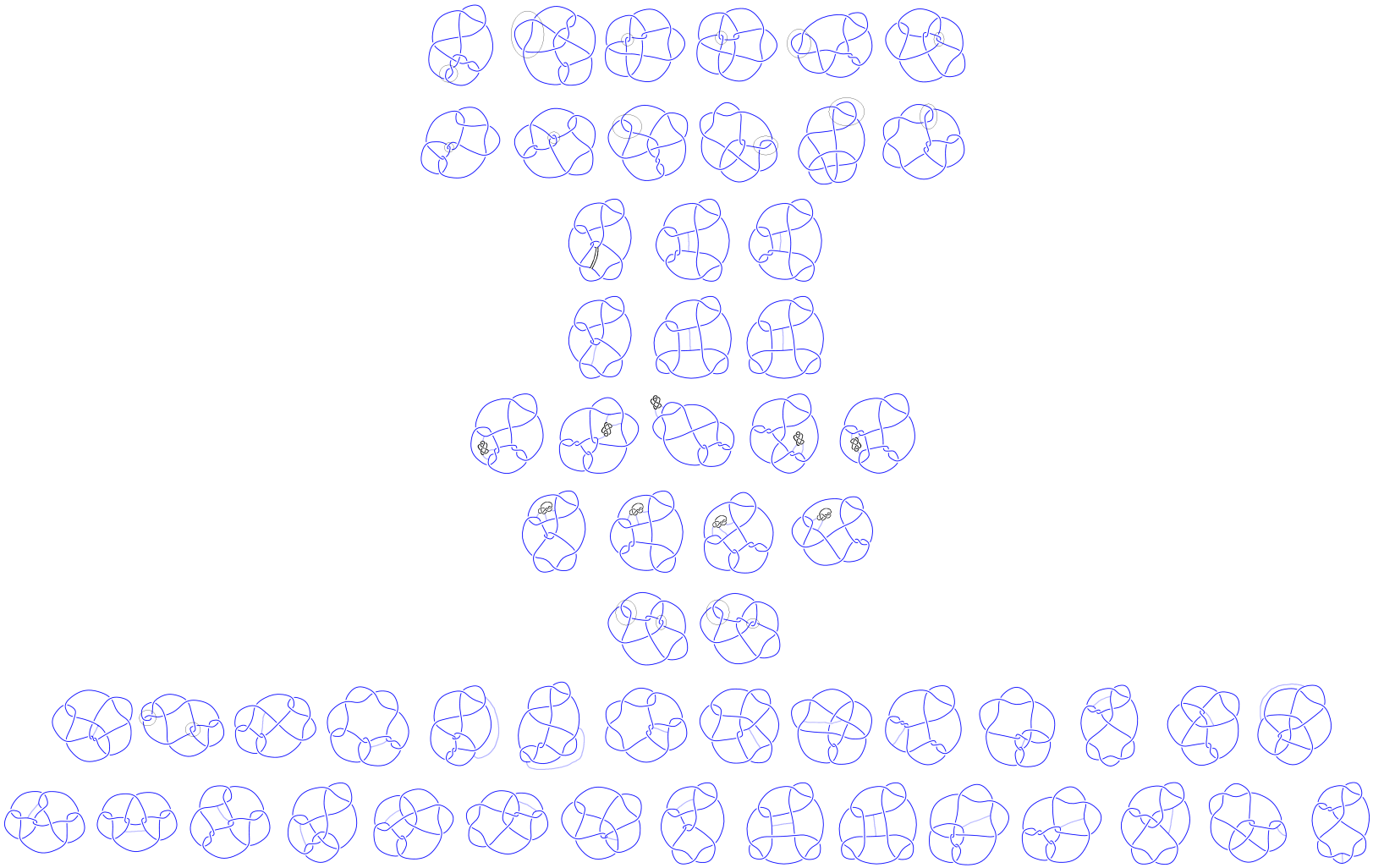
Some of this work can be aided through tricks such as:



Clasp move for concordance to the trefoil.



Clasp move for concordance to the figure eight knot.



Knot	g_c bounds	g_3	g_4	σ	Δ_K bound	possible concordance
$11a_6$	[2,3]	3	[1,2]	2	2	$3_1\#4_1$
$11a_8$	[2,3]	3	1	0	2	6_3
$11a_{67}$	[1,3]	3	[1,2]	0	1	4_1
$11a_{72}$	[2,4]	4	[1,2]	0	2	?
$11a_{108}$	[2,4]	4	[1,2]	2	2	6_2
$11a_{109}$	[2,4]	4	[1,2]	0	2	6_2
$11a_{135}$	[2,3]	3	[1,2]	0	2	?
$11a_{181}$	[2,3]	3	[1,2]	-2	2	6_2
$11a_{249}$	[2,3]	3	[1,2]	0	2	6_3
$11a_{264}$	[2,4]	4	1	-2	2	$3_1\#4_1$
$11a_{297}$	[1,3]	3	[1,2]	2	1	5_2
$11a_{305}$	[2,4]	4	[1,2]	2	2	$3_1\#4_1$
$11a_{332}$	[2,4]	4	[1,2]	0	2	7_7
$11a_{352}$	[2,3]	3	[1,2]	-2	2	$3_1\#4_1$
$11n_{34}$	[0,3]	3	[0,1]	0	0	slice?
$11n_{45}$	[1,3]	3	1	0	0	?
$11n_{66}$	[1,3]	3	[1,2]	-2	1	3_1
$11n_{145}$	[1,3]	3	1	0	0	?
$11n_{152}$	[2,3]	3	1	-2	2	8_6

Table: Unknown values

Definition The stable concordance genus of a knot is

$$g_{stc}(K) = \lim_{n\to\infty} \frac{g_c(nK)}{n}$$

Since g_c is subadditive and non-negative, this is well-defined and satisfies the following:

- $g_{stc}(K) \geq \frac{1}{2}|\sigma_\omega(K)|$
- $g_{stc}(K\#J) \leq g_{stc}(K) + g_{stc}(J)$
- $g_{stc}(nK) = ng_{stc}(K)$

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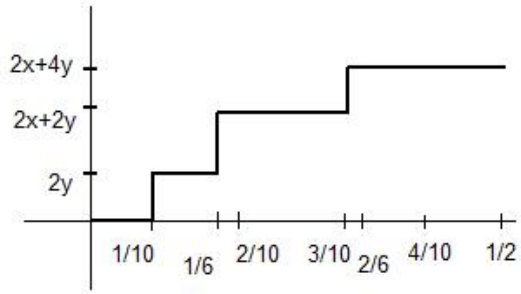
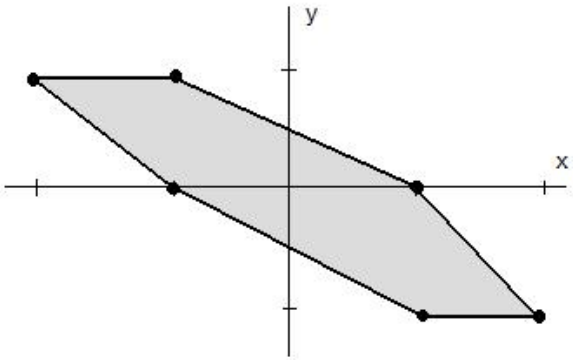
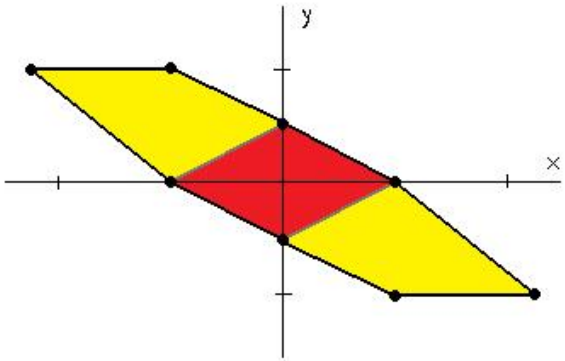
Example Let's examine knots of the form $K = xT_{2,3} + yT_{2,5}$. $\Delta_K(t) = (1-t+t^2)^{|x|}(1-t^2+t^3-t^4+t^5)^{|y|}$, $\sigma_\omega(K)$ jumps at roots of Δ_K, so here we have (left):





And with the corresponding inequalities and calculations of corner points, we confirm the g_{st} (the stable four genus) unit ball is above (right).

On the other hand, since signature is a concordance invariant, and jumps at the points above, for any K' concordant to K, Δ_K divides Δ'_K, so $g_c(K') \geq |x| + 2|y|$, and thus $g_{stc}(K) \geq |x| + 2|y|$.



Along with the fact that the corner points are 3_1 and 5_1, we find the g_{stc} unit ball (above). For instance, $g_c(2T_{2,3} + T_{2,5}) = 4$.

Theorem (K)

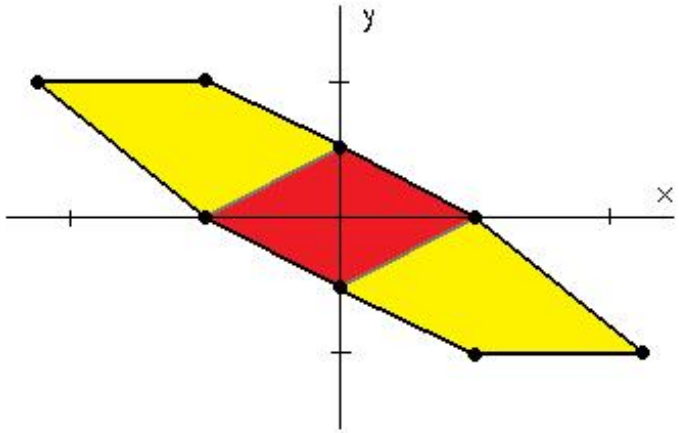
The stable concordance genus of knots of the form $xT_{2,n} + yT_{2,m}$ *is*

$$\frac{n-1}{2}|x| + \frac{m-1}{2}|y|$$

for any $n, m \in \mathbf{Z}$ *with* $n < m$, $n \neq km$.

Corollary

For any i, j, and k, for which $i \leq j \leq k$, there is a knot K for which $g_4(K) = i$, $g_c(K) = j$, and $g_3(K) = k$.



The difference between the unit balls for g_{st} and g_{stc} allow us to construct examples with the desired values of these invariants.

Thank you!