Math 301
HW 1. 1.1 2dhj, 3bdfjl, 4fjl, 5abde, 8abd, 9bd, 10bfik, 11abde.

1.1 2. d

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P\wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

hj

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	F
F	T	F	T	T	F	F
F	F	F	T	T	T	T
			h			j

P, Q, R (T)
S, T (F)

3. b. $Q \vee (R \wedge S) \Leftrightarrow T \vee (T \wedge F) \Leftrightarrow T \vee F \Leftrightarrow T$

d. $(\sim P) \vee (\sim Q) \vee ((\sim R) \vee (\sim S)) \Leftrightarrow F \vee F) \vee (F \vee T) \Leftrightarrow (F \vee T) \Leftrightarrow T$

f. $(\sim P) \vee (\sim Q) \Leftrightarrow (\sim T) \vee (\sim T) \Leftrightarrow F \vee F \Leftrightarrow F$

j. $(\sim T) \wedge P) \vee (T \wedge P) \Leftrightarrow (T \wedge T) \vee (F \wedge T) \Leftrightarrow T \vee F \Leftrightarrow T$

l. $(\sim R) \wedge (\sim S) \Leftrightarrow (\sim T) \wedge (\sim F) \Leftrightarrow F \wedge T \Leftrightarrow F$

4. f.

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	F
F	T	F	T	T	F	F
F	F	F	T	T	T	T

Not logically equivalent

j

$P \vee Q$	$\sim(P \vee Q)$
T	F
T	F
T	F
F	T

logically equivalent

l. Not logically equivalent.
e.g. if Q, P, false and R true we have:
$(P \wedge Q) \vee R \Leftrightarrow T$, but $P \vee (Q \wedge R) \Leftrightarrow F$

5abde, 8abd, 9bd, 10bfik, 11abcd

5abde

a) $A \vee \sim B$ (T) $T \vee F$

b) $\sim C \wedge \sim A$ or $\sim(C \vee A)$ (F) $T \wedge F$

d) $A \wedge \sim C$ (T) $T \wedge T$

e) $\sim A \wedge B \wedge \sim C$ (F) $(F \wedge T) \wedge T$

8abd

a)

P	Q	$P \wedge Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$
T	T	T	F	F	F	T
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	F	T	T	T	T

Neither a contradiction nor a tautology

b)

P	$\sim P$	$P \wedge \sim P$	$\sim(P \wedge \sim P)$
T	F	F	T
F	T	F	T

A tautology (always true)

d) A tautology. Since each proposition A, B must be true or false, $A \wedge B$, $A \wedge \sim B$, $\sim A \wedge \sim B$, $\sim A \wedge B$ cover all possibilities, so one of these four is $T \wedge T \Leftrightarrow T$. Since one part (at least) of $(A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B) \vee (\sim A \wedge \sim B)$ is true, and they are connected by or "$\vee$", then the entire proposition is true.

9. b. $A \wedge \sim B \Leftrightarrow T \wedge \sim F \Leftrightarrow T \wedge T \Leftrightarrow T$ (a tautology)

d. $\sim(\sim A \wedge B) \Leftrightarrow \sim(\sim T \wedge F) \Leftrightarrow \sim F \Leftrightarrow T$ (a tautology)

10. bfik

b. We will win neither the first game nor the second one.

f. $x \geq y$ and $m^2 \geq 1$

i. n is not even or n is a multiple of 5

K. At least one of x, y, and z is not prime.

11abcd

a. $\sim\sim P \vee \sim Q \wedge \sim S$

$\Leftrightarrow \sim(\sim P) \vee [(\sim Q) \wedge (\sim S)]$

b. $Q \wedge \sim S \vee \sim(\sim P \wedge Q)$

$\Leftrightarrow [Q \wedge (\sim S)] \vee [\sim((\sim P) \wedge Q)]$

c. $P \wedge \sim Q \vee \sim P \wedge \sim R \vee \sim P \wedge S$

$\Leftrightarrow ((P \wedge (\sim Q)) \vee ((\sim P) \wedge (\sim R))) \vee ((\sim P) \wedge S)$

d. $\sim P \vee Q \wedge \sim\sim P \wedge Q \vee R$

$\Leftrightarrow ((\sim P) \vee [(Q \wedge (\sim(\sim P))) \wedge Q]) \vee R$

3b. Prove $(\exists x)(\exists y)$ (x is irrational $\wedge$ y is irrational $\wedge$ $x+y$ is rational)

Proof: Let $x = \sqrt{2}$, $y = -\sqrt{2}$

These are both irrational (as shown in class $\sqrt{2} \neq a/b$ for any $a, b \in \mathbb{Z}$)

However $x+y = \sqrt{2} + (-\sqrt{2})$

$= 0 = \frac{0}{1}$ which is rational

HW # 2

1.2

1bcei, 2bcei, 4bdgh, 5bdefi, 8 all, 9ce (truth table, l.e.)
11. all, 13.bi, 14 all

1b. $\underbrace{\text{If the moon is made of cheese}}_{\text{antecedent}}$, then $\underbrace{\text{8 is an irrational number}}_{\text{consequent}}$

$p \Rightarrow q$

c. $\underbrace{\text{b divides 3}}_{\text{antecedent}}$ only if $\underbrace{\text{b divides 9}}_{\text{consequent}}$

$p \Rightarrow q$

e. A $\underbrace{\text{sequence } a \text{ is bounded}}_{\text{consequent}}$ whenever $\underbrace{a \text{ is convergent}}_{\text{antecedent}}$

$q \Rightarrow p$

i. A $\underbrace{\text{g.p.a of 3.0}}_{\text{antecedent}}$ is sufficient to $\underbrace{\text{graduate with honors}}_{\text{consequent}}$

$p \Rightarrow q$

2. b. Conv: If 8 is an irrational number, then the moon is made of green cheese.
$q \Rightarrow p$
Contrapos: If 8 is rational, then the moon is not made of green cheese
$\sim q \Rightarrow \sim p$

c. Conv: If b divides 9 then b divides 3
Contrapos: If b does not divide 9, then b does not divide 3.

e. Conv: If a is bounded, then a is convergent
Contrapos. If a is not bounded, then a is not convergent

i. Conv: If you graduate with honors, then you have a 3.0 g.p.a.
Contrapos: If you do not graduate with honors, then you don't have a 3.0 g.p.a.

4. b. $T \Rightarrow F$ (F)
d. $F \Rightarrow F$ (T)
g. $F \Rightarrow T$ (T)
h. $T \Rightarrow F$ (F)

5. b. $T \Leftrightarrow T$ (T)
d. $\begin{matrix} T & T \\ F & F \end{matrix}$ (T)
e. $T \Leftrightarrow F$ (F)
f. $F \Leftrightarrow F$ (T)
i. $\begin{matrix} T & F \\ T & T \end{matrix}$ (F)

8. a) [(f has a rel. min. at x_0) $\wedge$ (f diff at x_0)] $\Rightarrow$ ($f'(x_0)=0$)
b) (n is prime) $\Rightarrow$ [($n=2$) $\vee$ (n is odd)]
c) (x is irrational) $\Rightarrow$ [(x is real) $\wedge$ (x is not rational)]
d) [($x=1$) $\vee$ ($x=-1$)] $\Rightarrow$ ($|x|=1$)
e) (f has a critical point at x_0) $\Leftrightarrow$ [($f'(x_0)=0$) $\vee$ ($f'(x_0)$ does not exist)]
f) (S compact) $\Leftrightarrow$ ((S closed) $\wedge$ (S bounded))
g) (B invertible) $\Leftrightarrow$ ($\det B \neq 0$)
h) ($6 \geq n-3$) $\Rightarrow$ [($n \lrcorner 4$) $\vee$ ($n>10$)]

i) (X is Cauchy) $\Rightarrow$ (X is convergent)

j) $\left[\lim_{x \to x_0} f(x) = f(x_0)\right] \Rightarrow$ (f is continuous at x_0)

k) [(f diff. at x_0) $\wedge$ (f strictly increasing at x_0)] $\Rightarrow$ ($f'(x) > 0$)

9c.

$\sim P$	P	Q	R	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	$\sim R$	$\sim Q$	$\sim Q \vee \sim R$	$(\sim Q \vee \sim R) \Rightarrow \sim P$
F	T	T	T	T	T	F	F	F	T
F	T	T	F	F	F	T	F	T	F
F	T	F	T	F	F	F	T	T	F
F	T	F	F	F	F	T	T	T	F
T	F	T	T	T	T	F	F	F	T
T	F	T	F	F	T	T	F	T	T
T	F	F	T	F	T	F	T	T	T
T	F	F	F	F	T	T	T	T	T

thus, $[P \Rightarrow (Q \wedge R)]$ is logically equivalent to $[(\sim Q \vee \sim R) \Rightarrow \sim P]$

using logical equivalences:

$P \Rightarrow (Q \wedge R)$

$\Leftrightarrow \sim(Q \wedge R) \Rightarrow \sim P$ (contrapositive)

$\Leftrightarrow (\sim Q \vee \sim R) \Rightarrow \sim P$ (De Morgan's) $\therefore [P \Rightarrow (Q \wedge R)] \Leftrightarrow [(\sim Q \vee \sim R) \Rightarrow \sim P]$

e.

P	Q	R	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow R$	$P \wedge \sim Q$	$(P \wedge \sim Q) \vee R$
T	T	T	T	T	F	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	F	T
F	T	F	T	F	F	F
F	F	T	T	T	F	T
F	F	F	T	F	F	F

truth columns match, so they are logically equivalent

$(P \Rightarrow Q) \Rightarrow R$

$\Leftrightarrow \sim(P \Rightarrow Q) \vee R$ (since $A \Rightarrow B \Leftrightarrow \sim A \vee B$)

$\Leftrightarrow \sim(\sim P \vee Q) \vee R$ (since $A \Rightarrow B \Leftrightarrow \sim A \vee B$)

$\Leftrightarrow (\sim(\sim P) \wedge \sim Q) \vee R$ (De Morgan's law)

$\Leftrightarrow (P \wedge \sim Q) \vee R$ (Double Negation)

11. a. If 2 is even, then 3 is even. $T \Rightarrow F$ (F) Conv: $F \Rightarrow T$ (T)
d. Same as (a). Contrapos If 3 is odd, then 2 is odd. $T \Rightarrow F$ (F)

b. Not Possible since $T \Rightarrow F$ has a converse of $F \Rightarrow T$.
c. Not possible since a proposition and its contrapositive must have the same truth value.

10 all, 13. bi, 14 all

10 a. If n is even, n^2 is even.
b. If f is differentiable, then f is continuous. (e.g. $f(x) = |x|$)
c. Not possible since $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$
d. If n is even, then n^2 is even. (if n^2 is odd, then n is odd)

13. b.

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$	$P \Leftrightarrow (P \wedge (P \vee Q))$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	F	T

A tautology

i.

P	Q	$P \Leftrightarrow Q$	$P \wedge (P \Leftrightarrow Q)$	$\sim Q$	$(P \wedge (P \Leftrightarrow Q)) \wedge \sim Q$
T	T	T	T	F	F
T	F	F	F	T	F
F	T	F	F	F	F
F	F	T	F	T	F

a contradiction

Or note that for $P \wedge \sim Q$ to be true, P is T and Q is F, but then $P \Leftrightarrow Q$ is F. So, $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$ is F.

14. $P \Rightarrow Q$ inverse $\sim P \Rightarrow \sim Q$

a)

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim Q$	$\sim P \Rightarrow \sim Q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

Not equivalent.

b) $P \Rightarrow Q$ and $\sim P \Rightarrow \sim Q$ are both true when P and Q have the same truth values.

c) Converse $Q \Rightarrow P \Leftrightarrow \sim P \Rightarrow \sim Q$

$P \Rightarrow Q$ Inverse: $\sim P \Rightarrow \sim Q$, Contrapositive of inverse: $Q \Rightarrow P$

$P \Rightarrow Q$ Contrapositive: $\sim Q \Rightarrow \sim P$, inverse of contrapositive $Q \Rightarrow P$

Both the contrapositive of the inverse and the inverse of the contrapositive are equivalent to the converse of $P \Rightarrow Q$.

1.3. 1cdhjm, 2cdj, 3all, 5abcdghij, 6cfg, 7bceghj, 8be
Homework #3

1cdhmj

c) $(\exists x)((x \text{ is isosceles}) \wedge (x \text{ is a right triangle}))$

d) $(\forall x)(x \text{ is a right triangle} \Rightarrow x \text{ is not isosceles})$

or $\sim(\exists x)(x \text{ is a right triangle} \wedge x \text{ is isosceles})$

h) $(\forall x)(\forall y)((y > x) \Rightarrow (\exists z)(z \text{ is rational} \wedge x < z < y))$

m) ~~$(\forall z_1)(\exists! z_2)(z_1 z_2 = \pi)$~~

$(\forall z)((z \neq 0) \Rightarrow (\exists! w)(wz = \pi))$

✓ j) $\forall x \exists y (x \text{ loves } y)$

2cdj

c) $\sim(\exists x)((x \text{ is isosceles}) \wedge x \text{ is a right triangle}))$

$\Leftrightarrow \forall x (\sim(x \text{ is isosceles}) \vee \sim(x \text{ is right triangle}))$

$\forall x (x \text{ isosceles} \Rightarrow x \text{ is not a right triangle})$ $(P \Rightarrow Q \Leftrightarrow \sim P \vee Q)$

No isosceles triangle is a right triangle.

d) $\sim(\forall x)(x \text{ is a right triangle} \Rightarrow x \text{ is not isosceles})$

$\Leftrightarrow (\exists x) \sim (x \text{ is a right triangle} \Rightarrow x \text{ is not isosceles})$

$\Leftrightarrow (\exists x)(x \text{ is a right triangle} \wedge x \text{ is isosceles})$

There exists a right isosceles triangle.

or Some right triangles are isosceles. (at least one)

✓ j) $\sim(\forall x)(\exists y)(x \text{ loves } y)$

$\Leftrightarrow (\exists x) \sim (\exists y)(x \text{ loves } y)$

$\Leftrightarrow (\exists x)(\forall y) \sim (x \text{ loves } y)$

$(\exists x)(\forall y)(x \text{ does not love } y)$

There is someone who does not love anyone.

3.

				truth set is nonempty
✓a)	17 odd $\Rightarrow$ 17 > 8	T $\Rightarrow$ T	T	{17}
✓b)	6 odd $\Rightarrow$ 6 > 8	F $\Rightarrow$ F	T	{6}
✓c)	24 odd $\Rightarrow$ 24 > 8	F $\Rightarrow$ T	T	{24}
✓d)	2 odd $\Rightarrow$ 2 > 8	F $\Rightarrow$ F	T	{2, 4, 6, 24, 26}

truth set is nonempty for each $\mathcal{U}$

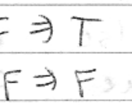
$\therefore (\exists x)(x \text{ is odd} \Rightarrow x > 8)$ is true

The sentence $(\exists x)(x \text{ is odd} \wedge x > 8)$ is only true in $\mathcal{U} = \{17\}$

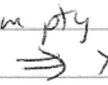
5. ✓a) False, eg. $x = -1$ gives us $-2 \geq -1$

✓b) True

✓c) False, eg $x = -1 \notin \mathbb{N}$



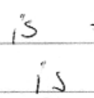
✓d) True



✓g) False e.g. $x = -2$

✓h) True.

$(x+2)^2 + 1$



✓i. True e.g. $x = 1$

✓j. False eg $x = 41$

6c. For all natural numbers x, if x is prime and x is not 2, then x is odd.
f. There is a unique real number x such that $x^2=0$.
g. For all natural numbers x, if x is odd, then x^2 is odd.

7. b. False — e.g. when $y=3$, x must be -3, but -3 does not work for all values y.
c. False — $x^2+y^2 \geq 0$ for all $x,y \in \mathbb{R}$
e. True — $x=0$ makes this true for all y, z
g. True — choose $x=y-1$
h. False — y exists, but is not unique e.g. $y=1$, $y=-2$
j. True — the unique value of x is precisely y^2

8b. Converse is $(\exists x)A(x) \Rightarrow (\exists! x)A(x)$ which is false
for example, $(\exists x)(x^2=4)$ is true since $2^2=4$
$(\mathcal{U}=\mathbb{R})$ but $(-2)^2=4$ is also true,
So there is not a unique x with $x^2=4$.

e. $\sim(\exists! x)A(x)$
$\Leftrightarrow \sim(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow (x=y))]$
$\Leftrightarrow \forall x\ (\sim A(x) \vee \sim(\forall y)(A(y) \Rightarrow (x=y)]$
$\Leftrightarrow \forall x\ (\sim(A(x)) \vee (\exists y)\sim(A(y) \Rightarrow (x=y)]$
$\Leftrightarrow \forall x\ (\sim(A(x)) \vee (\exists y)(A(y) \wedge x \neq y))$

For all x either $P(x)$ is false or there is at least one y for which $A(x)$ and $A(y)$ are both true.

OR Either $P(x)$ is false for all x or $P(x)$ is true for more than one x.

9a and its denial

9a. Recall f is continuous at $x=a$ if for all $\epsilon>0$, there is a $\delta>0$ such that if $|x-a|<\delta$ then $|f(x)-f(a)|<\epsilon$

$\lim_{x \to a} f(x) = f(a)$

$$\forall \epsilon>0\ \exists \delta>0\ \forall x(|x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon)$$

Denial $\exists \epsilon>0\ \forall \delta>0\ \exists x\ (|x-a|<\delta \wedge |f(x)-f(a)| \geq \epsilon)$

There is an $\epsilon>0$ ~~for which~~ such that for every $\delta>0$ there is an x within δ distance of a for which $f(x)$ and $f(a)$ differ by ϵ or more.

Due Monday 2-4-08 Homework #4

1.4 3cd, 5ae, 6d, 7aj, 8, 9ab

3c. Let A, B, and C be sets.
Suppose $A \subseteq B$ and $B \subseteq C$

⋮

Therefore $A \subseteq C$.
Thus, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

3d. Let $f(x)$ be differentiable on $[a,b]$.
Suppose $f(x)$ attains a maximum value at x_0.

⋮

Therefore $x_0 = a$, or $x_0 = b$, or $f'(x_0) = 0$.
Thus, if f is differentiable on $[a,b]$ and has a maximum at x_0, then $x_0 = a$, $x_0 = b$, or $f'(x) = 0$.

* This is part of the Extreme Value Theorem From Calculus.

5a. Let x and y be integers.
Suppose x and y are even.
Then there exist integers k and m such that $x = 2k$ and $y = 2m$.
Now $x + y = 2k + 2m = 2(k+m)$.
Since $k+m$ is an integer, $x+y$ is even.
Thus, if x and y are even integers, $x+y$ is an even integer.

5e. Let x and y be integers.
Suppose x is even and y is odd.
Then there are integers k and m such that $x = 2k$ and $y = 2m+1$.
Now $xy = (2k)(2m+1) = 2(2km+k)$, which is even
Thus, if x is even and y is odd, xy is even.

6d. Let a, b be real numbers.
Prove $|a+b| \leq |a|+|b|$

Case 1: $a \geq 0, b \geq 0$. Then $|a| = a$ and $|b| = b$

so, $|a+b| = a+b$ (since $a+b \geq 0$)

$= |a| + |b|$

Therefore $|a+b| \leq |a|+|b|$

Case 2: $a \geq 0, b < 0$. Then $|a| = a$ and $|b| = -b$

Case (2i) $|a| \geq |b|$.

Then $|a+b| = a+b$ (since $a+b \geq 0$)

$= |a| - |b|$

$\leq |a|+|b|$ (since $|b| \geq -|b|$)

Thus $|a+b| \leq |a|+|b|$

Case (2ii) $|a| < |b|$

Then $|a+b| = -(a+b)$ (since $a+b < 0$)

$= -a + -b$

$= -|a| + |b|$

$\leq |a| + |b|$ (since $-|a| \leq |a|$)

Thus, $|a+b| \leq |a|+|b|$

Case 3: $a < 0$, $b \geq 0$ (Same as case 2, just interchange a and b)

Case 4: $a < 0$ and $b < 0$. Then $|a| = -a$ and $|b| = -b$

Now $|a+b| = -(a+b)$ (since $a+b < 0$)

$= -a - b$

$= |a| + |b|$

Thus $|a+b| \leq |a|+|b|$

Thus, in all cases $|a+b| \leq |a|+|b|$

7a. Let a be an integer.

Then $2a-1 = 2(a+1-1) - 1$

$= 2(a-1) + 2 - 1$

$= 2(a-1) + 1$

which is odd since $a-1$ is an integer.

7j. Let a, b, c, d be integers.

Suppose a divides b and c divides d.

Then there are integers m, K with $b = am$ and $d = ck$

Now $bd = (am)(ck) = (ac)(mk)$, where mk is an integer.

Hence, ac divides bd. Thus, if a divides b and c divides d, then $ac \mid bd$.

8. Let n be a natural number.
(show n^2+n+3 is odd).

case 1: suppose n is even.

Then $n=2k$ for some integer k.

Now $(n^2+n+3) = (2k)^2 + 2k + 3$
$= 4k^2+2k+3$
$= 4k^2+2k+2+1$
$= 2(2k^2+2k+1)+1$
which is odd, since $2k^2+2k+1 \in \mathbb{Z}$

Case 2: Suppose n is odd

Then $n = 2m+1$ for some integer m.

Now $n^2+n+3 = (2m+1)^2 + (2m+1)+3$
$= 4m^2+4m+1+2m+1+2+1$
$= 4m^2+6m+4+1$
$= 2(2m^2+3m+2)+1$
which is odd, since $2m^2+3m+2 \in \mathbb{Z}$

Thus, if n is a natural number, n^2+n+3 is odd.

9 a. Let x and y be positive real numbers

Then $(x-y)^2 \geq 0$

$\Rightarrow x^2-2xy+y^2 \geq 0$

$\Rightarrow x^2-2xy+y^2+4xy \geq 4xy$

$\Rightarrow x^2+2xy+y^2 \geq 4xy$

$\Rightarrow (x+y)^2 \geq 4xy$

$\Rightarrow x+y \geq 2\sqrt{xy}$ (since $x+y>0$, $\sqrt{(x+y)^2} = x+y$)

$\Rightarrow \frac{x+y}{2} \geq \sqrt{xy}$

Thus, if x and y are positive real numbers, $\frac{x+y}{2} \geq \sqrt{xy}$

9b. Suppose a, b, c are integers such that $a|b$ and $a|(b+c)$.

Then $ak = b$ and $am = (b+c)$ for some $k, m \in \mathbb{Z}$.

Now $3c = 3(am - b)$
$= 3(am - ak)$
$= 3a(m-k)$
$= a(3m - 3k)$ where $3m - 3k \in \mathbb{Z}$.

Therefore $a|3c$.

Thus, If $a|b$ and $a|(b+c)$, then $a|3c$.

Homework #5

1.5 2ace, 3cf, 4a, 6bc, 7c, 10

2a. Suppose AB is not invertible

$\vdots$

Therefore A is not invertible or B is not invertible.

Thus, if A and B are invertible, then AB is invertible.

2c. Suppose A and B are invertible but AB is not invertible.

$\vdots$

$Q \wedge \sim Q$ (a contradiction)

Thus, if A and B are invertible, then AB is invertible.

2e. $\Rightarrow$ ① $(P \Rightarrow Q)$ Suppose A and B are invertible

$\vdots$

Therefore AB is invertible.

Thus if A and B are invertible, then AB is invertible.

$\Leftarrow$ ② $(Q \Rightarrow P)$ Suppose AB is invertible.

$\vdots$

Therefore A is invertible

Therefore B is invertible.

Thus, if AB is invertible, then A and B are invertible.

$(P \Leftrightarrow Q)$ ③ Hence, A and B are invertible iff. AB is invertible.

$(x \in \mathbb{Z})$

3c Prove "If x^2 is not divisible by 4, then x is odd"

proof:
Suppose x is even.
Then $x = 2k$ for some integer k.
So $x^2 = (2k)^2 = 4k^2$ where k^2 is an integer
Hence, 4 divides x^2.
Thus, if x is even then x^2 is divisible by 4
So, if x^2 is not divisible by 4, x is odd.

3f. Prove "If xy is odd then both x and y are odd" $(x, y \in \mathbb{Z})$

Proof: Suppose either x is even or y is even (or both).

Case 1: Suppose x is even.
Then $x = 2k$ for some integer k.
Now $xy = 2ky = 2(ky)$
which is even, since $ky \in \mathbb{Z}$

Case 2: Suppose y is even.
Then $y = 2k$ for some $k \in \mathbb{Z}$.
Now $xy = x(2k) = 2(kx)$
which is even since $ky \in \mathbb{Z}$.

In either case, if x or y is even, xy is even.
Hence, if xy is odd then x and y are odd.

4a. Prove "If $x^2+2x<0$, then $x<0$" $(x \in \mathbb{R})$

Suppose $x \geq 0$.
Then $x+2 \geq 2 > 0$, so $x+2>0$
Now $x(x+2) \geq 0$ since the product of two positive numbers is positive (and we get 0 when $x=0$)
Hence $x^2+2x \geq 0$.

Thus, if $x^2+2x<0$, then $x<0$.

6bc, 7c, 10

6b. Prove" If ab is odd then a and b are odd" $(a, b \in \mathbb{N})$

Proof: Suppose ab is odd, but at least one of a and b is even.

Case 1: Suppose a is even. Then $a = 2k$ for some integer k.
Thus $ab = (2k)b = 2(kb)$ which is even. $\Rightarrow\!\Leftarrow$
This is a contradiction since ab was assumed to be odd.

Case 2: Suppose b is even. then $b = 2k$ for some integer k.
Now $ab = a(2k) = 2(ak)$ which is even. $\Rightarrow\!\Leftarrow$
Again, this is a contradiction of the assumption that ab is odd.

Hence, if ab is odd, then a and b must be odd.

6c. Prove "If a is odd, then $a+1$ is even" $(a \in \mathbb{N})$
Let $a \in \mathbb{N}$.

Proof: Suppose a is odd and $a+1$ is odd.
Then $a = 2k+1$ for some $k \in \mathbb{Z}$
and $a+1 = 2m+1$ for some $m \in \mathbb{Z}$.
So $a = 2m$. Hence a is even. $\Rightarrow\!\Leftarrow$
This is a contradiction of the assumption that a is odd.
Thus, if a is odd, then $a+1$ is ~~od~~ even.

Proof: 7C. Prove "a is odd iff $a+1$ is even".

$\Rightarrow$ ① Suppose a is odd, then

$P \Rightarrow Q$ $a = 2K+1$ for some $K \in \mathbb{Z}$.

Now $a+1 = (2K+1)+1 = 2K+2$

$= 2(K+1)$ which is even

since $K+1$ is an integer.

Thus, if a is odd, $a+1$ is even.

$\Leftarrow$ ② Suppose a is even.

$Q \Rightarrow P$ Then $a = 2K$ for some $K \in \mathbb{Z}$.

via $\sim P \Rightarrow \sim Q$ Now $a+1 = (2K)+1 = 2K+1$

which is odd.

Thus, if a is even, then $a+1$ is odd.

So, if $a+1$ is even, then a is odd.

$P \Leftrightarrow Q$ ③ Hence a is odd iff $a+1$ is even.

10. Prove $\sqrt{5}$ is irrational.

Proof: Suppose $\sqrt{5}$ is rational

Then we can write $\sqrt{5} = a/b$

where $a, b \in \mathbb{Z}$, $b \neq 0$ and a and b

have no common factors other than ± 1.

(in other words, the fraction is in reduced form).

Now $\sqrt{5} = \frac{a}{b}$

$\Rightarrow 5 = a^2/b^2$

$\Rightarrow 5b^2 = a^2$

$\Rightarrow$ 5 divides a^2

$\Rightarrow$ 5 divides a, since 5 is prime.

$\Rightarrow a = 5K$ for some $K \in \mathbb{Z}$.

So, $\Rightarrow 5b^2 = a^2 = (5K)^2 = 25K^2$

$\Rightarrow b^2 = 5K^2$ (divide each side by 5)

$\Rightarrow$ 5 divides b. $\Rightarrow\Leftarrow$

Now 5 divides both a and b

which is a contradiction of

the assumption that a and

b had no common factors other than ± 1

Thus $\sqrt{5}$ is not rational.

Hence, $\sqrt{5}$ is irrational.

M301 Homework #6

1.6 1 bd, 2ad, 5def, 7e   1.7 1e, 2c, 3ab

1.6 | 1.b. $(\exists m)(\exists n)(15m+12n=3)$ $(\mathcal{U}=\mathbb{Z})$

Proof: let $m=1, n=-1$.
Then $15m+12n = 15(1)+12(-1)=3$.

d. $\sim(\exists m)(\exists n)(12m+15n=1) \Leftrightarrow (\forall m)(\forall n)(12m+15n \neq 1)$

Proof: By way of contradiction, suppose there exist integers m and n such that $12m+15n=1$. Then $3(4m+5n)=1$, where $4m+5n$ is an integer.
Thus, 3 divides 1. $\Rightarrow\!\Leftarrow$
This is a contradiction since the only integer factors of 1 are 1 and −1.
Thus, there do not exist integers m and n with $15m+12n=3$

2a. $(\forall a)(\forall b)(\forall c)(a|b \wedge a|c \Rightarrow (\forall x)(\forall y)(a|(bx+cy)))$

Proof. Let a,b,c be integers such that a divids b and a divides c.
Then $ak=b$ and $at=c$ for some integers k and t.
Now let x and y be integers.
$bx+cy = (ak)x+(at)y = a(kx+ty)$ where $kx+ty \in \mathbb{Z}$
So, a divides $bx+cy$.
Thus, for all integers a,b,c, if $a|b$ and $a|c$, then $a|(bx+cy)$ for all integers x and y.

proof:
2d. Suppose m and n are integers such that $am+bn=1$. Let d be an integer greater than 1. By way of contradiction, suppose d divides both a and b. Then $dk=a$ and $dt=b$ for some $k,t \in \mathbb{Z}$.
Now $1=am+bn = (dk)m+(dt)n = d(km+tn)$ where $km+tn \in \mathbb{Z}$. Thus $d|1$. But $d>1$, so d cannot divide 1. $\Rightarrow\!\Leftarrow$. Thus, if $am+bn=1$ and $d>1$ then $d \nmid a$ or $d \nmid b$.

Thus, if x is rational and y is irrational, then $x+y$

5d. Let $a, b, c \in \mathbb{Z}$. Prove or disprove: if $a|bc$, then $a|b$ or $a|c$.

Disprove: let $b=3$, $c=4$, $a=6$

$6|(3)(4)$ since $6 \times 2 = 12 = 3 \times 4$.

But $6 \nmid 3$ and $6 \nmid 4$.

e. Let $a,b,c,d \in \mathbb{Z}$. Prove or disprove: If $a|(b-c)$ and $a|(c-d)$, then $a|(b-d)$

Prove. Let $a,b,c,d \in \mathbb{Z}$ such that $a|(b-c)$ and $a|(c-d)$

Now $ak = b-c$ for some $k \in \mathbb{Z}$ and $at = (c-d)$ for some $t \in \mathbb{Z}$

thus, $b-d = b-c+c-d = (b-c)+(c-d)$

$= ak + at = a(k+t)$ where $k+t \in \mathbb{Z}$.

Hence, $a|(b-d)$.

Therefore if $a|(b-c)$ and $a|(c-d)$, then $a|(b-d)$

f. Prove or disprove: $(\forall x \in \mathbb{R})(x^2 - x \geq 0)$

Disprove: let $x = 1/2$, then $(1/2)^2 - (1/2) = 1/4 - 1/2 = -1/4 < 0$.

7e. There is no largest natural number

$\sim(\exists n)(\forall m)(n \geq m) \Leftrightarrow (\forall n)(\exists m)(n < m)$

Proof: Suppose, by way of contradiction that there is a largest natural number. Call it k.

Now consider $k+1$. $k+1 > k$ $\Rightarrow\Leftarrow$

This contradicts the fact that ~~there is no largest~~ ~~nat~~ k is the largest natural number.

Thus, there is no largest natural number.

1.7

1e. Proof: suppose x and y are rational numbers with $x<y$.

Now consider $\frac{x+y}{2}$. Then $x = \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2} = \frac{x+y}{2} < \frac{y}{2} + \frac{y}{2} = y$

And $\frac{x+y}{2}$ is rational since it is formed by adding two rational numbers and dividing by the rational number 2.

Thus there is a rational number between two rationals x and y with $x<y$.

1.7.

2c, 3ab

2c. Let l be the line $2x + ky = 3k$
Prove there is a unique $k \in \mathbb{R}$ so that l passes through $(1, 4)$.

proof: Suppose l is the line $2x + ky = 3k$, ~~where $k \in \mathbb{R}$~~.

Existence: Let $k = -2$ (which is in $\mathbb{R}$)

Now $2x + ky = 3k$
$\Rightarrow 2x - 2y = 3(-2)$
$\Rightarrow 2x - 2y = -6$
When $x = 1$ and $y = 4$, we get
$2(1) - 2(4) = 2 - 8 = -6$
Thus, if $k = -2$, $(1, 4)$ is on l.

Scratch work:
$2(1) + 4k = 3k$
$2 = -k$
$\boxed{k = -2}$

Uniqueness:
Suppose n and m are real numbers such that $(1, 4)$ is on l when $k = n$ and when $k = m$.
Then $2 + 4n = 3n$
and $2 + 4m = 3m$
$\Rightarrow 2 = -n$ and $2 = -m$
Hence, $-n = 2 = -m$
$\Rightarrow -n = -m$
$\Rightarrow n = m$.
Thus, the value of k is unique.

Therefore there is a unique $k \in \mathbb{R}$ so that $(1, 4)$ lies on the line $2x + ky = 3k$.

3a. Prove that if x is rational and y is irrational, then $x + y$ is irrational.

Proof: By way of contradiction,
Suppose x is rational and y is irrational, but $x + y$ is rational.
Then $x = a/b$ and $x + y = c/d$ for some $a, b, c, d \in \mathbb{Z}$ where $b \neq 0$, $d \neq 0$

Now $x + y = a/b + y = c/d$
Hence, $y = c/d - a/b = \dfrac{cb - ad}{bd}$, which is rational since $a, b, c, d \in \mathbb{Z}$, $bd \neq 0$

$\Rightarrow\!\Leftarrow$ This is a contradiction of the assumption that y was irrational.
Thus, if x is rational and y is irrational, then $x + y$ is irrational