An obstruction to knots bounding Möbius bands in B^4

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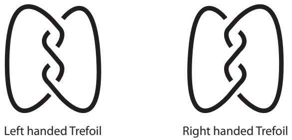
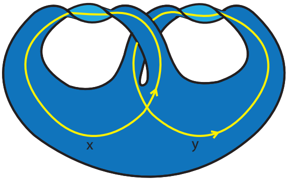
First, for the knot theory novices in the audience...



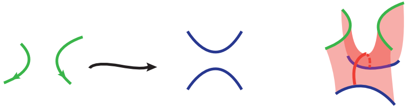
(a) Knots



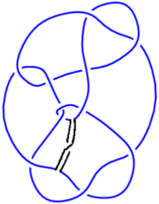
(b) Seifert surfaces

...we can find always find a Seifert surface for a given knot K (an orientable surface in S^3 with boundary K).

We can perform band moves (as illustrated below)..



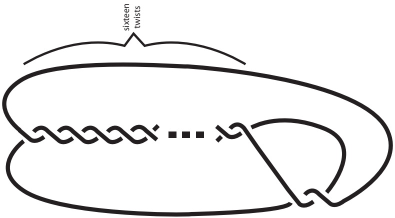
(c) band moves



(d) $11_{n69} \sim_C 5_1$

... to show that two knots are concordant ($K\# - J$ is slice), or that a knot is slice (K bounds a disk in B^4).

But these examples insist that our surfaces be orientable! Now let's consider the other case...

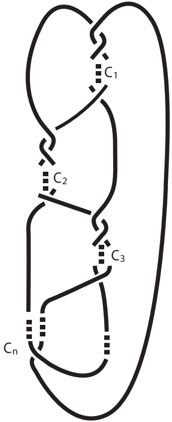


For a knot, K, we ask whether there is a smoothly embedded Möbius band, F, embedded in B^4, with boundary K in $S^3 = \partial B^4$.
We will show that the two-bridge knot $K_{31/2}$ cannot bound a Möbius band in B^4.

We will use the notation $K_{p/q}$ to refer to the two-bridge knot with associated continued fraction p/q.
A two-bridge knot is determined by a sequence of twist numbers $(c_1, c_2, c_3, \ldots, c_n)$, so we get the associated fraction:

$$p/q = c_1 + \cfrac{1}{c_2 + \cfrac{1}{\cdots + \cfrac{1}{c_n}}}$$

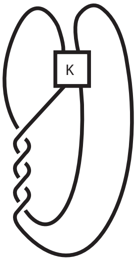
The two-fold branched cover of $K_{p/q}$ is the lens space $L(p, q)$.

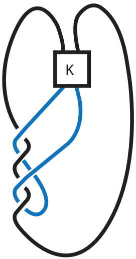


Many knots do bound Möbius bands in B^4. Any of the $T_{(2,n)}$ torus knots (and many other knots) bound Möbius bands in S^3, and consequently in B^4.

Theorem (K)

Any of the two-bridge knots $K_{(\pm 4k \pm 1)/4}$ or $K_{(\pm 8k \pm 1)/2k}$ bound Möbius bands in B^4.





From here out, K is a knot, F a surface in B^4, M is the two-fold branched cover of S^3 over K, and W is the two-fold branched cover of B^4 over F.

To show $K_{31/2}$ does not bound a Möbius band in B^4, we will:

- Use a theorem of Yasuhara to show that if such a Möbius band F exists, W has a positive definite intersection form.
- Use a theorem of Ozsváth and Szabó, and algebraic analysis of the results to show that if F exists, W must have a negative definite intersection form.
- Conclude that these can't both be true, and so F can't exist.

Theorem (Yasuhara)

If a knot $K \subset S^3$ bounds a smoothly embedded surface, F, in B^4, then for W the two-fold branched cover of B^4 branched over F,

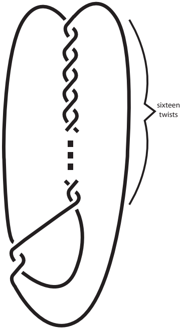
$$\sigma(K) + 4Arf(K) \equiv \sigma(W) + \beta(B^4, F) \mod 8.$$

For F a Möbius band, $\beta(B^4, F)$ (the Brown invariant), and $\sigma(W)$ are each ± 1. The evaluation of $\sigma(K) + \text{Arf}(K)$ may tell us one of a few things:

- If $\sigma(K) + \text{Arf}(K) \equiv 4 \mod 8$, then K does not bound a Möbius band. (Yasuhara's observation)
- If $\sigma(K) + \text{Arf}(K) \equiv 2 \mod 8$, then $\sigma(W) = 1$, and if K bounds a Möbius band, W has a (rank one) positive definite intersection form.

Example

- *$K_{31/2}$ has signature 2 and Arf invariant 0, so $\sigma(K_{31/2}) + Arf(K) \equiv 2 \mod 8$.*
- *This guarantees that if $K_{31/2}$ bounds a Möbius band, then W has a positive definite intersection form.*
- *Together with the fact that the two-fold branched cover of $K_{31/2}$ is $L(31, 2) = \partial W$ (the lens space given by $31/2$ surgery on the unknot), we conclude that the intersection form of W is $[31]$.*



Next, we consider $-W$ and the following theorem of Ozsváth and Szabó:

Theorem (Ozsváth-Szabó)

For M a rational homology three-sphere, fix a $Spin^c$ structure $\mathfrak{t}$ over M. Then for any smooth, negative definite four-manifold W with boundary M, and for any $Spin^c$ structure $\mathfrak{s} \in Spin^c(W)$ with $\mathfrak{s}|_M = \mathfrak{t}$, we have

$$c_1(\mathfrak{s})^2 + \beta_2(W) \leq 4d(M, \mathfrak{t}).$$

- $c_1(\mathfrak{s})^2$ is the square of the first Chern class.
- $c_1(\mathfrak{s} \cdot z) = c_1(\mathfrak{s}) + 2z$ for $\mathfrak{s} \in Spin^c(W)$ and $z \in H^2(W)$.
- $d(M, \mathfrak{t})$ is the d-invariant of Heegaard-Floer theory.

For a lens space, we can calculate the d-invariants as follows:

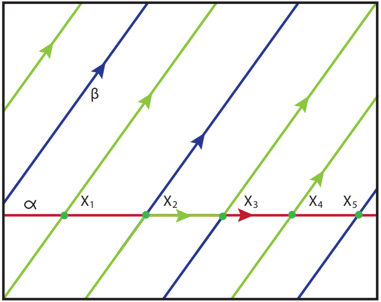
Theorem (Ozsváth-Szabó)

The d-invariant of the lens space $-L(p,q)$ on the Spin^c structure represented by i for $0 \leq i < p+q$ is given by

$$d(-L(p,q),i) = (\frac{pq-(2i+1-p-q)^2}{4pq}) - d(-L(q,r),j)$$

where $r \equiv p \mod q$ and $j \equiv i \mod q$ are the reductions modulo q of p and i.

"Wait!" You say, "...where did this *i* come from?"



Each consecutive pair of indices represent *Spin*^{c} structures on M which differ by a particular class $x \in H^2(M)$.

Now we calculate for $K_{31/2}$:

i=	0	1	2	3	4	5	6	7	8	9
$62d(-L(31,2),i) =$	-225	-225	-165	-169	-113	-121	-69	-81	-33	-49

i=	10	11	12	13	14	15	16	17	18	19
$62d =$	-5	-25	15	-9	27	-1	31	-1	27	-9
			*		*		*		*	

i=	20	21	22	23	24	25	26	27	28	29
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	*									

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We compare these values to the values of $\frac{31}{2}\left(\frac{-(2i+1)^2}{31}+1\right)$, which are:

$$15, 11, 3, -9, -25, -45, -69, -97, -129, -165, -205, -249, \ldots$$

To interpret these values, recall:

- We want $c_1(\mathfrak{s})^2 + \beta_2(W) \leq 4d(M, \mathfrak{t})$ for $\mathfrak{s}$ and $\mathfrak{t} = \mathfrak{s}|_M$
- We may calculate the left-hand side by $c_1(\mathfrak{s})^2 + \beta_2(W) = \frac{-(2i+1)^2}{p} + 1$
- If $\mathfrak{s}_0|_M = \mathfrak{t}$ and for $z \in H^2(W)$, $z|_M = x$, we have $\mathfrak{s}_0 \cdot z^i|_M = \mathfrak{t} \cdot x^i$.

So we verify $\frac{-(2i+1)^2}{p} + 1 \leq 4d(M, t \cdot x^i)$ for some $\mathfrak{t} \in Spin^c(M)$, some $x \in H^2(M)$, and all i, $0 \leq i < p$.

Adjusted for ease of comparison:

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Since the inequality cannot hold for all i for $K_{31/2}$, $-W$ cannot be negative definite. Thus $K_{31/2}$ cannot bound a Möbius band.

We may summarize this calculation with the following theorem:

Theorem (K)

Let $K = K_{p/q}$ bound a Möbius band F in B^4, with p square-free. Let W be the two-fold branched cover of B^4 over F and let M be the two-fold branched cover of S^3 over K. Suppose W is negative definite. Then for some $\mathfrak{t} \in Spin^c(M)$, $x \in H^2(M)$, and for all $0 \leq i < p$,

$$\frac{-(2i+1)^2}{p} + 1 \leq 4d(M, \mathfrak{t} \cdot x^i)$$

Thanks!

Any questions?