Introduction to Hopf Algebras

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The last twenty years has seen a number of advances in the area of Hopf algebras. Among these are the introduction of quantum groups, and the unification of group actions, actions of Lie Algebras, and graded algebras . As an introduction to Hopf algebras, this talk will cover basic definitions and some examples. We first briefly discuss the idea of a tensor product of vector spaces and then move on to algebras, coalgebras, Hopf algebras and finally quantum groups. In order to cover the amount of material, many of the details will be omitted and there will be no proofs, i.e., in order to present a view of the forest, there will be no pictures of pine needles. However, those interested can find most of the material in Montgomery[Mon91] and Kassel [Kas95]. If time permits, some quantum groups will also be discussed.

Tensor Product

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Algebras and Coalgebras

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Bialgebras and Hopf Algebras

↓

Quantum Groups

↓

Examples

Tensor Products

Let V and W be any two vector spaces over a field $\mathbb{F}$.

$V \otimes W$ can be thought of as the space of objects that look like

$$v_1 \otimes w_1 + v_2 \otimes w_2 + \cdots + v_k \otimes w_k$$

where

$$v_i \in V, \quad w_i \in W$$

and with the following bilinear relations

$$a_1(v_1 \otimes w) + a_2(v_2 \otimes w) = ((a_1 v_1 + a_2 v_2) \otimes w) \quad (1)$$

$$\textit{and}$$

$$a_1(v \otimes w_1) + a_2(v \otimes w_2) = (v \otimes (a_1 w_1 + a_2 w_2)) \quad (2)$$

where $a_i \in \mathbb{F}$, $v, v_i \in V$ and $w, w_i \in W$.

Remark 0.1 *If B_V is a basis for V and B_W is a basis for W, then*

$$B_{V\otimes W} = \{e \otimes f | e \in B_V \text{ and } f \in B_W\}$$

is a basis for $V \otimes W$

Example 0.1 *Consider the polynomials of one variable over* $\mathbb{C}$

$$\mathbb{C}[x]$$

These polynomials are a vector space with basis

$$\mathfrak{B} = \{1, x, x^2, x^3, x^4, \ldots\}$$

So examples of objects in $\mathbb{C}[x] \otimes \mathbb{C}[x]$ *are*

$$4x^2 \otimes 3x$$

$$x \otimes 3$$

$$(x^2 - 4) \otimes (x^3 - 2x^2 + x)$$

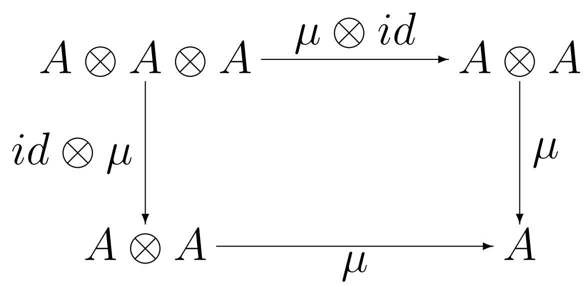
$$x^2 \otimes x^2 + x^2 \otimes 4x^3 + (x - 2x^2) \otimes x^3$$

Note: We may use the bilinear relations 1 and 2 to do the following:

$$\begin{aligned} x^2 \otimes x^2 + x^2 \otimes 4x^3 \ &+\ (x - 2x^2) \otimes x^3) \\ &= x^2 \otimes x^2 + 4x^2 \otimes x^3 \\ &\quad +(x - 2x^2) \otimes x^3 \\ &= x^2 \otimes x^2 + (4x^2 + x - 2x^2) \otimes x^3 \\ &= x^2 \otimes x^2 + (2x^2 + x) \otimes x^3 \end{aligned}$$

Algebras and Coalgebras

Definition 0.1 *An* **algebra** *over a field k is a vector space, A, together with two linear maps, a multiplication $\mu : A \otimes A \to A$, and a unit map $\eta : k \to A$ such that the following diagrams commute:*



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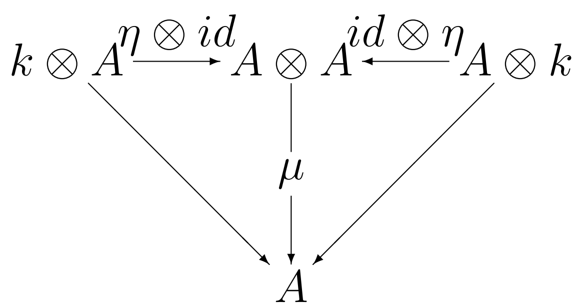


where the lower left and right maps are simply scalar multiplication.

Example 0.2 *Considering the example of* $\mathbb{C}[x]\otimes\mathbb{C}[x]$*, we define* μ *as the standard polynomial multiplication. E.g.*

$$\mu(x \otimes 3) = 3x$$

$$\mu(4x^2 \otimes 3x) = 12x^3$$

$$\mu\left((x^2 - 4) \otimes (x^3 - 2x^2 + x)\right) = x^5 - 2x^4 - 3x^3 + 8x^2 - 4x$$

$$\begin{aligned} \mu\left(x^2 \otimes x^2 + x^2 \otimes 4x^3 + (x - 2x^2) \otimes x^3\right) &= x^4 + 4x^5 + x^4 - 2x^5 \\ &= 2x^4 + 2x^5 \end{aligned}$$

Example 0.3 ($A(X)$) *Now consider the polynomial algebra*

$$A(X) = \mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}] \tag{3}$$

As a vector space, it's basis is

$$\{x_{1,1}^i x_{1,2}^j x_{2,1}^k x_{2,2}^l : i, j, k, l \geq 0 \in \mathbb{Z}\}$$

and examples of elements of $A(X)$

$$x_{1,2}x_{2,2} - 3x_{1,1}x_{2,2} + x_{2,2}$$

$$x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$$

If we think of

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}$$

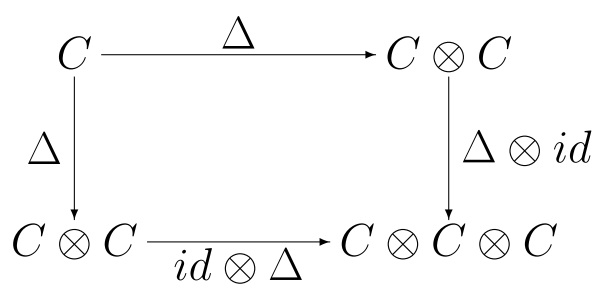
then we can think of the polynomials of $A(X)$ *as functions from* $Mat(2, \mathbb{C})$ *to* $\mathbb{C}$. *They are in fact often called the* **regular functions** *of* $Mat(2, \mathbb{C})$. *Let* $f = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$ *then*

$$f\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 1 \cdot 1 - 1 \cdot 0 = 1$$

Definition 0.2 *If A and B are algebras with respective multiplications μ_A and μ_B then*

> *A linear map $g : A \to B$ is an* **algebra morphism** *if $g \circ \mu_A = \mu_B \circ (g \otimes g)$ i.e. the following diagram commutes.*
>
> $$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ {\scriptstyle g \otimes g}\downarrow & & \downarrow{\scriptstyle g} \\ B \otimes B & \xrightarrow[\mu_B]{} & B \end{array}$$
>
> *I.e. if $a, b \in A$ then*
>
> $$g(ab) = g(a)g(b)$$

Definition 0.3 *A* **coalgebra** *is a vector space C together with two linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$, such that the following two diagrams commute.*



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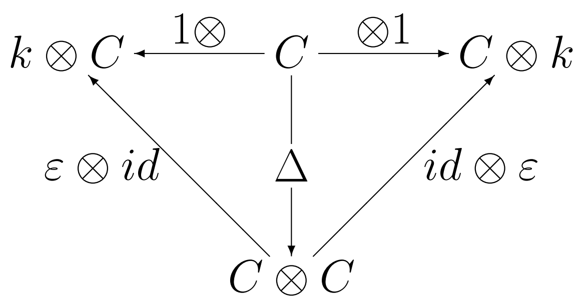


Example 0.4 ($A(X)$) *We see that if we define the comultiplication and counit maps on $A(X)$ in the following manner, then $A(X)$ is a coalgebra.*

$$\Delta(x_{i,j}) = x_{i,1} \otimes x_{1,j} + x_{i,2} \otimes x_{2,j} \tag{4}$$

$$\varepsilon(x_{i,j}) = \delta_{i,j} = \begin{Bmatrix} 1, i = j \\ 0, i \neq j \end{Bmatrix} \tag{5}$$

We extend the action of Δ to the rest of $A(X)$ by defining it to be an algebra morphism. I.e.

$$\Delta(x_{i,j}x_{k,l}) = \Delta(x_{i,j})\Delta(x_{k,l})$$

E.g.

$$\Delta(x_{1,1}) = x_{1,1} \otimes x_{1,1} + x_{1,2} \otimes x_{2,1}$$

$$\varepsilon(x_{1,1}) = 1$$

$$\varepsilon(x_{1,2}) = 0$$

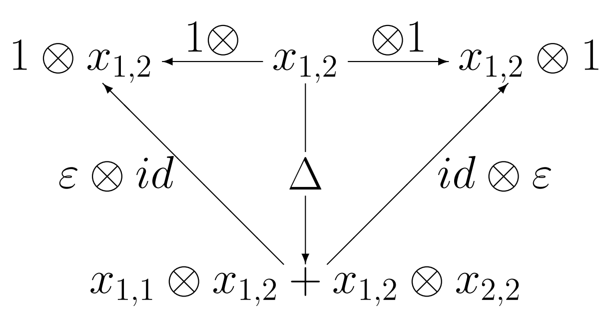
Example 0.5 (Coassociativity of Δ on $A(X)$)

$$\begin{aligned}(id \otimes \Delta) \circ \Delta(x_{1,2}) &= (id \otimes \Delta)(x_{1,1} \otimes x_{1,2} + x_{1,2} \otimes x_{2,2}) \\ &= (x_{1,1} \otimes (x_{1,1} \otimes x_{1,2} + x_{1,2} \otimes x_{2,2}) \\ &\quad + x_{1,2} \otimes (x_{2,1} \otimes x_{1,2} + x_{2,2} \otimes x_{2,2})) \\ &= x_{1,1} \otimes x_{1,1} \otimes x_{1,2} + x_{1,1} \otimes x_{1,2} \otimes x_{2,2} \\ &\quad + x_{1,2} \otimes x_{2,1} \otimes x_{1,2} + x_{1,2} \otimes x_{2,2} \otimes x_{2,2}\end{aligned}$$

and

$$\begin{aligned}(\Delta \otimes id) \circ \Delta(x_{1,2}) &= (\Delta \otimes id)(x_{1,1} \otimes x_{1,2} + x_{1,2} \otimes x_{2,2}) \\ &= (x_{1,1} \otimes x_{1,1} + x_{1,2} \otimes x_{2,1}) \otimes x_{1,2} \\ &\quad + (x_{1,1} \otimes x_{1,2} + x_{1,2} \otimes x_{2,2}) \otimes x_{2,2} \\ &= x_{1,1} \otimes x_{1,1} \otimes x_{1,2} + x_{1,2} \otimes x_{2,1} \otimes x_{1,2} \\ &\quad x_{1,1} \otimes x_{1,2} \otimes x_{2,2} + x_{1,2} \otimes x_{2,2} \otimes x_{2,2}\end{aligned}$$

Example 0.6 (Counit , ε, of $A(X)$)



Definition 0.4 *Let C be any coalgebra, and let $c \in C$. c is* **group-like** *if $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$. The set of group-like elements in C is denoted by $G(C)$.*

Example 0.7 *In $A(X)$ there is a special element*

$$det = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$$

It can be shown that det is group-like.

$$\Delta(det) = det \otimes det$$

$$
\begin{aligned}
\Delta(det) &= \Delta(x_{1,1}x_{2,2} - x_{1,2}x_{2,1}) \\
&= \Delta(x_{1,1})\Delta(x_{2,2}) - \Delta(x_{1,2})\Delta(x_{2,1}) \\
&= (x_{1,1} \otimes x_{1,1} + x_{1,2} \otimes x_{2,1})(x_{2,1} \otimes x_{1,2} + x_{2,2} \otimes x_{2,2}) \\
&\quad -(x_{1,1} \otimes x_{1,2} + x_{1,2} \otimes x_{2,2})(x_{2,1} \otimes x_{1,1} + x_{2,2} \otimes x_{2,1}) \\
&= (x_{1,1} \otimes x_{1,1})(x_{2,1} \otimes x_{1,2}) + (x_{1,1} \otimes x_{1,1})(x_{2,2} \otimes x_{2,2}) \\
&\quad +(x_{1,2} \otimes x_{2,1})(x_{2,1} \otimes x_{1,2}) + (x_{1,2} \otimes x_{2,1})(x_{2,2} \otimes x_{2,2}) \\
&\quad -(x_{1,1} \otimes x_{1,2})(x_{2,1} \otimes x_{1,1}) - (x_{1,1} \otimes x_{1,2})(x_{2,2} \otimes x_{2,1}) \\
&\quad -(x_{1,2} \otimes x_{2,2})(x_{2,1} \otimes x_{1,1}) - (x_{1,2} \otimes x_{2,2})(x_{2,2} \otimes x_{2,1}) \\
&= (x_{1,1}x_{2,1} \otimes x_{1,1}x_{1,2}) + (x_{1,1}x_{2,2} \otimes x_{1,1}x_{2,2}) \\
&\quad +(x_{1,2}x_{2,1} \otimes x_{2,1}x_{1,2}) + (x_{1,2}x_{2,2} \otimes x_{2,1}x_{2,2}) \\
&\quad -(x_{1,1}x_{2,1} \otimes x_{1,2}x_{1,1}) - (x_{1,1}x_{2,2} \otimes x_{1,2}x_{2,1}) \\
&\quad -(x_{1,2}x_{2,1} \otimes x_{2,2}x_{1,1}) - (x_{1,2}x_{2,2} \otimes x_{2,2}x_{2,1}) \\
&= (x_{1,1}x_{2,2} \otimes x_{1,1}x_{2,2}) - (x_{1,1}x_{2,2} \otimes x_{1,2}x_{2,1}) \\
&\quad +(x_{1,2}x_{2,1} \otimes x_{2,1}x_{1,2}) - (x_{1,2}x_{2,1} \otimes x_{2,2}x_{1,1}) \\
&= x_{1,1}x_{2,2} \otimes (x_{1,1}x_{2,2} - x_{1,2}x_{2,1}) \\
&\quad +x_{1,2}x_{2,1} \otimes (x_{2,1}x_{1,2} - x_{2,2}x_{1,1}) \\
&= x_{1,1}x_{2,2} \otimes (x_{1,1}x_{2,2} - x_{1,2}x_{2,1}) \\
&\quad -x_{1,2}x_{2,1} \otimes (x_{2,2}x_{1,1} - x_{2,1}x_{1,2}) \\
&= (x_{1,1}x_{2,2} - x_{1,2}x_{2,1}) \otimes (x_{1,1}x_{2,2} - x_{1,2}x_{2,1}) \\
&= det \otimes det
\end{aligned}
$$

Bialgebras and Hopf Algebras

Definition 0.5 *Given a space B, B is a* **bialgebra** *if (B, Δ, ε) is a coalgebra, (B, μ, η) is an algebra and either of the following equivalent conditions is true:*

1. *Δ and ε are algebra morphisms*
2. *μ and η are coalgebra morphisms*

This bialgebra structure is often denoted by $(B, \mu, \eta, \Delta, \varepsilon)$.

Remark 0.2 *$A(X)$ is a bialgebra.*

Definition 0.6 (Convolution) *Given an algebra* (A, μ, η), *a coalgebra* (C, Δ, ε) *and two linear maps* $f, g : C \to A$ *then the* **convolution** *of* f *and* g *is the linear map* $f \star g : C \to A$ *defined by*

$$(f \star g)(c) = \mu \circ (f \otimes g) \circ \Delta(c), \qquad c \in C$$

Definition 0.7 (Antipode and Hopf Algebra) *Let* $(H, \mu, \eta, \Delta, \varepsilon)$ *be a bialgebra. An endomorphism* S *of* H *is called an* **antipode** *for the bialgebra* H *if*

$$id_H \star S = S \star id_H = \eta \circ \varepsilon$$

A **Hopf algebra** *is a bialgebra with an antipode.*

Example 0.8 **($A(G)$)** *Unfortunately, $A(X)$ has no antipode, so we construct a new space.*

$$A(G) = \mathbb{C}\left[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, det^{-1}\right] \tag{6}$$

The comultiplication and counit are defined as with $A(X)$

$$\Delta(x_{i,j}) = \sum_{k=1}^{2} x_{i,k} \otimes x_{k,j} \tag{7}$$

$$\varepsilon(x_{i,j}) = \delta_{i,j} \tag{8}$$

Then

$$\begin{aligned} S(x_{1,1}) &= x_{2,2}det^{-1} \\ S(x_{1,2}) &= -x_{1,2}det^{-1} \\ S(x_{2,1}) &= -x_{2,1}det^{-1} \\ S(x_{2,2}) &= x_{1,1}det^{-1} \end{aligned}$$

defines the antipode for $A(G)$ and makes $A(G)$ a Hopf algebra.

Quantum Groups

Example 0.9 *Let $q \in \mathbb{C}$ such that it is not a root of unity, then define*

$$A_q(X) = \mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}] \tag{9}$$

with added relations:

$$\begin{aligned}
x_{1,1}x_{1,2} &= qx_{1,2}x_{1,1} \\
x_{2,1}x_{2,2} &= qx_{2,2}x_{2,1} \\
x_{1,1}x_{2,1} &= qx_{2,1}x_{1,1} \\
x_{1,2}x_{2,2} &= qx_{2,2}x_{1,2}
\end{aligned}$$

$$x_{1,2}x_{2,1} = x_{2,1}x_{1,2}$$

$$x_{1,1}x_{2,2} = x_{2,2}x_{1,1} + \left(q - \frac{1}{q}\right)x_{1,2}x_{2,1}$$

Now if we define Δ and ε as we did for $A(X)$, $A_q(X)$ becomes a bialgebra.

Definition 0.8 *We define the quantum determinant by*

$$det_q = x_{1,1}x_{2,2} - qx_{1,2}x_{2,1} \in A_q(X) \qquad (10)$$

Example 0.10 ($A_q(Gl(n))$, a quantum Hopf algebra)

$$A_q(Gl(n)) = \mathbb{C}\left[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, {det_q}^{-1}\right] \tag{11}$$

The comultiplication and counit are defined as with $A(X)$ Then

$$\begin{aligned}
S(x_{1,1}) &= det_q^{-1} x_{2,2} \\
S(x_{1,2}) &= -q det_q^{-1} x_{1,2} \\
S(x_{2,1}) &= -\frac{1}{q} det_q^{-1} x_{2,1} \\
S(x_{2,2}) &= det_q^{-1} x_{1,1}
\end{aligned}$$

defines an antipode.

More Examples

Example 0.11 (Divided Powers) *If we let $C = \mathbb{C}[t]$ be the polynomials of one variable over $\mathbb{C}$, then we can define a comultiplication and counit by*

$$\Delta(t^n) = \sum_{p+q=n} t^p \otimes t^q$$

$$\varepsilon(t^n) = \delta_{n0}$$

We can demonstrate the coassociativity (for $n = 2$) with the following calculations:

$$\begin{aligned}
(id \otimes \Delta) \circ \Delta(t^2) &= (id \otimes \Delta)(t^2 \otimes 1 + t \otimes t + 1 \otimes t^2) \\
&= t^2 \otimes 1 \otimes 1 + t \otimes 1 \otimes t + t \otimes t \otimes 1 \\
&\quad +1 \otimes t^2 \otimes 1 + 1 \otimes t \otimes t + 1 \otimes 1 \otimes t^2 \\
&= (\Delta \otimes id)(t^2 \otimes 1 + t \otimes t + 1 \otimes t^2) \\
&= (\Delta \otimes id) \circ \Delta(t^2)
\end{aligned}$$

To show that $\varepsilon(t^n) = \delta_{n0}$ defines a counit map we check that $(id \otimes \varepsilon) \circ \Delta(t^n) = t^n \otimes 1$ and that $(\varepsilon \otimes id) \circ \Delta(t^n) = 1 \otimes t^n$

$$\begin{aligned}
(id \otimes \varepsilon) \circ \Delta(t^n) &= (id \otimes \varepsilon) \left(\sum_{p+q=n} t^p \otimes t^q \right) \\
&= t^n \otimes 1
\end{aligned}$$

Similarly, $(\varepsilon \otimes id) \circ \Delta(t^n) = 1 \otimes t^n$.

Example 0.12 *If we let G be a group then $B = \mathbb{C}G$, the associated group algebra, becomes a bialgebra with the following defined maps*

$$\Delta(g) = g \otimes g, \quad \forall g \in G$$

$$\varepsilon(g) = 1, \quad \forall g \in G$$

and a Hopf algeba with the antipode defined by

$$S(g) = g^{-1}$$

Remark 0.3 *The set of group-like elements of this Hopf algebra is the original group G.*

Example 0.13 ($U(\mathfrak{sl}(2))$) *Consider the universal enveloping algebra of* $\mathfrak{sl}(2)$*,* $U(\mathfrak{sl}(2))$*. One can think of* $U(\mathfrak{sl}(2))$ *as the polynomial algebra of three generators* e*,* f*, and* h*, with the added relations*

$$[e, f] = h, \ \ [h, e] = 2x, \ \ [h, f] = -2y \tag{12}$$

Also, note that the set $\{e^i f^j h^k : i, j, k \geq 0 \in \mathbb{Z}_+\}$ *is a basis of* $U(\mathfrak{sl}(2))$ *as a result of the Poincaré-Birkhoff-Witt theorem [Kas95]. If we define the comultiplication and counit maps on* $U(\mathfrak{sl}(2))$ *in the following manner, then it has a bialgebra structure.*

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \ \ \varepsilon(x) = 0, \ \ \forall x \in \mathfrak{sl}(2) \tag{13}$$

$$S(x) = -x \tag{14}$$

Infact, the enveloping algebra of any Lie algebra is a Hopf algebra with the above definitions (13 and 14).

Example 0.14 (Quantum Plane) *Choose q to be an invertible element in $\mathbb{C}$ and define*

$$\mathbb{C}_q[x,y] = \mathbb{C}\{x,y|xy = qyx\} \tag{15}$$

With comultiplication and counits defined as

$$\Delta(x) = x \otimes x, \;\; \Delta(y) = y \otimes 1 + 1 \otimes y \tag{16}$$

and

$$\varepsilon(x) = 1, \;\; \varepsilon(y) = 0 \tag{17}$$

$\mathbb{C}_q[x,y]$ *is a bialgebra.*

Example 0.15 *A four dimensional noncommutative, noncocomutative Hopf algebra [Mon91].*

$$H_4 = \left\{1, g, x, gx | g^2 = 1, x^2 = 0, xg = -gx\right\} \qquad (18)$$

with structure maps

$$\begin{aligned} \Delta g &= g \otimes g \\ \Delta x &= 1 \otimes x + x \otimes 1 \\ \varepsilon(g) &= 1 \\ \varepsilon(x) &= 0 \\ S(g) &= g = g^{-1} \\ S(x) &= -gx \end{aligned}$$

References

[Kas95] Christian Kassel. *Quantum Groups*. Springer-Verlag New York, Inc., 1995.

[Mon91] Susan Montgomery. Hopf algebras and their actions on rings. In *CBMS Regional Conference Series in Mathematics*, volume 82. Conference Board of the Mathematical Sciences, American Mathematical Society, 1991.