# The Stable Concordance Genus

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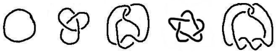
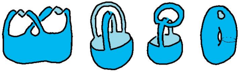
A knot is a smooth embedding of $S^1$ in $S^3$.



Knots are considered up to isotopy.
A surface is a two-dimensional manifold.

If a surface with boundary is embedded in $S^3$ (or $B^4$), the boundary is a knot. We wish to examine the relationship between such surfaces and their boundaries.



For any knot, $K$, there exists an orientable surface embedded in $S^3$ with boundary $K$.

There are a variety of knot invariants that can be determined by the relationship between knots and surfaces. For a given knot $K$, with a surface $F \hookrightarrow S^3$, $\partial F = K$, we define a quadratic form, known as the Seifert form, by $V = \left[ lk(x_i, x_j^+) \right]$. From this, we define the signature,

$$\sigma'_t(K) = \text{signature}((1 - e^{2\pi it})V + (1 - e^{-2\pi it})V^T)$$

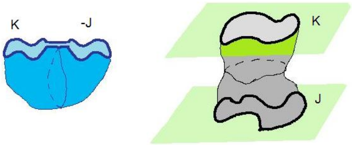
and the Alexander polynomial,

$$(1 - t^{-1})^n \Delta_K(t) = det((1 - t^{-1})V + (1 - t)V^t),$$

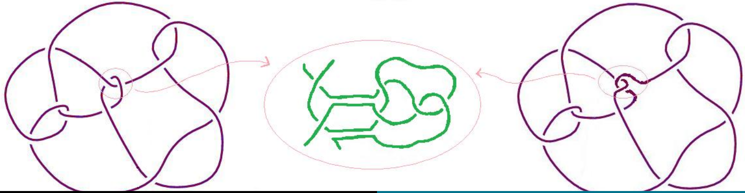
as well as many other invariants.

## Definition

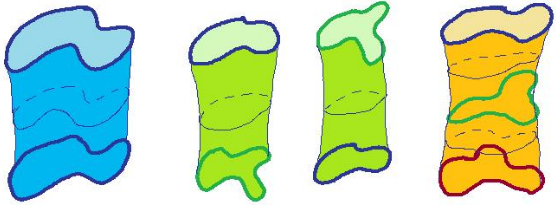
*We call a knot slice if it bounds a disk in $B^4$. Two knots, $K$ and $J$, are called concordant if $K \# -J$ is slice, or equivalently, if $K \cup J$ is the boundary of a cylinder in $S^3 \times I$.*




**Example** $11_{a104}$ is concordant to $4_1$.



Notice that concordance is an equivalence relation!



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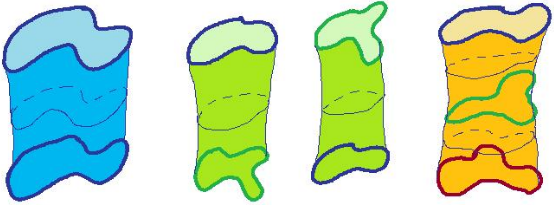

## Definition

*Knots, under the equivalence relation of concordance, form a group called the concordance group,* $\mathcal{C}$*.*

- The identity is the equivalence class of the unknot (slice knots).
- Addition in this group is the connect sum, $\#$.
- The inverse of a knot, $K$ is $-K$.

Many people have studied and continue to study the structure and properties of the concordance group. It is known to have a quotient group isomorphic to $Z^{\infty} \oplus Z_2^{\infty} \oplus Z_4^{\infty}$, called the algebraic concordance group. One of the main goals of the study of concordance is to understand the kernel of this map, a subgroup consisting of the algebraically slice knots.

In particular, it is known that there is 2-torsion in the concordance group. It is not known whether there is any other kind of torsion in the concordance group.

- $g_3(K) := \min\{g(F) \mid F \hookrightarrow S^3, \partial F = K\}$.
- $g_4(K) := \min\{g(F) \mid F \hookrightarrow B^4, K = \partial F \hookrightarrow S^3 = \partial B^4\}$.
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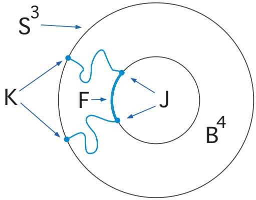



**Fact** $g_4(K) \leq g_c(K) \leq g_3(K)$. If $K$ is slice, $g_c(K) = g_4(K) = 0$.

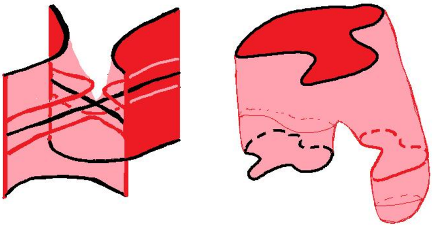
Aside from $g_3(K)$ and $g_4(K)$, other invariants bound $g_c(K)$, and the value of $g_c(K)$ can be determined for many examples just by examining the bounds.

- $\frac{1}{2}|\sigma_t(K)| \leq g_4(K) \leq g_c(K)$
- $2g_3(K) \geq deg(\Delta_K(t))$
- For slice $K$, $\Delta_K(t) = f(t)f(t^{-1})$ for some polynomial $f(t)$
- As a consequence, if $\Delta_K(t)$ is irreducible, the degree also bounds $g_c(K)$

Here are some examples:

| | Alexander Polynomial | Signature | $g_3$ | $g_4$ | $g_c$ |
|---|---|---|---|---|---|
| unknot | $1$ | $0$ | $0$ | $0$ | $0$ |
| $3_1$ | $1-t+t^2$ | $-2$ | $1$ | $1$ | $1$ |
| $4_1$ | $1-3t+t^2$ | $0$ | $1$ | $1$ | $1$ |
| $5_1$ | $1-t+t^2-t^3+t^4$ | $-4$ | $2$ | $2$ | $2$ |
| $5_2$ | $2-3t+2t^2$ | $-2$ | $1$ | $1$ | $1$ |
| $2(-3_1)\#5_1$ | $(1-t+t^2)^2(1-t+t^2-t^3+t^4)$ | $0$ | $4$ | $1$ | $4$ |

In some cases, to find the concordance genus, you must find a concordance to a simpler knot.



You can construct a concordance using band moves.

For example, in the case of $11_{n69}$, the $g_3(11_{n69} = 3$, but $\sigma(11_{n69}) = -4$, $\Delta_{11_{n69}}(t) = (-2+t)(-1+2t)(1-t+t^2-t^3+t^4)$, and $g_4(11_{n69}) = 2$, so bounds don't give us enough information. However, the Alexander polynomial indicates a possible concordance to $5_1$.

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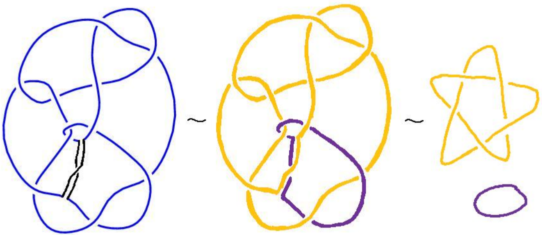

In fact, $11_{n69}$ is concordant to $5_1$, which has 3-genus 2. So the concordance genus of both knots is 2.

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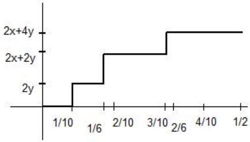
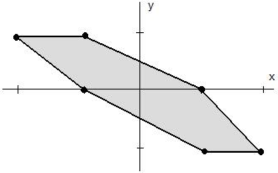
Since $g_c$ is subadditive and non-negative, this is well-defined and satisfies the following:

- $\underset{\rightarrow}{g_c}(K) \geq \frac{1}{2}|\sigma_\omega(K)|$
- $\underset{\rightarrow}{g_c}(K\#J) \leq \underset{\rightarrow}{g_c}(K) + \underset{\rightarrow}{g_c}(J)$
- $\underset{\rightarrow}{g_c}(nK) = n\underset{\rightarrow}{g_c}(K)$

In particular, $\underset{\rightarrow}{g_c}$ is a semi-norm. So we can understand it by way of understanding unit balls and extending by linearity.

**Example** Let's examine knots of the form $K = xT_{2,3} + yT_{2,5}$. $\Delta_K(t) = (1 - t + t^2)^{|x|}(1 - t^2 + t^3 - t^4 + t^5)^{|y|}$, $\sigma_\omega(K)$ jumps at roots of $\Delta_K$, so here we have (left):



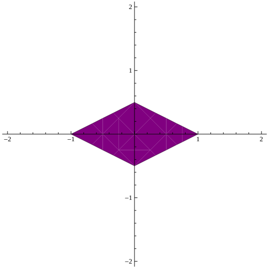




And with the corresponding inequalities and calculations of corner points, we confirm the $\underrightarrow{g_4}$ (the stable four genus) unit ball is above (right).

## Proposition

*If a knot, $K$, has Alexander polynomial $\Delta_K(t) = f(t)^x g(t)$ and $\mathrm{j}_\rho(K) = \pm 2x$ for where $f(t)$ is the minimal polynomial for $\rho$ in $\mathbb{Z}[t, t^{-1}]$, then for any $J$ concordant to $K$, $f(t)^x$ is a factor of $\Delta_J(t)$.*

On the other hand, since signature is a concordance invariant, and jumps at the points above, for any $K'$ concordant to $K$, $\Delta_K$ divides $\Delta'_K$, so $g_c(K') \geq |x| + 2|y|$, and thus $\underrightarrow{g_c}(K) \geq |x| + 2|y|$.




Along with the fact that the corner points are $3_1$ and $5_1$, we find the $\underrightarrow{g_c}$ unit ball (above).

## Theorem

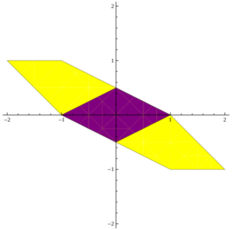
*The stable concordance genus of knots of the form $xT_{2,n} + yT_{2,m}$ is*

$$\frac{n-1}{2}|x| + \frac{m-1}{2}|y|$$

*for any $n, m \in \mathbf{Z}$ with $n < m$, $n \neq km$.*

## Conjecture

*For any $i$, $j$, and $k$, for which $i \leq j \leq k$, there is a knot $K$ for which $g_4(K) = i$, $g_c(K) = j$, and $g_3(K) = k$.*




The difference between the unit balls for $\underrightarrow{g_4}$ and $\underrightarrow{g_c}$ allow us to construct examples with the desired values of these invariants.

What I can actually prove...

**Theorem**

*For any $j, k \in \mathbb{Q}$, for which $1 \leq j \leq k$, there is some $K \in \mathcal{C} \otimes \mathbb{Q}$ for which $\underset{\rightarrow}{g_4}(K) = j$, $\underset{\rightarrow}{g_c}(K) = k$. Furthermore, if $K \in \mathcal{C}$, given any $l \geq k$ then for some knot $K'$ in the concordance class of $K$, $g_3(K') = l$.*

Some other cool things to think about:

- Livingston gives an example of a knot with rational (non-integer) stable four genus. On the other hand, there are no known knots with rational (non-integer) stable concordance genus.
- In all of the examples calculated so far, if $\underset{\longrightarrow}{g_c}(K) = k$, then for some integer multiple of $K$, $\frac{g_c(nK)}{n} = k$.
- A special case of the previous question: Does there exist a knot $K$ which is not finite order in the concordance group but $\underset{\longrightarrow}{g_c}(K) = 0$?
- We observed that if $g_4(K) = 0$ then $g_c(K) = 0$. Does the same hold for $\underset{\longrightarrow}{g_4}$ and $\underset{\longrightarrow}{g_c}$?

Thank You!