1. We want \[ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \]. Performing the multiplications gives

\[ \begin{bmatrix} a + 3c \\ 2a + 4c \end{bmatrix} = \begin{bmatrix} b + 3d \\ 2b + 4d \end{bmatrix} = \begin{bmatrix} a + 2b \\ c + 2d \end{bmatrix} \begin{bmatrix} 3a + 4b \\ 3c + 4d \end{bmatrix} \]

and equating entries gives

\[ a + 3c = a + 2b \]
\[ 2a + 4c = c + 2d \]
\[ b + 3d = 3a + 4b \]
\[ 2b + 4d = 3c + 4d \]

Pulling all terms to one side in each and rearranging equations produces the system

\[
\begin{align*}
3a + 3b - 3d &= 0 \\
2a + 3c - 2d &= 0 \\
2b - 3c &= 0 \\
2b - 3c &= 0
\end{align*}
\]

which has matrix

\[
\begin{bmatrix}
3 & 3 & 0 & -3 \\
2 & 0 & 3 & -2 \\
0 & 2 & -3 & 0 \\
0 & 2 & -3 & 0
\end{bmatrix}
\]

Note that it is not necessary to augment this with a columns of 0's...those 0's will remain intact throughout the row reduction procedure. You should be able to row reduce this matrix to

\[
\begin{bmatrix}
1 & 1 & 0 & -1 \\
0 & 1 & 0 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We take \( c \) and \( d \) to be free variables.

Second row says that \( b = \frac{3}{2} c \) while first row says that \( a + b - d = 0 \) so \( a = d - b = d - \frac{3}{2} c \).

So our matrix \( \begin{bmatrix} a \\ c \\ d \end{bmatrix} \) now looks like \( \begin{bmatrix} d - \frac{3}{2} c \\ \frac{3}{2} c \\ \frac{3}{2} c \end{bmatrix} \) which has \( \begin{bmatrix} -\frac{3}{2} & 3/2 & 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \)
2. If \( A^2 = 0 \) and \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) or

\[
\begin{bmatrix}
  a^2 + bc & ab + bd \\
  ac + dc & bc + d^2
\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
equating components gives us

\[
\begin{align*}
a^2 + bc &= 0 \\
ab + bd &= 0 \\
ac + dc &= 0 \\
bc + d^2 &= 0
\end{align*}
\]
Second and third equations are \( b(a + d) = 0 \) and \( c(a + d) = 0 \). If we want \( a, b, c, d \neq 0 \), it must be that \( a + d = 0 \) so \( a = -d \). Choose \( d = -1 \) so that \( a = 1 \). With these choices both first and third equation become \( bc = -1 \). Choose \( b = 1, c = -1 \).

You should be able to easily check that

\[
\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

3. We want to row reduce matrix \( A \) to see if we can convert it to \( I_3 \), the \( 3 \times 3 \) identity matrix. I would be inclined to first switch rows 1 and 2 \((R_1 \leftrightarrow R_2)\) to produce

\[
\begin{bmatrix} 1 & 3 & 2 \\ 3 & \lambda & 6 \\ 2 & 7 & \lambda
\end{bmatrix}.
\]
Next applying \(-3R_1 + R_2 \rightarrow R_2\) and \(-2R_1 + R_3 \rightarrow R_3\) brings us to

\[
\begin{bmatrix}
  1 & 3 & 2 \\
  0 & 1 & \lambda - 4 \\
  0 & \lambda - 9 & 0
\end{bmatrix}.
\]
Then applying \(- (\lambda - 9)R_2 + R_3 \rightarrow R_3\) give us

\[
\begin{bmatrix}
  1 & 3 & 2 \\
  0 & 1 & \lambda - 4 \\
  0 & 0 & (\lambda - 4)(\lambda - 9)
\end{bmatrix}.
\]
In order to successfully convert \( A \) into \( I_3 \), we need a pivot in each row. This will not be the case if \( \lambda = 4 \) or \( \lambda = 9 \).

So, \( A \) will not be invertible if \( \lambda = 4 \) or \( \lambda = 9 \).

4a. Many possible examples; here's one:

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \text{ are both symmetric, but their product } AB = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \text{ is not.}
\]

4b. To show \( AB \) is symmetric, look at \((AB)^t\).

\[
(AB)^t = B^t A^t = BA \text{ (since } A, B \text{ are symmetric, } A^t = A \text{ and } B^t = B) = AB \text{ (given that } AB = BA). \text{ Since } (AB)^t = AB, \text{ } AB \text{ is symmetric.} \]
5. 

\[ [A(BC)]_{ij} = \sum_{k=1}^{n} A_{ik}(BC)_{kj} = \sum_{k=1}^{n} A_{ik} \left( \sum_{l=1}^{p} B_{kl}C_{lj} \right) = \sum_{k=1}^{n} \sum_{l=1}^{p} A_{ik}B_{kl}C_{lj} \] 

(here \(A_{ik}\) is a constant as far as the second summation is concerned so it can be multiplied by that sum by multiplying each term in the sum, effectively bringing \(A_{ik}\) inside the second summation).

Likewise

\[ [(AB)C]_{ij} = \sum_{l=1}^{p} (AB)_{il}C_{lj} = \sum_{l=1}^{p} \left( \sum_{k=1}^{n} A_{ik}B_{kl} \right)C_{lj} = \sum_{l=1}^{p} \sum_{k=1}^{n} A_{ik}B_{kl}C_{lj} \]

showing that these are equal so \(A(BC) = (AB)C\).