Knots, Concordance, and More

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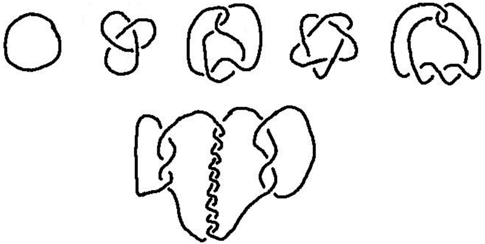

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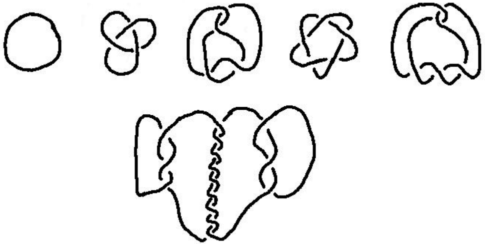
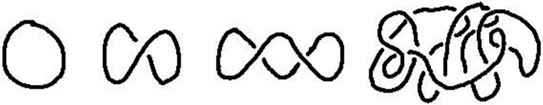


Knot Theory asks the question: When are two knots isotopic?

Definition An invariant is a "function" from knots to (real numbers, polynomials, groups, etc), which is well-defined. That is, no matter what diagram you use to evaluate, you always get the same thing.



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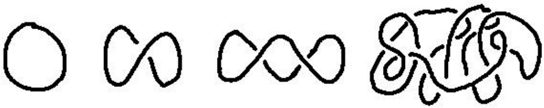

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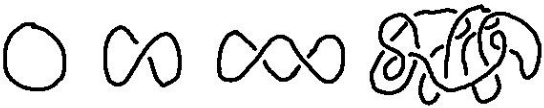
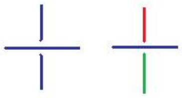


For example, the number of crossings in a diagram depends on the diagram.
But if we define the "crossing number" to be the least number of crossings in any diagram of the knot, we get an invariant. Unfortunately this one is hard to compute.

Another example of a knot invariant is 3-colorability. A knot is 3-colorable if it can be colored with three colors, using at least two different colors, so that all crossings have either only one color or all three.



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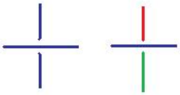
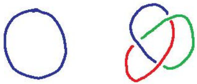


For example, the unknot is not 3-colorable, but the trefoil is:



The Alexander polynomial and Signature are two more invariants. Here are there values on the knots we saw above:

	Alexander Polynomial	Signature
unknot	1	0
3_1	$1 - t + t^2$	-2
4_1	$1 - 3t + t^2$	0
5_1	$1 - t + t^2 - t^3 + t^4$	-4
5_2	$3 - 2t + 3t^2$	-2
$2(-3_1)\#9_1$	$(1 - t + t^2)(1 - t + t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8)$	4

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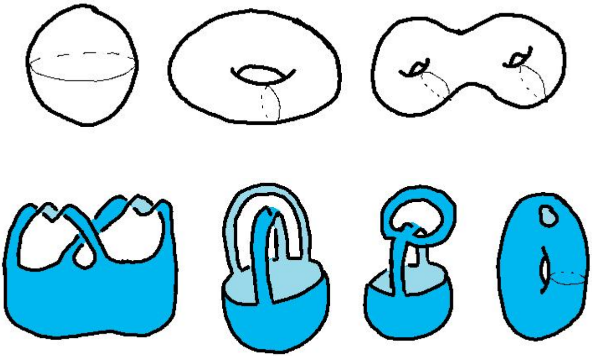
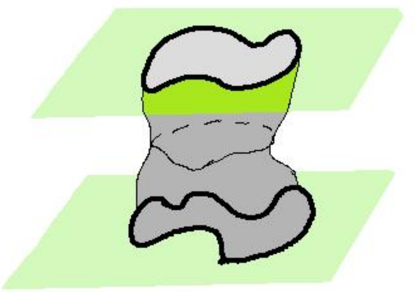
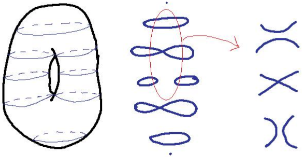


Definition Two knots, K and J, are called concordant if $K \cup -J$ is the boundary of an embedded $S^1 \times I$ in $S^3 \times I$.



To discover and analyze concordances visually, we draw "movies".



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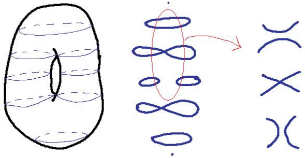
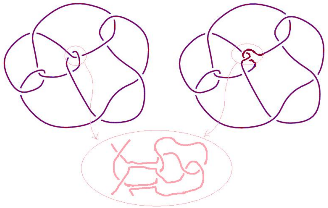
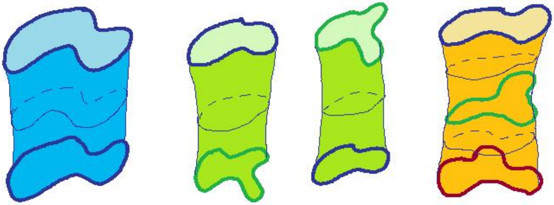


Example 11_{a104} is concordant to 4_1.



Notice that concordance is an equivalence relation!



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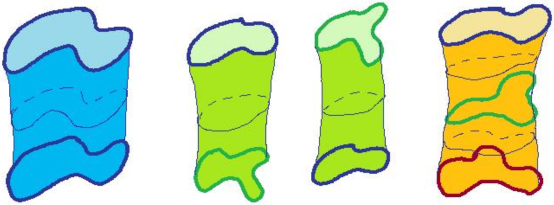

Definition Knots, under the equivalence relation of concordance, form a group called the concordance group, $\mathcal{C}$.

- The identity is the equivalence class of the unknot (slice knots).
- Addition in this group is the connect sum, $\#$.
- The inverse of a knot, K is $-K$.

There is a surjection $\mathcal{C} \to \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$.
An invariant is a concordance invariant if whenever $K \sim J$, they have the same value for the invariant. The surjection above is given by looking at a collection of concordance invariants.

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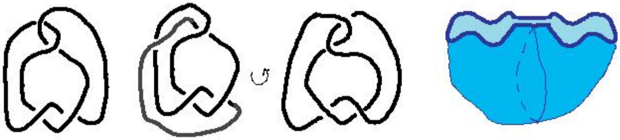
Notice that there are finite order elements in the concordance group! Many knots, such as 4_1 are amphicheiral ($K = -K$), so $K \# K = K \# -K$ is slice.



Recall the Alexander polynomial and signature, which we saw earlier. The Alexander polynomial is a concordance mod squares (that is, the Alexander polynomial of a slice knot is of the form $f(t)f(t^-1)$). Signature is a concordance invariant, and is additive under connect sum.

The $(2, k)$ torus knots (for k odd) all have different Alexander polynomials, and positive signature. So, the subgroup generated by these knots surjects to $\mathbb{Z}^\infty$.

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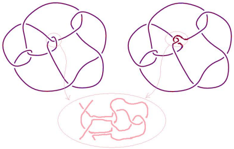
The genus (or 3–genus) of a knot, K, is the least genus of a surface with boundary K.

The concordance genus of K is the least 3-genus of a knot J, concordant to K.

In some cases, to find the the concordance genus, you must find a concordance to a simpler knot. For example, in the case of 11_{a104}, the 3-genus is 3, but 11_{a104} is concordant to 4_1, which has 3-genus 1. So the concordance genus of both knots is 1.



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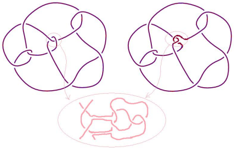
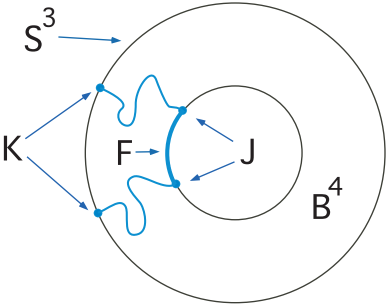


Luckily, we don't always have to do this. There are both upper and lower bounds on the concordance genus (for instance, the 3-genus, signature, and the degree of the Alexander polynomial give bounds). So in many cases, we can determine the concordance genus by evaluating other invariants, which are easier to calculate.

The main goal of my research is to understand, from a variety of viewpoints, the relationship between knots and surfaces. A main tool to do this is genus. I study:

- $g_4(K)$
- $g_c(K)$
- $g_3(K)$
- $\underset{\rightarrow}{g_4}(K)$
- $\underset{\rightarrow}{g_c}(K)$
- $h(K)$

Here's a picture to give you a sense of what these are:



And to give a sense of the state of the art:

- $g_3(K)$ and $g_4(K)$ are relatively well understood. I use them to get information about $g_c(K)$, since $g_4(K) \leq g_c(K) \leq g_3(K)$.
- I can calculate $g_c(K)$ for almost all knots with 11 or fewer crossings, and other special cases.
- Livingston has calculated some examples of $\underset{\rightarrow}{g_4}(K)$. This is a lower bound for $\underset{\rightarrow}{g_c}(K)$.
- I can calculate $\underset{\rightarrow}{g_c}(K)$ for $xT_{2,n}\#yT_{2,m}$, as well as most knots of 8 or fewer crossings. Comparison with $\underset{\rightarrow}{g_4}(K)$ gives me interesting examples of knots with differing values of g_4, g_c, g_3.
- $h(K)$ is hard. I can sometimes tell if $h(K) = 1$.

References

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- C. Livingston. Knot Theory.
- D. Rolfsen. Knots and Links.