Knots, Concordance, and More

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Knot Theory asks the question: When are two knots isotopic?
Definition An invariant is a "function" from knots to (real numbers, polynomials, groups, etc), which is well-defined. That is, no matter what diagram you use to evaluate, you always get the same thing.
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For example, the number of crossings in a diagram depends on the diagram.
But if we define the "crossing number" to be the least number of crossings in any diagram of the knot, we get an invariant. Unfortunately this one is hard to compute.
Another example of a knot invariant is 3-colorability. A knot is 3-colorable if it can be colored with three colors, using at least two different colors, so that all crossings have either only one color or all three.
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For example, the unknot is not 3-colorable, but the trefoil is:
The Alexander polynomial and Signature are two more invariants. Here are their values on the knots we saw above:

<table>
<thead>
<tr>
<th>Knot</th>
<th>Alexander Polynomial</th>
<th>Signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>unknot</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$3_1$</td>
<td>$1 - t + t^2$</td>
<td>-2</td>
</tr>
<tr>
<td>$4_1$</td>
<td>$1 - 3t + t^2$</td>
<td>0</td>
</tr>
<tr>
<td>$5_1$</td>
<td>$1 - t + t^2 - t^3 + t^4$</td>
<td>-4</td>
</tr>
<tr>
<td>$5_2$</td>
<td>$3 - 2t + 3t^2$</td>
<td>-2</td>
</tr>
<tr>
<td>$2(-3_1)#9_1$</td>
<td>$(1 - t + t^2)(1 - t + t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8)$</td>
<td>4</td>
</tr>
</tbody>
</table>
Surfaces are two dimensional manifolds.
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**Definition** Two knots, $K$ and $J$, are called concordant if $K \cup -J$ is the boundary of an embedded $S^1 \times I$ in $S^3 \times I$. 
To discover and analyze concordances visually, we draw "movies".
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**Example** $11_{a104}$ is concordant to $4_1$. 
Notice that concordance is an equivalence relation!
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**Definition** Knots, under the equivalence relation of concordance, form a group called the concordance group, $C$.

- The identity is the equivalence class of the unknot (slice knots).
- Addition in this group is the connect sum, $\#$.
- The inverse of a knot, $K$ is $-K$. 
There is a surjection $\mathcal{C} \to \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$. An invariant is a concordance invariant if whenever $K \sim J$, they have the same value for the invariant. The surjection above is given by looking at a collection of concordance invariants.
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Notice that there are finite order elements in the concordance group! Many knots, such as 4_1, are amphicheiral ($K = -K$), so $K \# K = K \# -K$ is slice.
Recall the Alexander polynomial and signature, which we saw earlier. The Alexander polynomial is a concordance mod squares (that is, the Alexander polynomial of a slice knot is of the form $f(t)f(t^{-1})$). Signature is a concordance invariant, and is additive under connect sum.

The $(2, k)$ torus knots (for $k$ odd) all have different Alexander polynomials, and positive signature. So, the subgroup generated by these knots surjects to $\mathbb{Z}^\infty$. 
The genus of a surface is the "number of holes" in it. For instance, the surfaces below have genus 0, 1, and 2.
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The genus (or 3-genus) of a knot, K, is the least genus of a surface with boundary K. The concordance genus of K is the least 3-genus of a knot J, concordant to K.
In some cases, to find the concordance genus, you must find a concordance to a simpler knot. For example, in the case of $11_{a104}$, the 3-genus is 3, but $11_{a104}$ is concordant to $4_1$, which has 3-genus 1. So the concordance genus of both knots is 1.
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Luckily, we don’t always have to do this. There are both upper and lower bounds on the concordance genus (for instance, the 3-genus, signature, and the degree of the Alexander polynomial give bounds). So in many cases, we can determine the concordance genus by evaluating other invariants, which are easier to calculate.
The main goal of my research is to understand, from a variety of viewpoints, the relationship between knots and surfaces. A main tool to do this is genus. I study:

- $g_4(K)$
- $g_c(K)$
- $g_3(K)$
- $g_4(K)$
- $g_c(K)$
- $h(K)$
Here’s a picture to give you a sense of what these are:
And to give a sense of the state of the art:

- $g_3(K)$ and $g_4(K)$ are relatively well understood. I use them to get information about $g_c(K)$, since $g_4(K) \leq g_c(K) \leq g_3(K)$.

- I can calculate $g_c(K)$ for almost all knots with 11 or fewer crossings, and other special cases.

- Livingston has calculated some examples of $g_4(K)$. This is a lower bound for $g_c(K)$.

- I can calculate $g_c(K)$ for $xT_{2,n} \# yT_{2,m}$, as well as most knots of 8 or fewer crossings. Comparison with $g_4(K)$ gives me interesting examples of knots with differing values of $g_4, g_c, g_3$.

- $h(K)$ is hard. I can sometimes tell if $h(K) = 1$. 
References

- KnotInfo. http://www.indiana.edu/~knotinfo/
- D. Rolfsen. Knots and Links.