Hopf Algebras, definitions and examples

March 26, 2002

1 Introduction

This presentation, as the title indicates, is designed to be an introduction to Hopf algebras, covering basic definitions and some examples. Most of the material is from [Mon91] and [Kas95]. Only a couple of the simpler "proofs" are mine. These are generally the type of proof that were omitted from the source document, being considered trivial or obvious. In general, such things are not so obvious to me, so to clarify the material for myself, I wrote out the explanations.

If time permits I would like to also talk about how Hopf algebras are related to what I am studying.

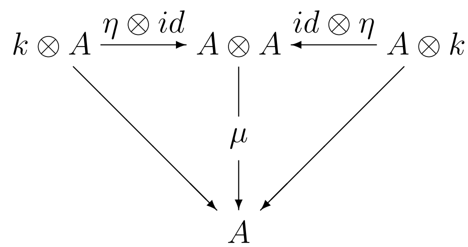
2 Algebras and Coalgebras

Before stating the definition of a Hopf algebra, we should first go over a few preliminary definitions. First recall the definition of an algebra over a field k.

Definition 2.1 *An* **algebra** *over a field k is a vector space, A, together with two linear maps, a multiplication $\mu : A \bigotimes A \to A$, and a unit map $\eta : k \to A$ such that the following diagrams commute:*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes id} & A \otimes A \\ {\scriptstyle id \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

and



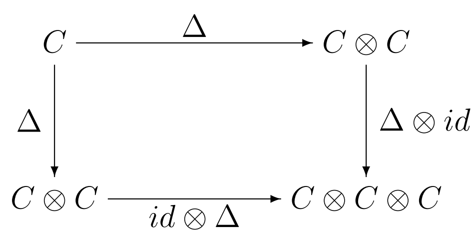
where the lower left and right maps are simply scalar multiplication.

Note: the second diagram implies

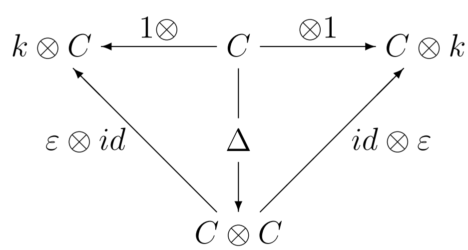
$$1_A = u(1_k)$$

Examples of algebras are the polynomial algebras and the matrix algebras.

Definition 2.2 *A* **coalgebra** *is a vector space C together with two linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$, such that the following two diagrams commute.*



and



Definition 2.3 *If C and D are coalgebras with respective comultiplication maps Δ_C and Δ_D, and respective counit maps ε_C and ε_D then*

1. *A map $f : C \to D$ is a* **coalgebra morphism** *if $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ i.e. the following diagram commutes.*

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ \Delta_C \big\downarrow & & \big\downarrow \Delta_D \\ C \otimes C & \xrightarrow[f \otimes f]{} & D \otimes D \end{array}$$

2. *A subspace $I \subseteq C$ is a* **coideal** *if $\Delta I \subseteq I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$*

Example 2.1 $(A(X))$ *As an example of a coalgebra, consider the polynomials of four variables over the field $\mathbb{C}$*

$$C = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}]$$

We can define the algebra morphisms Δ and ε on the generators of C by

$$\Delta(x_{ij}) = \sum_{k=1}^{2} x_{ik} \otimes x_{kj}$$

$$\varepsilon(x_{ij}) = \delta_{ij}$$

For later use, we will define $\det = x_{11}x_{22} - x_{12}x_{21} \in C$. Also, this algebra will be referred to as $A(X)$ where X is the set of all 2×2 matrices with complex coefficients. This algebra is also referred to as the algebra of regular functions of X.

Example 2.2 (Divided Powers) *If we let $C = \mathbb{C}[t]$ be the polynomials of one variable over $\mathbb{C}$, then we can define a comultiplication and counit by*

$$\Delta(t^n) = \sum_{p+q=n} t^p \otimes t^q$$

$$\varepsilon(t^n) = \delta_{n0}$$

We can demonstrate the coassociativity (for $n = 2$) with the following calculations:

$$\begin{aligned} (id \otimes \Delta) \circ \Delta(t^2) &= (id \otimes \Delta)(t^2 \otimes 1 + t \otimes t + 1 \otimes t^2) \\ &= t^2 \otimes 1 \otimes 1 + t \otimes 1 \otimes t + t \otimes t \otimes 1 \\ &\quad +1 \otimes t^2 \otimes 1 + 1 \otimes t \otimes t + 1 \otimes 1 \otimes t^2 \\ &= (\Delta \otimes id)(t^2 \otimes 1 + t \otimes t + 1 \otimes t^2) \\ &= (\Delta \otimes id) \circ \Delta(t^2) \end{aligned}$$

To show that $\varepsilon(t^n) = \delta_{n0}$ defines a counit map we check that $(id\otimes\varepsilon)\circ\Delta(t^n) = t^n \otimes 1$ and that $(\varepsilon \otimes id) \circ \Delta(t^n) = 1 \otimes t^n$

$$
\begin{aligned}
(id \otimes \varepsilon) \circ \Delta(t^n) &= (id \otimes \varepsilon) \left(\sum_{p+q=n} t^p \otimes t^q \right) \\
&= t^n \otimes 1
\end{aligned}
$$

Similarly, $(\varepsilon \otimes id) \circ \Delta(t^n) = 1 \otimes t^n$.

Remark 2.1 (Construction of an Algebra from a Coalgebra) *Given an coalgebra C over a field k, we may consider the dual space $C^* = Hom(C, k)$. We see here that there is a natural way of dualizing the coalgebra structure of C into an algebra structure on C^*. Associated with the coalgebra structure of C we may define the following multiplication and unit maps on C^**

$$\mu(f \otimes g)(c) = (f \otimes g) \circ \Delta(c)$$

$$\eta(\alpha)(c) = \alpha\varepsilon(c)$$

$$f, g \in C^*, \quad c \in C, \quad \alpha \in k$$

3 Bialgebras and Hopf Algebras

We see in example 2.1 that we defined a coalgebra structure on top of an existing algebra. This leads us to the following definition which combines both ideas.

Definition 3.1 *Given a space B, B is a* **bialgebra** *if (B, Δ, ε) is a coalgebra, (B, μ, η) is an algebra and either of the following equivalent conditions is true:*

1. *Δ and ε are algebra morphisms*
2. *μ and η are coalgebra morphisms*

This bialgebra structure is often denoted by $(B, \mu, \eta, \Delta, \varepsilon)$

Proposition 3.1 *The conditions in the previous definition are indeed equivalent. (Here we will simply show that Δ and μ are compatible, leaving it to the interested reader to verify the conditions for the unit and counit maps).*

Proof: Here, we must make use of the algebra and coalgebra structures of $B \otimes B$, which are defined respectively by the maps $\mu_{B\otimes B}$ and $\Delta_{B\otimes B}$ [Kas95]:

$$\mu_{B\otimes B} = \mu_B \otimes \mu_B \circ (id \otimes \tau \otimes id)$$
$$\Delta_{B\otimes B} = (id \otimes \tau \otimes id) \circ (\Delta_B \otimes \Delta_B)$$

Where $\tau : B \otimes B \to B \otimes B$ defined by $\tau(a \otimes b) = b \otimes a$.
Now, if μ_B is a coalgebra morphism we have

$$\begin{aligned} \Delta_B \circ \mu_B &= (\mu_B \otimes \mu_B) \circ \Delta_{B\otimes B} \\ &= (\mu_B \otimes \mu_B) \circ [(id \circ \tau \circ id) \circ (\Delta_B \otimes \Delta_B)] \end{aligned}$$

and if Δ_B is an algebra morphism then

$$\begin{aligned} \Delta_B \circ \mu_B &= \mu_{B\otimes B} \circ (\Delta_B \otimes \Delta_B) \\ &= [(\mu_B \otimes \mu_B) \circ (id \circ \tau \circ id)] \circ (\Delta_B \otimes \Delta_B) \end{aligned}$$

which are equivalent because the composition of maps is associative.

In Example 2.1, Δ (and ε) were defined to be algebra morphisms, thus the defined structure is automatically a bialgebra. The following are a few more examples of bialgebras:

Example 3.1 *If we let G be a group then $B = \mathbb{C}G$, the associated group algebra, becomes a bialgebra with the following defined maps*

$$\Delta(g) = g \otimes g, \quad \forall g \in G$$

$$\varepsilon(g) = 1, \quad \forall g \in G$$

Definition 3.2 *If C is a any coalgebra, then for $c \in C$, we say that c is* **grouplike** *if $\Delta(c) = c \otimes c$ and if $\varepsilon(c) = 1$. The set of all grouplike elements of a coalgebra is denoted $G(C)$.*

An interesting point to mention here is that if B is a group algebra, then $G(B) = G$, the original group. [Mon91]. Also, it can be shown through direct calculation that $det = x_{11}x_{22} - x_{12}x_{21}$ from the bialgebra of Example

2.1 is grouplike. As follows:

$$\begin{aligned}
\Delta(det) &= \Delta(x_{11}x_{22} - x_{12}x_{21}) \\
&= \Delta(x_{11})\Delta(x_{22}) - \Delta(x_{12})\Delta(x_{21}) \\
&= (x_{11} \otimes x_{11} + x_{12} \otimes x_{21})(x_{21} \otimes x_{12} + x_{22} \otimes x_{22}) \\
&\quad -(x_{11} \otimes x_{12} + x_{12} \otimes x_{22})(x_{21} \otimes x_{11} + x_{22} \otimes x_{21}) \\
&= (x_{11}x_{22} - x_{12}x_{21}) \otimes (x_{11}x_{22} - x_{12}x_{21}) \\
&= det \otimes det
\end{aligned}$$

Example 3.2 ($U(\mathfrak{sl}(2))$) *Consider the universal enveloping algebra of $\mathfrak{sl}(2)$, $U(\mathfrak{sl}(2))$. One can think of $U(\mathfrak{sl}(2))$ as the polynomial algebra of three generators e, f, and h, with the added relations*

$$[x, y] = H, \quad [h, x] = 2x, \quad [h, y] = -2y$$

Also, note that the set $\{x^i y^j h^k | i, j, k \in \mathbb{Z}_+\}$ is a basis of $U(\mathfrak{sl}(2))$ as a result of the Poincaré-Birkhoff-Witt theorem [Kas95]. If we define the comultiplication and counit maps on $U(\mathfrak{sl}(2))$ in the following manner, then it has a bialgebra structure.

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad \forall x \in \mathfrak{sl}(2)$$

Definition 3.3 *Given an algebra (A, μ, η), a coalgebra (C, Δ, ε) and two linear maps $f, g : C \to A$ then the* **convolution** *of f and g is the linear map $f \star g : C \to A$ defined by*

$$f \star g(c) = \mu \circ (f \otimes g) \circ \Delta(c), \qquad c \in C$$

Definition 3.4 *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. An endomorphism S of H is called an* **antipode** *for the bialgebra H if*

$$id_H \star S = S \star id_H = \eta \circ \varepsilon$$

A **Hopf algebra** *is a bialgebra with an antipode.*

Example 3.3 *The (group) bialgebra B, of Example 3.1 is a Hopf algebra with antipode S defined by*

$$S(g) = g^{-1}$$

We can show that S is an antipode by

$$\begin{aligned} id \star S(g) &= \mu \circ (id \otimes S) \circ \Delta(g) \\ &= \mu \circ (id \otimes S)(g \otimes g) \\ &= \mu(g \otimes g^{-1}) \\ &= 1_B \\ &= \eta \circ \varepsilon(g) \end{aligned}$$

Proposition 3.2 *Given a Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon)$, with antipode S, then for any grouplike element $g \in H$, $S(g) = g^{-1}$.*

Proof. Using the definition of the antipode S, we have

$$\begin{aligned} id \star S(g) &= \eta \circ \varepsilon(g) \\ \mu \circ (id \otimes S) \circ \Delta(g) &= 1_H \\ \mu(g \otimes S(g)) &= 1_H \\ g \cdot S(g) &= 1_G \end{aligned}$$

Similarly, $S(g) \cdot g = 1$

Example 3.4 *An example of a bialgebra that is* **not** *a Hopf algebra, refer back to Example 2.1, $A(X)$. If it were a Hopf algebra (i.e. if it had an antipode S) then $S(det) = det^{-1}$, because det is grouplike. However, det is not invertible, so $A(X)$ is not a Hopf algebra.*

However, we can use this bialgebra to construct the next example of a Hopf algebra.

Example 3.5 *Consider the polynomial algebra $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}][1/det]$. This is obtained by simply adjoining the inverse of det to the previous bialgebra. Other notations found in the literature for describing the same algebra are $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}, det^{-1}]$ and $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}][t]/(t(det) - 1)$. Now we are free to define the map S in the following manner [Kas95].*

$$S(x_{11}) = det^{-1}x_{22}, \quad S(x_{12}) = -det^{-1}x_{12}$$

$$S(x_{22}) = det^{-1}x_{11}, \quad S(x_{21}) = -det^{-1}x_{21}$$

This is often referred to as the algebra of regular functions on the group $GL(2, \mathbb{C})$, and is denoted as $A(GL(2, \mathbb{C}))$.

Example 3.6 *Here, we let q be some complex number that is not a root of unity. We will now construct $A_q(X)$ as the polynomial algebra (similar to $A(X)$ from Example 2.1). However, unlike $A(X)$ we will require that the generators of $A_q(X)$ no longer commute. In fact, we define the relations on the generators to be*

$$x_{12}x_{11} = qx_{11}x_{12}, \quad x_{22}x_{12} = qx_{12}x_{22}$$

$$x_{21}x_{11} = qx_{11}x_{21}, \quad x_{22}x_{21} = qx_{21}x_{22}$$

$$x_{12}x_{21} = x_{21}x_{12}, \quad x_{11}x_{22} - x_{22}x_{11} = (q^{-1} - q)x_{12}x_{21}$$

Also, we will define the element $det_q = x_{11}x_{22} - q^{-1}x_{12}x_{21} = x_{22}x_{11} - qx_{12}x_{21}$ and we also define the algebra morphisms (the same as for $A(X)$)

$$\Delta(x_{ij}) = \sum_{k=1}^{2} x_{ik} \otimes x_{kj}$$

$$\varepsilon(x_{ij}) = \delta_{ij}$$

Just as $A(X)$ is a bialgebra, so is $A_q(X)$. And it can also be shown that

$$\Delta(det_q) = det_q \otimes det_q$$

and is thus grouplike. As in the previous case, we can show that $A_q(Gl(2,n)) = C[x_{11}, x_{12}, x_{21}, x_{22}, det_q^{-1}]$, with the relations of $A_q(X)$, is a Hopf algebra if we define the antipode S by

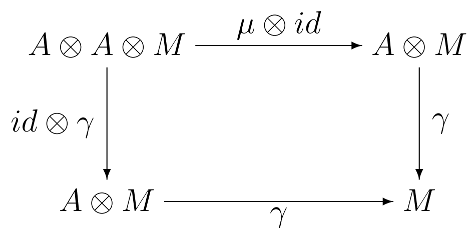
$$S(x_{11}) = det_q^{-1}x_{22}, \quad S(x_{12}) = -det^{-1}qx_{12}$$

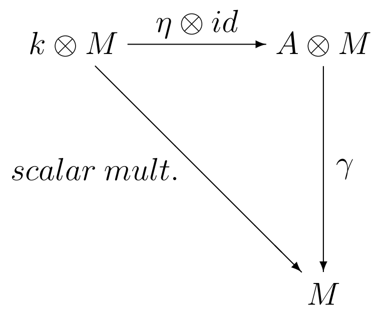
$$S(x_{22}) = det_q^{-1}x_{11}, \quad S(x_{21}) = -det^{-1}q^{-1}x_{21}$$

4 Modules and Comodules

Recall the definition of a module over an algebra A.

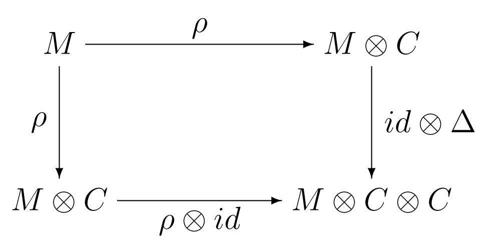
Definition 4.1 *For an algebra A (over a field k), a (left) A-module is a space M with a linear map $\gamma : A \otimes M \to M$ such that the following diagrams commute:*

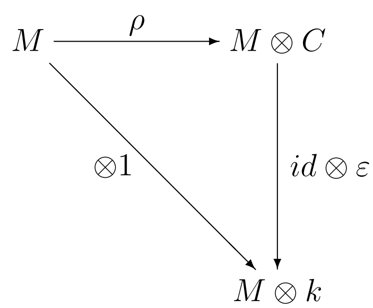




The catagory of left A-modules is denoted ${}_A\mathcal{M}$. Right modules are defined similarly.

Definition 4.2 *For a coalgebra C (over a field k), a (right) C-comodule is a space M with a linear map $\rho : M \to M \otimes C$ such that the following diagrams commute:*





The category of right C-modules is denoted $\mathcal{M}^C$. And the left comodules are defined similarly.

If C is a coalgebra and we have a right C-comodule V (resp. a left C-comodule W) with structure map $R_C : V \to V \otimes C$ (resp. $L_C : W \to C \otimes W$). Then V and W have natural left and right C^* module structures defined by the following

$$a.v = (id \otimes a)R_G(v), \quad a \in C^*, v \in V$$

$$w.a = (a \otimes id)L_G(w), \quad a \in C^*, w \in W$$

where C^* is the dual space to C.

This is a very convenient way to construct a module in certain cases. For example, given a bialgebra B, B is itself a B-comodule, using the comultiplication as the comodule structure map. As such, we know that B is a B^*-module.

References

[Kas95] Christian Kassel. *Quantum Groups*. Springer-Verlag New York, Inc., 1995.

[Mon91] Susan Montgomery. Hopf algebras and their actions on rings. In *CBMS Regional Conference Series in Mathematics*, volume 82. Conference Board of the Mathematical Sciences, American Mathematical Society, 1991.