Abstract

RAY, ROBERT. Quantum Symmetric Spaces and Quantum Symplectic Invariants. (Under the direction of Dr. Naihuan Jing).

Although subgroups of the general linear group, $GL(n, \mathbb{C})$, are well understood, properties of quantum analogs of the subgroups of $GL_q(n, \mathbb{C})$ have been a little more elusive. This is due in part to the fact that these sets are not groups and there does not appear to be a natural way to embed them in $GL_q(n, \mathbb{C})$. With respect to the matrix multiplication, these sets generally fail to be closed, and the elements do not all have inverses. However, the associated quantized regular functions and quantized universal enveloping Lie algebras still retain Hopf algebra structures. It is this structure that was used by Jing and Yamada [7] in 1994 to construct q-analogs of the orthogonal group and the associated q-orthogonal invariants (quantum symmetric algebra).

The first two chapters of this text provide the basic background information for the main thesis topic. In chapter 1 , we review some basic definitions and properties of Hopf algebras, first discussing algebras and coalgebras (drawing mostly from material

by Montgomery [6] and Kassel [3]). Here the definitions of the regular functions of $GL(n, \mathbb{C})$ are recalled and other examples of Hopf algebras are given to illustrate some of the properties. In chapter 2, quantum versions of these Hopf algebras are presented, thus defining the quantum version of $GL_q(n, \mathbb{C})$.

In their paper, Jing and Yamada [7] use a differential method of defining q-orthogonal invariants of the action of $O_q(n, \mathbb{C})$ on $A_q(X)$. In other words, the q-orthogonal invariant subspace is defined as the subspace of $A_q(X)$ that is annihilatad by a q-analog of $U(\mathfrak{so}(n, \mathbb{C}))$. In the third chapter, a q-analog of $U(\mathfrak{sp}(n, \mathbb{C}))$ is constructed and the q-symplectic invariants in $A_q(X)$ are defined relative to the left and right action of $U_q(\mathfrak{sp}(n, \mathbb{C}))$, in a differential fashion similar to [7], where we require n be even. The space of these q-symplectic invariants is then decomposed into right and left irreducible modules and several properties are discussed and we show how these q-symplectic invariants define quantum antisymmetric matrices.

QUANTUM SYMMETRIC SPACES AND QUANTUM SYMPLECTIC INVARIANTS

by

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Biography

Rob Ray was born in Caldwell, Idaho on November 20, 1963 and spent most of his elementary school and high school years in Yakima, Washington. Rob graduated from Albertson College of Idaho in 1986 with a Bachelor of Science in mathmatics/computer science with a minor in physics. He graduated from Western Washington University in 1992 with a Master of Science in mathematics. While attending Western Washington University, he and Robin McDonald were married and in 1997 their first child, Maggie was born. Rob and his family moved to North Carolina in 1998 where Rob began his Doctorate degree at North Carolina State University. Rob and Robin's son Thomas was born in 2001.

Rob finished his Doctor of Philosophy in mathematics in 2005. He and his family have moved to Spokane, Washington, where he will pursue a career in education.

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And for their patience, I thank the members of my committee: Dr. Ernest Stitzinger, Dr. Kailash Misra, and Dr. Tom Lada. As educators and mathematicians, they have been quintessential role models.

I would like to express my eternal gratitude to my wife, Robin. Without her inspiration, I would not have returned to school, and without her support and encouragement I could not have stayed in school. Her strength became my strength.

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Chapter 1

Hopf Algebras

In this text, a quantum version of the classical matrix group, $GL(n, \mathbb{C})$ and some of its quantum "subgroups" are discussed. The quantum versions of these groups are defined in terms of their regular functions and these definitions rely on the Hopf algebra structure of these functions. Therefore, before discussing these quantum groups, some background in Hopf algebras will be presented. The examples used in this section will generally be associated with the classical groups and the quantum versions of these will be discussed in the following chapter. Much of this information can be found in Kassel [3] and Montgomery [6].

1.0.1 Algebras

First recall the definition of an algebra over a field $\mathbb{F}$.

Definition 1.1. *[6] An* **algebra** *over a field $\mathbb{F}$ is a vector space, V, together with two linear maps, a multiplication $\mu : V \otimes V \to V$, and a unit map $\eta : \mathbb{F} \to V$ such that the following diagrams (Figure 1.1 and Figure 1.2) commute:*

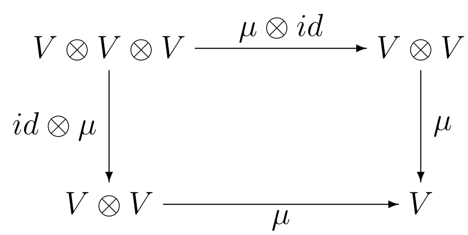

Figure 1.1: Algebra associativity

and

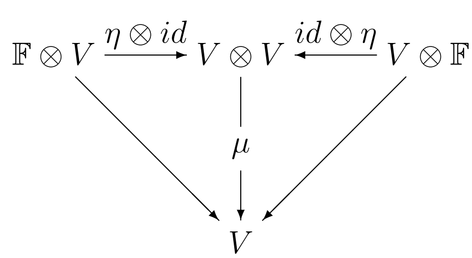

Figure 1.2: Unit map

where the lower left and right maps are simply scalar multiplication. This algebra is sometimes denoted, (V, μ, η).

Example 1.1. *The $n \times n$ matrices with entries in a field $\mathbb{F}$, denoted $Mat(n, \mathbb{F})$ is an algebra with μ defined as the standard matrix multiplication and unit map,*

$\eta : \alpha \to \alpha I_n$ *for* $\alpha \in \mathbb{F}$.

Example 1.2. *For any group* G *with identity* e *we may construct its* **group algebra**, *denoted* $\mathbb{C}[G]$. *We consider the elements of the group as linearly independent vectors and* $\mathbb{C}[G]$ *to be their span. The group multiplication induces a multiplication on* $\mathbb{C}[G]$ *and, together with the unit map* $c \to ce$, *defines an algebra. [6]*

Example 1.3. *Another example is* $A(X)$, *the algebra of polynomials of* n^2 *variables over the field* $\mathbb{C}$, *denoted* $\mathbb{C}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ *with* μ *being the standard polynomial multiplication and* η *being the injection map, taking the field* $\mathbb{C}$ *to the constant polynomials. [6]*

We also refer to $A(X)$ as the **regular functions** of $Mat(n, \mathbb{C})$. In other words, for a polynomial $f(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \in A(X)$, f can be viewed as a function $f : Mat(n, \mathbb{C}) \to \mathbb{C}$. We evaluate f at a matrix, $(a_{i,j})$ as $f(a_{1,1}, a_{1,2}, \ldots, a_{n,n})$.

1.0.2 Coalgebras

Definition 1.2. *[6] A* **coalgebra** *is a vector space* V *together with two linear maps, comultiplication* $\Delta : V \to V \otimes V$ *and counit* $\varepsilon : V \to \mathbb{F}$, *such that the following diagrams (Figure 1.3 and Figure 1.4) commute.*

This coalgebra is sometimes denoted, (V, Δ, ε).

Example 1.4. *The group algebra,* $\mathbb{C}[G]$, *becomes a coalgebra with the maps:*

$$\Delta(g) = g \otimes g \tag{1.1}$$

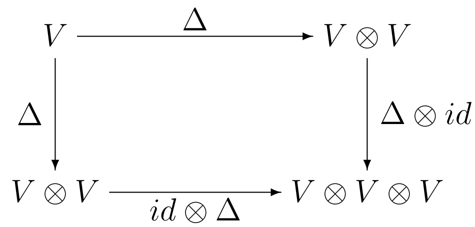

Figure 1.3: Coalgebra coassociativity

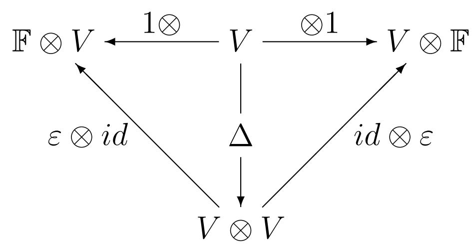

Figure 1.4: Counit map

$$\varepsilon(g) = 1 \tag{1.2}$$

for all $g \in G$.

Example 1.5. *Again, considering $A(X)$ (of Example 1.3). If we define Δ by*

$$\Delta(x_{i,j}) = \sum_{k=1}^{n} x_{i,k} \otimes x_{k,j} \tag{1.3}$$

and ε by

$$\varepsilon(x_{ij}) = \delta_{ij} \tag{1.4}$$

then $A(X)$ becomes a coalgebra.

1.0.3 Bialgebras

Definition 1.3. *[6] A vector space V over a field $\mathbb{F}$, is a* **bialgebra** *if (V, μ, η) is an algebra, (V, Δ, ε) is a coalgebra, and either of the following (equivalent) conditions holds:*

1. *Δ and ε are algebra morphisms*

2. *μ and η are coalgebra morphisms*

This bialgebra structure is often denoted by $(V, \mu, \eta, \Delta, \varepsilon)$.

Example 1.6. *The previous example of a coalgebra (Example 1.3), $\mathbb{C}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$, by its definition is a bialgebra. Recall that the coalgebra structure maps were defined on the generators and then extended to the rest of the space by defining them to be algebra morphisms.*

Example 1.7. *Additionally, any group algebra with the above defined coalgebra maps (Example 1.4), is also a bialgebra.*

1.0.4 Hopf Algebras

Before stating the definition of a Hopf algebra there are a couple preliminary definitions that need to be covered.

Definition 1.4. *Given an algebra (V, μ, η), a coalgebra (W, Δ, ε) and two linear maps $f, g : W \to V$ then the* **convolution** *of f and g is the linear map $f \star g : W \to V$*

defined by

$$(f \star g)(c) = (\mu \circ (f \otimes g) \circ \Delta)(c), \qquad c \in W \tag{1.5}$$

It is now convenient to mention another relationship between $Mat(n, \mathbb{C})$ and $A(X)$. Since any algebra morphism of $A(X)$ is determined on its generators, an algebra morphism from $A(X)$ to $\mathbb{C}$ can be identified with an n^2-tuple of elements from $\mathbb{C}$. Thus we have a natural bijection between $Hom_{Alg}(A(X), \mathbb{C})$ and $Mat(n, \mathbb{C})$. And, if we let $f, g \in Hom_{Alg}(A(X), \mathbb{C})$ then convolution, $f \star g$, corresponds to the familiar matrix multiplication and as algebras,

$$Mat(n, \mathbb{C}) \simeq Hom_{Alg}(A(X), \mathbb{C}) \tag{1.6}$$

Definition 1.5. *Let $(V, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. An endomorphism S of V is called an* **antipode** *for the bialgebra V if*

$$id_V \star S = S \star id_V = \eta \circ \varepsilon \tag{1.7}$$

Definition 1.6. *A* **Hopf algebra** *is a bialgebra with an antipode.*

Example 1.8. *In the case of the group algebra $\mathbb{C}[G]$, the antipode is given by*

$$S(g) = g^{-1} \quad \forall g \in G \tag{1.8}$$

and $\mathbb{C}[G]$ becomes a Hopf algebra.

The bialgebra $A(X)$ from example 1.3, is not a Hopf algebra, however, a Hopf algebra can be constructed from $A(X)$.

Definition 1.7. *[7] Let I and J be two subsets of $\{1, 2, \ldots, n\}$ such that $\#I = \#J = r$. Now arrange the elements of I and J so that they are in increasing order, $I = \{i_1 < i_2 < \ldots < i_r\}$ and $J = \{j_1 < j_2 < \ldots < j_r\}$, then we define the* **r-minor determinant**, *ζ^I_J by*

$$\zeta^I_J = \sum_{\sigma \in \mathcal{S}_r} sgn(\sigma) x_{i_1, j_{\sigma(1)}} \ldots x_{i_r, j_{\sigma(r)}} \in A(X) \tag{1.9}$$

Where $\mathcal{S}_r$ is the symmetric group of permutations of r objects.

The n-minor determinant is simply denoted det. By adjoining the element det^{-1} to $A(X)$ we construct

$$A(GL(n, \mathbb{C})) = \mathbb{C}\left[x_{1,1}, x_{1,2}, \ldots x_{n,n}, det^{-1}\right] = A(X)\left[det^{-1}\right] \tag{1.10}$$

also referred to as $A(G)$ or as the regular functions of $GL(n, \mathbb{C})$. We note that the algebra morphisms from $A(G)$ to $\mathbb{C}$ is the group $GL(n, \mathbb{C})$. That is,

$$Hom_{Alg}(A(G), \mathbb{C}) \simeq GL(n, \mathbb{C}) \tag{1.11}$$

We are now able to define the antipode map for $A(G)$ by

$$S(x_{i,j}) = (-1)^{i-j} det^{-1} \zeta_{\hat{i}}^{\hat{j}} \tag{1.12}$$

where $\hat{k} = \{1, \ldots, k-1, k+1, \ldots, n\}$. Thus, $A(G)$ is a Hopf algebra.

1.0.5 Modules and Comodules

Recall the definition of a module over an algebra A.

Definition 1.8. *[6] For an algebra A (over a field $\mathbb{F}$), a (left) A-module is a space M with a linear map $\gamma : A \otimes M \to M$ such that the following diagrams commute:*

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mu \otimes id} & A \otimes M \\ {\scriptstyle id \otimes \gamma}\downarrow & & \downarrow{\scriptstyle \gamma} \\ A \otimes M & \xrightarrow[\gamma]{} & M \end{array} \tag{1.13}$$

$$\begin{array}{ccc} \mathbb{F} \otimes M & \xrightarrow{\eta \otimes id} & A \otimes M \\ & {\scriptstyle scalar\ mult.}\searrow & \downarrow{\scriptstyle \gamma} \\ & & M \end{array} \tag{1.14}$$

The category of left A-modules is denoted ${}_A\mathcal{M}$. Right modules are defined similarly.

Definition 1.9. *For a coalgebra C (over a field $\mathbb{F}$), a (right) C-comodule is a space*

M with a linear map $\rho : M \to M \otimes C$ such that the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ \rho \downarrow & & \downarrow id \otimes \Delta \\ M \otimes C & \xrightarrow[\rho \otimes id]{} & M \otimes C \otimes C \end{array} \tag{1.15}$$

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ & \searrow^{\otimes 1} & \downarrow id \otimes \varepsilon \\ & & M \otimes \mathbb{F} \end{array} \tag{1.16}$$

The category of right C-modules is denoted $\mathcal{M}^C$. And the left comodules are defined similarly.

Example 1.9. *If C is a coalgebra and we have a right C-comodule V (resp. a left C-comodule W) with structure map $R_C : V \to V \otimes C$ (resp. $L_C : W \to C \otimes W$). Then V and W have natural left and right C^* module structures defined by the following*

$$a.v = (id \otimes a)R_C(v), \quad a \in C^*, v \in V \tag{1.17}$$

$$w.a = (a \otimes id)L_C(w), \quad a \in C^*, w \in W \tag{1.18}$$

where C^* is the dual space to C.

This is a very convenient way to construct a module in certain cases. For example, given a bialgebra such as $A(G)$, $A(G)$ is itself a $A(G)$-comodule, using the comultiplication as the comodule structure map. This defines $A(G)$ as a $A(G)^*$-module.

Chapter 2

Quantum Groups

2.1 $A_q(X)$, $A_q(G)$), $GL_q(n, \mathbb{C})$ and $U_q(\mathfrak{g})$

Here we introduce quantum analogs of some of the bialgebras and Hopf algebras that were discussed in the previous chapters. These constructions are found in Jing and Yamada [7] and in Noumi, Yamada, and Mimachi [4]. In general, the quantum versions of these algebras can be described as single parameter deformations of the classical algebras. That is, we will describe $A_q(X)$ to be like the classical algebra $A(X)$ except with noncommuting relations imposed upon its generators. These imposed relations involve a fixed $q \in \mathbb{C}$ such that, if $q = 1$ then $A_q(X) = A(X)$. For $q \neq 1$ we require that q not be a root of unity, i.e. $q^m \neq 1$ for all $m \in \mathbb{Z}$

2.1.1 $A_q(X)$, $A_q(G)$ and $GL_q(n,\mathbb{C})$

We first define the algebra of functions $A_q(X)$ on $X = Mat_q(n,\mathbb{C})$ (see [7]) as

$$A_q(X) = \mathbb{C}_q\left[x_{1,1}, x_{1,2}, \ldots, x_{n,n}\right] \tag{2.1}$$

This is the non-commuting $\mathbb{C}$-algebra generated by $x_{1,1}, x_{1,2}, \ldots, x_{n,n}$ and with relations

$$x_{i,k}x_{j,k} = qx_{j,k}x_{i,k} \tag{2.2}$$

$$x_{k,i}x_{k,j} = qx_{k,j}x_{k,i} \tag{2.3}$$

$$x_{i,l}x_{j,k} = x_{j,k}x_{i,l} \tag{2.4}$$

$$x_{i,k}x_{j,l} - x_{j,l}x_{i,k} = \left(q - \frac{1}{q}\right) x_{i,l}x_{j,k} \tag{2.5}$$

where $i < j$ and $k < l$. To help visualize these relations with respect to the matrix X, we construct the following diagram with a “square” of generators [7]:

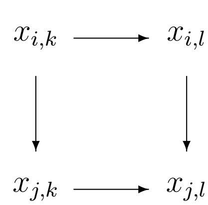

Figure 2.1: $A_q(X)$ Relations

The convention here will be that $x \to y$ implies $xy = qyx$. Additionally, by letting

$A_q(X)$ inherit the coproduct and counit maps as defined on $A(X)$, we see that $A_q(X)$ is also a bialgebra.

Let I and J be two subsets of $\{1, 2, \ldots, n\}$ with $\#I = \#J = r$ with ordered elements, i.e. $i_1 < i_2 < \ldots < i_r \in I$ and $j_1 < j_2 < \ldots j_r \in J$. Then the **quantum r-minor determinants** are defined as

$$\xi_J^I = \xi_{j_1,\ldots,j_r}^{i_1,\ldots,i_r} = \sum_{\sigma \in \mathcal{S}_r} (-q)^{l(\sigma)} x_{i_1 j_{\sigma(1)}} x_{i_2 j_{\sigma(2)}} \ldots x_{i_r j_{\sigma(r)}} \tag{2.6}$$

where $l(\sigma)$ denotes the number of pairs (i, j) with $i < j$ and $\sigma(i) > \sigma(j)$ [7]. There is a unique quantum n-minor determinant, and it is denoted by det_q [7]. To construct $A_q(G)$ we simply adjoin det_q^{-1} to $A_q(X)$

$$A_q(G) = \left[x_{1,1}, x_{1,2}, x_{1,3}, \ldots, x_{n,n}, det_q^{-1}\right] \tag{2.7}$$

Also, we keep all of the relations of $A_q(X)$ and add the following relations [4]

$$x_{ij} \cdot det_q^{-1} = det_q^{-1} \cdot x_{ij} \tag{2.8}$$

$$det_q^{-1} \cdot det_q = det_q \cdot det_q^{-1} = 1 \tag{2.9}$$

With det_q^{-1} the following algebra morphism S may be defined

$$S(x_{i,j}) = (-q)^{i-j} \xi_{\hat{i}}^{\hat{j}} \cdot det_q^{-1} \quad 1 \leq i, j \leq n, \tag{2.10}$$

where $\hat{k} = \{1, \ldots, k-1, k+1, \ldots, n\}$ [7]. It can be verified that S is the antipode for $A_q(G)$. As such, $A_q(G)$ is a Hopf algebra. Finally, $GL_q(n, \mathbb{C})$ is defined as the algebra morphisms from $A_q(G)$ to $\mathbb{C}$, i.e.

$$GL_q(n, \mathbb{C}) \simeq Hom_{Alg}\left(A_q(X), \mathbb{C}\right) \tag{2.11}$$

and is the quantum analog of $GL(n, \mathbb{C})$.

2.1.2 Other Quantum Groups

In addition to the above mentioned quantum groups, we will need to define some additional subgroups of $G = GL_q(n, \mathbb{C})$. These subgroups are discussed by Jing and Yamada [7] and by Noumi, Yamada, and Mimachi [4]. The Borel subgroups B_+ and B_- of G consist of the upper and lower triangular matrices are defined in terms of their associated Hopf algebras

$$A(B_+) = \mathbb{C}\left[x_{i,j}(i \le j), x_{1,1}, \ldots, x_{n,n}\right] \tag{2.12}$$

$$A(B_-) = \mathbb{C}\left[x_{i,j}(i \ge j), x_{1,1}, \ldots, x_{n,n}\right] \tag{2.13}$$

These algebras have relations induced from $A_q(G)$ and we note that the diagonal elements $x_{1,1}, \ldots, x_{n,n}$ commute with each other. [4]

The diagonal subgroup H_n of $GL_q(n,\mathbb{C})$ is defined by its regular functions

$$A(H_n) = \mathbb{C}\left[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}\right] \tag{2.14}$$

2.1.3 $U_q(\mathfrak{g})$

Next we will introduce the quantum universal enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ in the same manner as [4] and [7]. Let L_n be the free $\mathbb{Z}$-module of rank n with the canonical basis $\{\epsilon_1, \ldots, \epsilon_n\}$, i.e. $L_n = \bigoplus_{k=1}^{n} \mathbb{Z}\epsilon_k$, endowed with the symmetric bilinear form $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. We will define $\alpha_k = \epsilon_k - \epsilon_{k+1}$. Additionally, we will identify a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n$ with $\lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n \in L_n$. We will refer to such an element of L_n as a dominant integral weight. The fundamental weights are defined by $\Lambda_k = \epsilon_1 + \cdots + \epsilon_k$ (see [7]). Now we define $U_q(\mathfrak{g})$ as the $\mathbb{C}$-algebra with generators e_k, f_k $(1 \le k < n)$ and q^λ $\left(\lambda \in \frac{1}{2}L_n\right)$ with the following relations [4]:

$$q^0 = 1, \quad q^\lambda q^\mu = q^{\lambda+\mu}, \quad \left(\lambda, \mu \in \frac{1}{2}L_n\right), \tag{2.15}$$

$$q^\lambda e_k q^{-\lambda} = q^{\langle \lambda, \alpha_k \rangle} e_k \quad \left(\lambda \in \frac{1}{2}L_n, 1 \le k < n\right), \tag{2.16}$$

$$q^\lambda f_k q^{-\lambda} = q^{-\langle \lambda, \alpha_k \rangle} f_k \quad \left(\lambda \in \frac{1}{2}L_n, 1 \le k < n\right), \tag{2.17}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}} \quad (1 \le i, j < n), \tag{2.18}$$

$$e_i^2 e_j - \left(q + q^{-1}\right) e_i e_j e_i + e_j e_i^2 = 0 \quad (|i-j| = 1), \tag{2.19}$$

$$f_i^2 f_j - \left(q + q^{-1}\right) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i-j| = 1), \tag{2.20}$$

$$e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i \quad (|i-j| > 1). \tag{2.21}$$

If we define a coproduct, Δ_U, and a counit, ε_U, on the generators in the following manner

$$\Delta_U(q^\lambda) = q^\lambda \otimes q^\lambda, \quad \varepsilon(q^\lambda) = 1, \tag{2.22}$$

$$\Delta_U(e_k) = e_k \otimes q^{-\alpha_k/2} + q^{\alpha_k/2} \otimes e_k, \quad \varepsilon(e_k) = 0, \tag{2.23}$$

$$\Delta_U(e_k) = f_k \otimes q^{-\alpha_k/2} + q^{\alpha_k/2} \otimes f_k, \quad \varepsilon(f_k) = 0, \tag{2.24}$$

we see that $U_q(\mathfrak{g})$ is a bialgebra. Additionally, with the antipode S_U defined by

$$S_U(q^\lambda) = q^{-\lambda} \tag{2.25}$$

$$S_U(e_k) = -q^{-1} e_k \tag{2.26}$$

$$S_U(f_k) = -q f_k \tag{2.27}$$

it becomes a Hopf algebra.

2.1.4 $A_q(G), U_q(\mathfrak{g})$ Duality

As stated in [4], there exists a unique pairing of Hopf algebras $U_q(\mathfrak{g})$ and $A_q(G)$

$$(\quad,\quad) : U_q(\mathfrak{g}) \times A_q(G) \to \mathbb{C} \tag{2.28}$$

satisfying the following relations:

$$\big(q^{\lambda}, x_{i,j}\big) = \delta_{i,j} q^{\langle \lambda, \varepsilon_i \rangle}, \quad \lambda \in \frac{1}{2} L_n, \quad 1 \leq i, j \leq n \tag{2.29}$$

$$(e_k, x_{i,j}) = \delta_{i,k}\delta_{j,k+1}, \quad 1 \leq i, j \leq n \tag{2.30}$$

$$(f_k, x_{i,j}) = \delta_{i,k+1}\delta_{j,k}, \quad 1 \leq i, j \leq n \tag{2.31}$$

$$\big(q^{\lambda}, det_q^m\big) = q^{m\langle \lambda, \varepsilon_1, \ldots, \varepsilon_n \rangle} \quad m \in \mathbb{Z} \tag{2.32}$$

$$\big(e_k, det_q^m\big) = \big(f_k, det_q^m\big) = 0 \quad m \in \mathbb{Z} \tag{2.33}$$

For our purpose, we will switch to a functional notation used in [7]. In other words,

$$q^{\lambda}(x_{i,j}) = \delta_{i,j} q^{\langle \lambda, \varepsilon_i \rangle}, \quad \lambda \in \frac{1}{2} L_n, \quad 1 \leq i, j \leq n \tag{2.34}$$

$$e_k(x_{i,j}) = \delta_{i,k}\delta_{j,k+1}, \quad 1 \leq i, j \leq n \tag{2.35}$$

$$f_k(x_{i,j}) = \delta_{i,k+1}\delta_{j,k}, \quad 1 \leq i, j \leq n \tag{2.36}$$

$$q^{\lambda}(det_q^m) = q^{m\langle \lambda, \varepsilon_1, \ldots, \varepsilon_n \rangle} \quad m \in \mathbb{Z} \tag{2.37}$$

$$e_k(det_q^m) = f_k(det_q^m) = 0 \quad m \in \mathbb{Z} \tag{2.38}$$

and then extend these to the rest of the algebras by

$$a(\varphi\psi) = \Delta_U(a)(\varphi \otimes \psi) \tag{2.39}$$

$$a(1) = \varepsilon_U(a) \tag{2.40}$$

$$(ab)(\varphi) = (a \otimes b)\Delta(\varphi) \tag{2.41}$$

$$1(\varphi) = \varepsilon(\varphi) \tag{2.42}$$

$$a, b \in U_q(\mathfrak{g}), \quad \varphi, \psi \in A_q(G)$$

Additionally, we have

$$S_U(a).\psi = a.S(\psi) \quad a \in U_q(\mathfrak{g}), \psi \in A_q(G) \tag{2.43}$$

These relations realize a duality between the two Hopf algebras and allows us to regard the elements of $U_q(\mathfrak{g})$ as linear functionals on $A_q(G)$ (see [4]). As discussed previously, this duality allows any right $A_q(G)$-comodule V (resp. left $A_q(G)$-comodule W) with structure map $R_G : V \to V \otimes A_q(G)$ (resp. $L_G : W \to A_q(G) \otimes W$) to become a left (resp. right) $U_q(\mathfrak{g})$-module with the following defined action

$$a.v = (id \otimes a)R_G(v), \quad a \in U_q(\mathfrak{g}), v \in V \tag{2.44}$$

$$w.a = (a \otimes id)L_G(v), \quad a \in U_q(\mathfrak{g}), w \in W \tag{2.45}$$

More specifically, we already know $A_q(X)$ is a completely reducible two-sided $A_q(G)$-comodule using the comultiplication, Δ, as the comodule structure map. As such, it becomes a completely reducible left and right $U_q(\mathfrak{g})$-module [7, 4]. We can describe the left module action of the generators of $U_q(\mathfrak{g})$ on the generators of $A_q(X)$ by

$$q^\lambda.x_{i,j} = x_{i,j}q^{\langle\lambda,\varepsilon_j\rangle} \tag{2.46}$$

$$e_k.x_{i,j} = x_{i,j-1}\delta_{j,k+1} \tag{2.47}$$

$$f_k.x_{i,j} = x_{i,j+1}\delta_{j,k} \tag{2.48}$$

and the right module action as

$$x_{i,j}.q^\lambda = x_{i,j}q^{\langle\lambda,\varepsilon_i\rangle} \tag{2.49}$$

$$x_{i,j}.e_k = x_{i+1,j}\delta_{k,i} \tag{2.50}$$

$$x_{i,j}.f_k = x_{i-1,j}\delta_{k+1,i} \tag{2.51}$$

Throughout the rest of this text, this will be the action of $U_q(\mathfrak{g})$ on $A_q(X)$.

2.1.5 Relative Invariants

In [7, 4], the following subsets of $A_q(X)$ are defined. For an element $\lambda = \sum_{k=1}^{n} \lambda_k \epsilon_k \in L_n$, let $z^\lambda = \prod_{k=1}^{n} z_{kk}^{\lambda_k} \in A(B_\pm)$ and $t^\lambda = \prod_{k=1}^{n} t_k^{\lambda_k} \in A(H)$. We define the spaces of relative invariants with respect to the subgroups $B_\pm$ by

$$A(G/B_+; z^\lambda) = \left\{\varphi \in A_q(G); (id \otimes \pi_{B_+})\Delta(\varphi) = \varphi \otimes z^\lambda\right\} \tag{2.52}$$

$$A(B_-\backslash G; z^\lambda) = \left\{\varphi \in A_q(G); (\pi_{B_-} \otimes id)\Delta(\varphi) = z^\lambda \otimes \varphi\right\} \tag{2.53}$$

where the restrictions maps $\pi_\pm : A_q(G) \to A_q(B_\pm)$ are defined by $\pi_{B_+}(x_{i,j}) = z_{i,j}$ $(1 \le i \le j \le n)$, $\pi_{B_+}(x_{i,j}) = 0$ $(i > j)$, and $\pi_{B_-}(x_{i,j}) = z_{i,j}$ $(1 \le j \le i \le n)$, $\pi_{B_-}(x_{i,j}) = 0$ $(i < j)$.

$A(G/B_+; z^\lambda)$ (resp. $A(B_-\backslash G; z^\lambda)$) is a left (resp. right) $A_q(G)$-subcomodule of $A_q(G)$ with structure mapping Δ. It is proved in [4] that, for a dominant integral weight $\lambda \in P_n$, the space $A(G/B_+; z^\lambda)$ (resp. $A(B_-\backslash G; z^\lambda)$) gives a realization of the irreducible left (resp. right) $A_q(G)$-subcomodule $V_q^L(\lambda)$ (resp. $V_q^R(\lambda)$) of $A_q(X)$, with highest weight λ. This fact will be used extensively in many proofs of this text.

Chapter 3

Quantum Symplectic Invariants

In this section we define the space of quantum symplectic invariants in the subalgebra of $A_q(X)$. This is done in a differential manner, similar to Jing and Yamada [7], by defining a subalgebra of $A_q(X)$ that is annihilated by a quantum version of $U(\mathfrak{sp}(n, \mathbb{C}))$. In other words, it will be defined as the subspace of $A_q(X)$ that is annihilated by $U_q(\mathfrak{sp}(n, \mathbb{C})) \subset U_q(\mathfrak{g})$ by the module action previously defined. For consistency of notation we require from here on that n be even, so rather than use the notation $\mathfrak{sp}(2n)$ we will use $\mathfrak{sp}(n)$.

3.0.6 $U_q(\mathfrak{sp}(n, \mathbb{C}))$

We will first describe a subalgebra of $U_q(\mathfrak{g})$ that is a quantum deformation of $U_q(\mathfrak{sp}(n, \mathbb{C}))$, ($n$ even). Relative to the standard n dimensional representation of $U_q(\mathfrak{g})$, we identify the generators e_k of $U_q(\mathfrak{g})$ with $E_{k,k+1}$ and f_k with $E_{k+1,k}$. The diagonal generators

$q^{c\epsilon_i}$ are represented by

$$E_{1,1} + E_{2,2} + \cdots + q^c E_{i,i} + \cdots + E_{n,n} \quad c \in \frac{1}{2}\mathbb{Z} \tag{3.1}$$

For example

$$q^{\frac{\alpha_2}{2}} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & q^{1/2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & q^{-1/2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{3.2}$$

By associating the generators of $U_q(\mathfrak{g})$ with elements of the form $E_{i,j}$, subalgebras of $U_q(\mathfrak{g})$ may be defined in terms of these $E_{i,j}$. More specifically, the following "off diagonal" elements $E_{i,j}$, where $|i - j| > 1$, can be inductively, generated by the elements of the form $E_{i,i\pm1}$ by

$$E_{i,j} = E_{i,k}E_{k,j} - E_{k,j}E_{i,k} \tag{3.3}$$

$$E_{j,i} = E_{j,k}E_{k,i} - E_{k,i}E_{j,k} \tag{3.4}$$

where $i < k < j$ and $E_{i,j}$ and $E_{j,i}$ are independent of our choice of k [7]. As an example,

$$\begin{aligned} E_{1,4} = & E_{1,2}E_{2,3}E_{3,4} - E_{1,2}E_{3,4}E_{2,3} - E_{2,3}E_{3,4}E_{1,2} + E_{3,4}E_{2,3}E_{1,2} \\ = & e_1e_2e_3 - e_1e_3e_2 - e_2e_3e_1 + e_3e_2e_1 \\ = & [e_1, [e_2, e_3]] \end{aligned} \tag{3.5}$$

We define a subalgebra $U_q(\mathfrak{sp}(n, \mathbb{C}))$ of $U_q(\mathfrak{g})$ (n even) as the subalgebra generated by the following elements:

$$sp_e(i,j) = E_{2i-1,2j} + q^{2(i-j)}E_{2j-1,2i} \qquad 1 \le i \ne j \le n/2 \tag{3.6}$$

$$sp_e(i,i) = E_{2i-1,2i} \qquad 1 \le i \le n/2 \tag{3.7}$$

$$sp_f(i,j) = E_{2i,2j-1} + q^{2(i-j)}E_{2j,2i-1} \qquad 1 \le i \ne j \le n/2 \tag{3.8}$$

$$sp_f(i,i) = E_{2i,2i-1} \qquad 1 \le i \le n/2 \tag{3.9}$$

$$sp_h(i,j) = E_{2i-1,2j-1} - q^{2(i-j)}E_{2j,2i} \qquad 1 \le i, j \le n/2 \tag{3.10}$$

with $i, j \le n/2$. Some of the relations of these generators are:

$$sp_e(j,i) = q^{2(i-j)}sp_e(i,j) \quad i < j \tag{3.11}$$

$$sp_f(j,i) = q^{2(i-j)}sp_f(i,j) \quad i < j \tag{3.12}$$

$$sp_h(j,i) = q^{2(i-j)}sp_h(i,j)^T \quad i < j \tag{3.13}$$

$$[sp_h(i,j), sp_h(k,l)] = \delta_{j,k} sp_h(i,l) - \delta_{i,l} sp_h(j,k) \tag{3.14}$$

$$[sp_e(i,j), sp_f(k,l)] = sp_h(i,l) \quad i \leq j = k \leq l \tag{3.15}$$

$$[sp_h(i,j), sp_e(k,l)] = sp_e(i,l) \quad i \leq j = k \leq l \tag{3.16}$$

$$[sp_h(k,l), sp_f(i,j)] = -sp_f(i,l) \quad i \leq j = k \leq l \tag{3.17}$$

Using these relations (primarily Equations 3.15, 3.16 and 3.17) it is easy to show that the elements of the form

$$sp_e(j,j),\ sp_f(j,j), \quad \text{where } 1 \leq j \leq n/2 \tag{3.18}$$

$$sp_e(i,i+1),\ sp_f(i,i+1), \quad 1 \leq i \leq n/2-1 \tag{3.19}$$

generate $U_q(\mathfrak{sp}(n,\mathbb{C}))$. For example, we see

$$sp_e(1,3) = [sp_h(1,2), sp_e(2,3)] \ \text{(by 3.16)} \tag{3.20}$$

$$= [[sp_e(1,1), sp_f(1,2)], sp_e(2,3)] \ \text{(by 3.15)} \tag{3.21}$$

3.0.7 Spaces of q-symplectic invariants

For a given left (resp. right) $U_q(\mathfrak{g})$-module V (resp. W) we define the q-symplectic invariants by [7]

$$V^K = \{v \in V; sp_e(i,j).v = 0, sp_f(i,j).v = 0 \quad 1 \leq i,j \leq n/2\} \tag{3.22}$$

$$ {}^{K}W = \{w \in W; w.sp_e(i,j) = 0, w.sp_f(i,j) = 0 \quad 1 \le i,j \le n/2\} \tag{3.23} $$

Using the fact that $A_q(X)$ is a two-sided $U_q(\mathfrak{g})$-module (see 2.44, 2.45) we define the left and right quantum symplectic invariants in $A_q(X)$ as

$$ A_q(X)^K = \{\varphi \in A_q(X); sp_e(i,j).\varphi = 0, sp_f(i,j).\varphi = 0 \quad 1 \le i,j \le n/2\} \tag{3.24} $$

$$ {}^{K}A_q(X) = \{\varphi \in A_q(X); \varphi.sp_e(i,j) = 0, \varphi.sp_f(i,j) = 0 \quad 1 \le i,j \le n/2\} \tag{3.25} $$

The spaces $A_q(X)^K$ and ${}^{K}A_q(X)$ are subalgebras of $A_q(X)$. Additionally, we see that $A_q(X)^K$ is a left $A_q(G)$-subcomodule of $A_q(X)$ (similarly ${}^{K}A_q(X)$ is a right $A_q(G)$-subcomodule of $A_q(X)$). Equivalently, $A_q(X)^K$ is a right $U_q(\mathfrak{g})$-submodule of $A_q(X)$ and ${}^{K}A_q(X)$ is a left $U_q(\mathfrak{g})$-submodule of $A_q(X)$.

Definition 3.1. *For n even, the following quadratic elements of $A_q(X)$ may be defined*

$$ z^L_{i,j} = \sum_{k=1}^{n/2} q^{(i+j+1-4k)/2} \left(x_{i,2k-1}x_{j,2k} - q x_{i,2k}x_{j,2k-1}\right) \tag{3.26} $$

$$ = \sum_{k=1}^{n/2} q^{(i+j+1-4k)/2} \xi^{i,j}_{2k-1,2k} \tag{3.27} $$

and

$$ z^R_{i,j} = \sum_{k=1}^{n/2} q^{-(i+j+1-4k)/2} \left(x_{2k-1,i}x_{2k,j} - q x_{2k,i}x_{2k-1,j}\right) \tag{3.28} $$

$$ = \sum_{k=1}^{n/2} q^{-(i+j+1-4k)/2} \xi^{2k-1,2k}_{i,j} \tag{3.29} $$

Example 3.1. *If we let* $n = 6$ *then*

$$\begin{aligned} z^L_{3,4} = &\ q^2 x_{3,1}x_{4,2} - q^3 x_{3,2}x_{4,1} + x_{3,3}x_{4,4} - q x_{3,4}x_{4,3} + \frac{1}{q^2}x_{3,5}x_{4,6} - \frac{1}{q}x_{3,6}x_{4,5} \\ = &\ q^2\xi^{3,4}_{1,2} + \xi^{3,4}_{3,4} + \frac{1}{q^2}\xi^{3,4}_{5,6} \end{aligned} \tag{3.30}$$

Lemma 3.1. $z^R_{i,j} \in {}^K A_q(X)$ *and* $z^L_{i,j} \in A_q(X)^K$

Proof. To show $z^R_{i,j}$ (resp. $z^L_{i,j}$) are annihilated by all $sp_e(k,l)$ and $sp_f(k,l)$, it is sufficient to show they are annihilated by $sp_e(k,k)$, $sp_e(k,k+1)$, $sp_f(k,k)$ and $sp_f(k,k+1)$ (by 3.18). First,

$$\begin{aligned} sp_e(k,k).z^L_{i,j} = &\ e_{2k-1}.z^L_{i,j} \text{ for } k \le n/2 \\ = &\ 0 \text{ (see A.3)} \end{aligned} \tag{3.31}$$

$$sp_e(k,k+1).z^L_{i,j} = 0 \text{ (see A.5)} \tag{3.32}$$

$$\begin{aligned} sp_f(k,k).z^L_{i,j} = &\ f_{2k-1}.z^L_{i,j} \text{ for } k \le n/2 \\ = &\ 0 \text{ (see A.4)} \end{aligned} \tag{3.33}$$

$$sp_f(k,k+1).z^L_{i,j} = 0 \text{ (see A.6)} \tag{3.34}$$

Thus $z^L_{i,j} \in A_q(X)^K$. Similar calculations show $z^R_{i,j} \in {}^K A_q(X)$. □

We denote the subalgebra of $A_q(X)^K$ (resp. ${}^K A_q(X)$) by $A^L_q(\mathcal{A})$ (resp. $A^R_q(\mathcal{A})$) generated by $z^L_{i,j}$ (resp. $z^R_{i,j}$). $A^L_q(\mathcal{A})$ is a left $A_q(G)$-subcomodule of $A_q(X)^K$ and $A^R_q(\mathcal{A})$ is a right $A_q(G)$-subcomodule of $A_q(X)^K$.

Theorem 3.1. *The algebras $A^L_q(\mathcal{A})$ and $A^R_q(\mathcal{A})$ are isomorphic to the algebra $A_q(\mathcal{A})$ generated by $z_{i,j}$ $(1 \leq i, 1 \leq j)$ with the following relations:*

$$z_{i,j} = -\frac{1}{q} z_{j,i} \quad i < j \tag{3.35}$$

$$z_{i,l} z_{j,k} = z_{j,k} z_{i,l} \quad i < j < k < l \tag{3.36}$$

$$z_{i,j} z_{i,k} = q z_{i,k} z_{i,j} \quad j < k \tag{3.37}$$

$$z_{i,j} z_{j,k} = q z_{j,k} z_{i,j} \quad i < j < k \tag{3.38}$$

$$z_{i,k} z_{j,k} = q z_{j,k} z_{i,k} \quad i < j \tag{3.39}$$

$$z_{i,k} z_{j,l} - z_{j,l} z_{i,k} = \left(q - \frac{1}{q}\right) z_{i,l} z_{j,k} \tag{3.40}$$

$$z_{i,j} z_{k,l} - z_{k,l} z_{i,j} = \left(q - \frac{1}{q}\right) z_{i,k} z_{j,l} - q \left(q - \frac{1}{q}\right) z_{i,l} z_{j,k} \tag{3.41}$$

Using 3.40 we may rewrite 3.41 as

$$z_{i,j} z_{k,l} - z_{k,l} z_{i,j} = q z_{j,l} z_{i,k} - \frac{1}{q} z_{i,k} z_{j,l} \tag{3.42}$$

Proofs of these relations are in the appendix. □

The definitions of these generators also imply

$$z_{i,i} = 0 \tag{3.43}$$

3.0.8 Relation Diagrams of $A_q(\mathcal{A})$

In the same way that we provide a visual perspective for the generators of $A_q(X)$ (see Figure 2.1.1), we may also provide pictures of the relations for $A_q(\mathcal{A})$. In fact, the "square" inherits similar relations of $A_q(X)$, except, due to the antisymmetric nature of the generators, there are some significant differences in the relations when the "square" crosses the diagonal of $Z = (z_{i,j})$. These modified relations can be seen in the following diagrams.

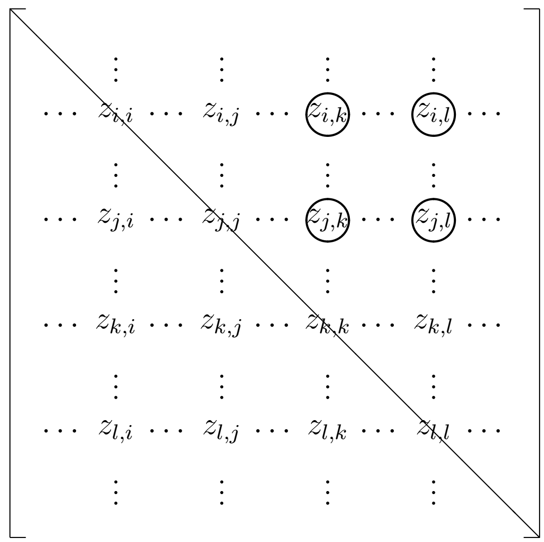

Figure 3.1: “Square” Above Diagonal

Here, the square does not lie across the diagonal. We see the original relations of the generators for $A_q(X)$ apply here. (see 3.36, 3.37, 3.39, and 3.40)

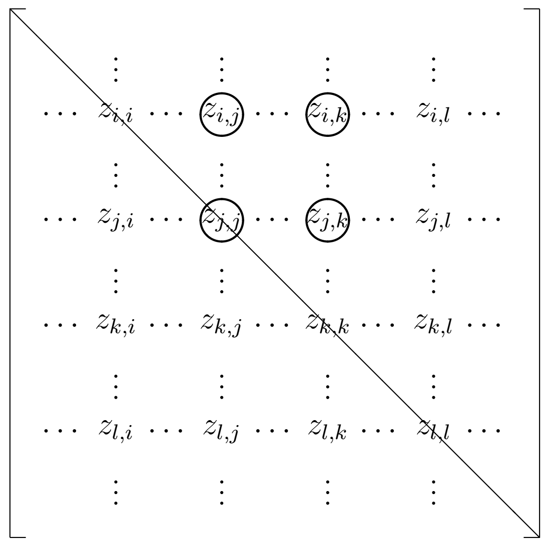

Figure 3.2: Corner of “Square” on Diagonal

In this diagram the square has the original relations of $A_q(X)$ (see 3.36, 3.37 and 3.39) however, we note that $z_{j,j} = 0$ and instead of using 3.40 to relate $z_{i,j}$ and $z_{j,k}$, we use 3.38.

$$z_{i,j}z_{j,k} = qz_{j,k}z_{i,j} \tag{3.44}$$

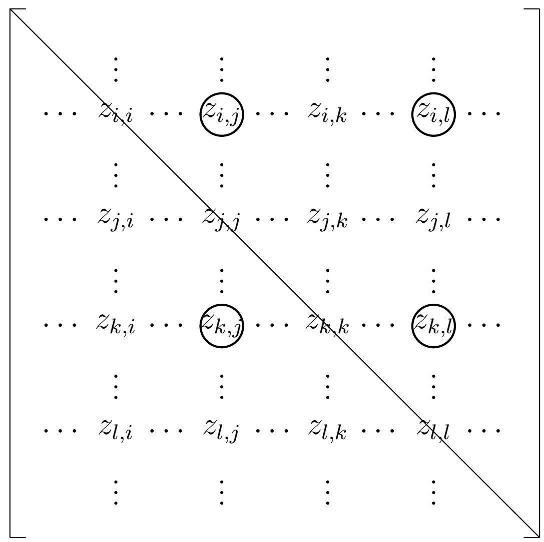

Figure 3.3: Corner of "Square" Across Diagonal

In this diagram, one of the corners of the square lies across the diagonal. These generators have some of the same relations (3.37 and 3.39). Specifically, two generators in the same row or in the same column still have the same relations.

$$\begin{aligned} z_{i,l}z_{k,j} &= -qz_{i,l}z_{j,k} \\ &= -qz_{j,k}z_{i,l} \\ &= z_{k,j}z_{i,l} \end{aligned} \tag{3.45}$$

Instead of using 3.40 to relate $z_{i,j}$ and $z_{k,l}$, we use 3.42.

$$z_{i,j}z_{k,l} - z_{k,l}z_{i,j} = qz_{j,l}z_{i,k} - \frac{1}{q}z_{i,k}z_{j,l} \tag{3.46}$$

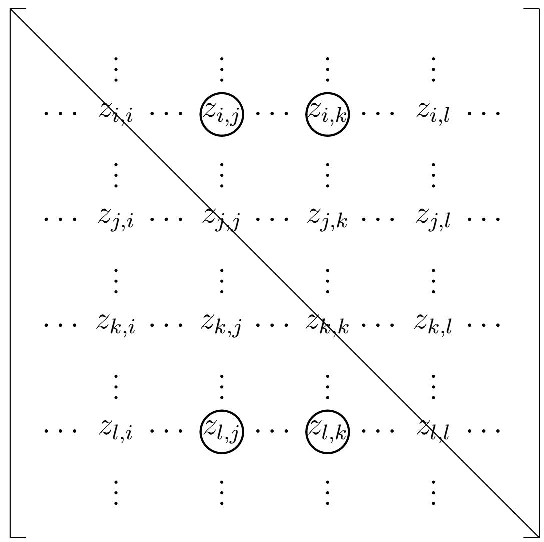

Figure 3.4: “Square” Straddles Diagonal

In this diagram the square has the relations 3.37 and 3.39 to relate generators in the same row or same column. However, to relate the opposing corners of the square we use relation 3.35 to get

$$\begin{aligned} z_{i,k}z_{l,j} - z_{l,j}z_{i,k} &= -q(z_{i,k}z_{j,l} - z_{j,l}z_{i,k}) \\ &= -q\left(q - \frac{1}{q}\right) z_{i,l}z_{j,k} \\ &= \left(1 - q^2\right) z_{i,l}z_{j,k} \end{aligned} \tag{3.47}$$

and relation 3.42 to get

$$\begin{aligned} z_{i,j}z_{l,k} - z_{l,k}z_{i,j} &= -q(z_{i,j}z_{k,l} - z_{k,l}z_{i,j}) \\ &= -q\left(qz_{j,l}z_{i,k} - \frac{1}{q}z_{i,k}z_{j,l}\right) \text{ by } 3.42 \end{aligned}$$

$$= qz_{l,j}z_{i,k} - \frac{1}{q}z_{i,k}z_{l,j} \tag{3.48}$$

3.0.9 Quantum Antisymmetric Matrices

If we denote by $\mathcal{A}$, the vector space of $n \times n$ antisymmetric matrices with basis

$$B_{\mathcal{A}} = \{E_{i,j} - E_{j,i} \,|1 < i < j \leq n\} \tag{3.49}$$

then $dim(\mathcal{A}) = n(n-1)/2$. We observe that $Hom_{Alg}(A_q(\mathcal{A}), \mathbb{C})$ is the set of $n \times n$ matrices with restrictions imposed by the relations 3.35 and 3.43. If we denote $Hom_{Alg}(A_q(\mathcal{A}), \mathbb{C})$ by $\mathcal{A}_q$, and treat it as a vector space (in other words we are ignoring multiplication) we see its basis is

$$B_{\mathcal{A}_q} = \{E_{i,j} - qE_{j,i} \,|1 < i < j \leq n\} \tag{3.50}$$

where $dim(\mathcal{A}_q) = n(n-1)/2$ and we have $\mathcal{A}_q \simeq \mathcal{A}$ as vector spaces. We may think of $\mathcal{A}_q$ as the quantum analog of the antisymmetric matrices.

3.0.10 Quantum Pfaffian

In the classical case, if $A = (a_{i,j}) \in Mat(n, \mathbb{C})$ is an antisymmetric matrix, it can be written as

$$A = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,n} \\ -a_{1,2} & 0 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1,n} & -a_{2,n} & \cdots & 0 \end{bmatrix} \tag{3.51}$$

Also, we know there exists $f \in \mathbb{Z}[x_{i,j}]$ such that $f^2(A) = det(A)$, (see Jacobson [2] and Godsil [1]). This polynomial is called the **Pfaffian**, denoted Pf, and we write

$$Pf^2(A) = det(A) \tag{3.52}$$

Moreover, if $B = (b_{i,j}) \in Mat(n, \mathbb{C})$ and we define A by

$$a_{i,j} = det \begin{bmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{bmatrix} + det \begin{bmatrix} b_{i,3} & b_{i,4} \\ b_{j,3} & b_{j,4} \end{bmatrix} + \cdots + det \begin{bmatrix} b_{i,n-1} & b_{i,n} \\ b_{j,n-1} & b_{j,n} \end{bmatrix} \tag{3.53}$$

then A is antisymmetric and we have $Pf(A) = det(B)$, see Jacobson [2].

To construct an explicit formula for Pf we define an index set Π, which consists of all ordered 2-partitions of n. In other words,

$$\Pi = \left\{(i_1, j_1)(i_2, j_2)\ldots(i_{n/2}, j_{n/2}) \ ; \ i_k < j_k \text{ and } i_k < i_{k+1}\right\} \tag{3.54}$$

For example, if $n = 4$ we have

$$\Pi = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \tag{3.55}$$

Notice, we can associate the elements of Π with the subset of the symmetric group

$\mathcal{S}_n$ in the following manner

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \\ i_1 & j_1 & i_2 & j_2 & \cdots & j_{n/2} \end{bmatrix} \in \mathcal{S}_n \tag{3.56}$$

for $\pi = \{(i_1, j_1)(i_2, j_2) \ldots (i_{n/2}, j_{n/2})\}$. This allows us to define $sgn(\pi)$ as $sgn(\sigma)$ and $l(\pi)$ as $l(\sigma)$. Now, if $A = (a_{i,j})$ is an antisymmetric matrix, and we let $a_\pi = a_{i_1,j_1} a_{i_2,j_2} \ldots a_{i_{n/2},j_{n/2}}$, we can write

$$Pf(A) = \sum_{\pi \in \Pi} sgn(\pi) a_\pi \quad \text{(see [5])} \tag{3.57}$$

As an example, when $n = 4$

$$Pf(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3} \tag{3.58}$$

Before we construct a quantum analog of the Pfaffian, we note that the quantum antisymmetric generators $z^L_{i,j}$ (resp. $z^R_{i,j}$), defined by Equation 3.26 (resp. 3.28), are in fact quantum analogs of Equation 3.53. Additionally, we have already noted that $Z = (z^L_{i,j})$ is a quantum antisymmetric matrix with the relation $z^L_{i,j} = -\frac{1}{q} z^L_{j,i}$ for $i < j$. We now use the same index set Π, to define the **quantum Pfaffian** as

$$Pf_q(Z) = \sum_{\pi \in \Pi} (-q)^{l(\pi)} z^L_\pi \tag{3.59}$$

using the convention that $z^L_\sigma = z^L_{i_1,j_1} z^L_{i_2,j_2} \cdots z^L_{i_{n/2},j_{n/2}}$.

Example 3.2. *If* $n = 4$

$$Pf_q(Z) = z^L_{1,2}z^L_{3,4} - qz^L_{1,3}z^L_{2,4} + q^2 z^L_{1,4}z^L_{2,3} \tag{3.60}$$

Example 3.3. *As a larger example, when* $n = 6$

$$\begin{aligned} Pf_q(Z) = {} & z^L_{1,2}z^L_{3,4}z^L_{5,6} - qz^L_{1,2}z^L_{3,5}z^L_{4,6} + q^2 z^L_{1,2}z^L_{3,6}z^L_{4,5} \\ & - qz^L_{1,3}z^L_{2,4}z^L_{5,6} + q^2 z^L_{1,3}z^L_{2,5}z^L_{4,6} - q^3 z^L_{1,3}z^L_{2,6}z^L_{4,5} \\ & + q^2 z^L_{1,4}z^L_{2,3}z^L_{5,6} - q^3 z^L_{1,4}z^L_{2,5}z^L_{3,6} + q^4 z^L_{1,4}z^L_{2,6}z^L_{3,5} \\ & - q^3 z^L_{1,5}z^L_{2,3}z^L_{4,6} + q^4 z^L_{1,5}z^L_{2,4}z^L_{3,6} - q^5 z^L_{1,5}z^L_{2,6}z^L_{3,4} \\ & + q^4 z^L_{1,6}z^L_{2,3}z^L_{4,5} - q^5 z^L_{1,6}z^L_{2,4}z^L_{3,5} + q^6 z^L_{1,6}z^L_{2,5}z^L_{3,4} \end{aligned} \tag{3.61}$$

Regarding the notation, we will usually denote this quantum Pfaffian as Pf_q. However, in the case of possible confusion regarding the notation, we will sometimes denote the quantum Pfaffian as $Pf_q(Z)$ to emphasize the fact that Pf_q is a polynomial of the generators of $A^L_q(\mathcal{A})$. If it is not clear from the context, we will also write $det_q(X)$ instead of det_q to remind ourselves that det_q is a polynomial of the generators of $A_q(X)$.

In addition to the quantum Pfaffian, Pf_q, we will extend the notation slightly to define some additional polynomials. First, let $I = \{i_1, i_2, \ldots, i_r\}$ be a subset of

$\{1, 2, 3, \ldots, n\}$ with r being even. We define Π^I to be the set of ordered 2-partitions of I. For example, we have

$$\Pi^{\{3,5,8,9\}} = \{(3,5)(8,9), (3,8)(5,9), (3,9)(5,8)\} \tag{3.62}$$

Then we may define

$$Pf_q^I = \sum_{\pi \in \Pi^I} (-q)^{l(\pi)} z_\pi^L \tag{3.63}$$

Remark 3.1. *We defined the quantum Pfaffian as a polynomial of the left q-symplectic invariants generated by* $\left\{z_{i,j}^L\right\}$*, then in the following discussion we examine the right action of* $U_q(\mathfrak{g})$ *on* Pf_q*. We could have analogously constructed* Pf_q *using the generators of* $A_q^R(\mathcal{A})$ *and then proceeded to examine the left action of* Uq *on* Pf_q *However, to simplify the explanation, it was chosen to express everything as the former.*

When $n = 2$ it is easy to see, $Pf_q(Z) = \xi_{1,2}^{1,2} = det_q(X)$. However, this is not so obvious for larger values of n.

Proposition 3.1. *For every positive, even* n*,* $Pf_q(Z) = det_q(X)$

Proof. To show this equality, we will prove that Pf_q is simultaneously a highest and lowest weight vector for the right action of $U_q(\mathfrak{g})$. This will show Pf_q to be a scalar multiple of $(det_q)^c$ for some $c \in \mathbb{Z}_+$. One way of viewing this fact is to note that if Pf_q is annihilated by all generators of $U_q(\mathfrak{g})$ then it is a basis of a one dimensional irreducible $U_q(\mathfrak{g})$ module, $V_q^R(\lambda)$, which is realized by $A(B_-\backslash G\ ;\ z^\lambda)$. But we observe that the only such modules are indexed by $\lambda = c\Lambda_n$ for some positive integer c. Then by comparing degree and coefficients we will see $Pf_q(Z) = det_q(X)$.

Let k be a positive integer such that $1 \leq k < n$. Recall the right action of generators of $U_q(\mathfrak{g})$ on products of elements of $A_q(X)$ (see Jing and Yamada [7]),

$$\phi\psi.e_k = (\phi \otimes \psi).(e_k \otimes q^{-a_k/2} + q^{a_k/2} \otimes e_k) \tag{3.64}$$

$$\phi\psi.f_k = (\phi \otimes \psi).(f_k \otimes q^{-a_k/2} + q^{a_k/2} \otimes f_k) \tag{3.65}$$

we may expand this notation to describe the following right action of e_k on the components of Pf_q as

$$\begin{aligned} z^L_{a_1b_1} z^L_{a_2b_2} \cdots z^L_{a_{n/2}b_{n/2}}.e_k = {} & z^L_{a_1b_1}.e_k \otimes z^L_{a_2b_2}.q^{-\alpha_k/2} \otimes \cdots \otimes z^L_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2} \\ & + z^L_{a_1b_1}.q^{\alpha_k/2} \otimes z^L_{a_2b_2}.e_k \otimes \cdots \otimes z^L_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2} \\ & \vdots \\ & + z^L_{a_1b_1}.q^{\alpha_k/2} \otimes z^L_{a_2b_2}.q^{\alpha_k/2} \otimes \cdots \otimes z^L_{a_{n/2}b_{n/2}}.e_k \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} z^L_{a_1b_1} z^L_{a_2b_2} \cdots z^L_{a_{n/2}b_{n/2}}.f_k = {} & z^L_{a_1b_1}.f_k \otimes z^L_{a_2b_2}.q^{-\alpha_k/2} \otimes \cdots \otimes z^L_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2} \\ & + z^L_{a_1b_1}.q^{\alpha_k/2} \otimes z^L_{a_2b_2}.f_k \otimes \cdots \otimes z^L_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2} \\ & \vdots \end{aligned}$$

$$+ z^L_{a_1 b_1}.q^{\alpha_k/2} \otimes z^L_{a_2 b_2}.q^{\alpha_k/2} \otimes \cdots \otimes z^L_{a_{n/2} b_{n/2}}.f_k \tag{3.67}$$

Additionally, each of these $A^L_q(\mathcal{A})$ generators is a sum of quantum 2-minor determinants (see Equation 3.26) in which the indices i and j of z^L_{ij} define the rows for each of these quantum 2-minor determinants . As such, the right action of e_k and f_k on these generators can be described by the following,

$$z^L_{i,j}.e_k = q^{-1/2}\left(\delta_{i,k} z^L_{k+1,j} + \delta_{j,k} z^L_{j,k+1}\right) \tag{3.68}$$

$$z^L_{i,j}.f_k = q^{1/2}\left(\delta_{i,k+1} z^L_{k,j} + \delta_{j,k+1} z^L_{j,k}\right) \tag{3.69}$$

and the right action of $q^{\alpha/2}$ and $q^{-\alpha/2}$ are described by

$$z^L_{i,j}.q^{\alpha_k/2} = q^{\frac{1}{2}(\delta_{i,k}-\delta_{i,k+1}+\delta_{j,k}-\delta_{j,k+1})} z^L_{i,j} \tag{3.70}$$

$$z^L_{i,j}.q^{-\alpha_k/2} = q^{\frac{1}{2}(-\delta_{i,k}+\delta_{i,k+1}-\delta_{j,k}+\delta_{j,k+1})} z^L_{i,j} \tag{3.71}$$

For example

$$z^L_{3,4}.q^{\alpha_4/2} = q^{1/2} z^L_{3,4} \tag{3.72}$$

Before we give a detailed description of the action of e_k on Pf_q, we show how the components of Π may be paired, relative to the value of k. Since the components of Pf_q are indexed by all of these ordered 2-partitions, this will allow us to group the components of Pf_q in a way that the right action of e_k (and f_k) will annihilate the pairs.

We first fix $k \in \mathbb{Z}$ such that $1 \leq k < n$. Now if we choose any of the ordered 2-partitions, say $\pi = (a_1, b_1)(a_2, b_2) \cdots (a_{n/2}, b_{n/2})$, it must have an index r, containing k and an index s containing $k+1$. In other words, there exist r and s such that

$$k \in (a_r, b_r) \tag{3.73}$$

$$k+1 \in (a_s, b_s) \tag{3.74}$$

This fixes r and s. Also contained in these (a_r, b_r) and (a_s, b_s) pairs are two other integers, u and v such that $u < v$. If it happens that $r = s$, in other words, there exists (a_r, b_r) such that $(a_r, b_r) = (k, k+1)$ then we will not pair it with another 2-partition. We will show later how the right action of e_k and f_k already annihilate it.

Example 3.4. *Suppose $n = 8$ and we fix $k = 5$. One of the ordered 2-partitions of Π is $(1,3)(2,6)(4,8)(5,7)$. In this case we see that $r = 4$ and $s = 2$. We then designate $u = 2$ and $v = 7$.*

Now, with r and s still fixed, and for the designated u and v, there are precisely three possibilities describing how k, $k+1$, u and v can be ordered. These are:

$$k < k+1 < u < v \tag{3.75}$$

$$u < k < k+1 < v \tag{3.76}$$

$$u < v < k < k+1 \tag{3.77}$$

For each of these possibilities we have the following,

- $k < k+1 < u < v$

  In this situation, if $r \neq s$, there is another 2-partition, $\hat{\pi}$ identical to π except in the r^{th} and s^{th} pairs, u and v are switched.

$$\pi = (a_1, b_1) \cdots (k, u)(k+1, v) \cdots (a_{n/2}, b_{n/2}) \tag{3.78}$$

$$\hat{\pi} = (a_1, b_1) \cdots (k, v)(k+1, u) \cdots (a_{n/2}, b_{n/2}) \tag{3.79}$$

  If $r = s$ then we have

$$\pi = (a_1, b_1) \cdots (k, k+1) \cdots (u, v) \cdots (a_{n/2}, b_{n/2}) \tag{3.80}$$

- $u < k < k+1 < v$

  In this situation, if $r \neq s$, there is a second partion $\hat{\pi}$ identical to π except in the r^{th} and s^{th} pairs, k and $k+1$ are switched.

$$\pi = (a_1, b_1) \cdots (u, k) \cdots (k+1, v) \cdots (a_{n/2}, b_{n/2}) \tag{3.81}$$

$$\hat{\pi} = (a_1, b_1) \cdots (u, k+1) \cdots (k, v) \cdots (a_{n/2}, b_{n/2}) \tag{3.82}$$

If $r = s$ then we have

$$\pi = (a_1, b_1) \cdots (u, v) \cdots (k, k+1) \cdots (a_{n/2}, b_{n/2}) \tag{3.83}$$

- $u < v < k < k+1$

  In this situation, if $r \neq s$, there is a second partion $\hat{\pi}$ identical to π except in the r^{th} and s^{th} pairs, k and $k+1$ are switched.

$$\pi = (a_1, b_1) \cdots (u, k) \cdots (v, k+1) \cdots (a_{n/2}, b_{n/2}) \tag{3.84}$$

$$\hat{\pi} = (a_1, b_1) \cdots (u, k+1) \cdots (v, k) \cdots (a_{n/2}, b_{n/2}) \tag{3.85}$$

  If $r = s$ then we have

$$\pi = (a_1, b_1) \cdots (u, v) \cdots (k, k+1) \cdots (a_{n/2}, b_{n/2}) \tag{3.86}$$

Example 3.5. *Continuing with the previous example (Example 3.4), with* $n = 8$, $k = 5$ *and* 2*-partition* $(1,3)(2,6)(4,8)(5,7)$*, the other* 2*-partition with which this would be paired is* $(1,3)(2,5)(4,8)(6,7)$.

Using this construction, we see that after fixing k, we may exhaustively list all of the ordered 2-partitions of Π, identifying each 2-partition as containing a pair $(k, k+1)$ or as being one of the pairs just described.

This allows us to write Pf_q as a sum of components of the form

$$(-q)^* z^L_{a_1,b_1} \cdots z^L_{k,k+1} \cdots z^L_{a_{n/2},b_{n/2}} \tag{3.87}$$

or appear in pairs such as

$$\begin{aligned} (-q)^* z^L_{a_1,b_1} \cdots z^L_{k,u} z^L_{k+1,v} \cdots z^L_{a_{n/2},b_{n/2}} \\ (-q)^{*+1} z^L_{a_1,b_1} \cdots z^L_{k,v} z^L_{k+1,u} \cdots z^L_{a_{n/2},b_{n/2}} \end{aligned} \tag{3.88}$$

or

$$\begin{aligned} (-q)^* z^L_{a_1,b_1} \cdots z^L_{u,k} \cdots z^L_{k+1,v} \cdots z^L_{a_{n/2},b_{n/2}} \\ (-q)^{*+1} z^L_{a_1,b_1} \cdots z^L_{u,k+1} \cdots z^L_{k,v} \cdots z^L_{a_{n/2},b_{n/2}} \end{aligned} \tag{3.89}$$

or

$$\begin{aligned} (-q)^* z^L_{a_1,b_1} \cdots z^L_{u,k} \cdots z^L_{v,k+1} \cdots z^L_{a_{n/2},b_{n/2}} \\ (-q)^{*+1} z^L_{a_1,b_1} \cdots z^L_{u,k+1} \cdots z^L_{v,k} \cdots z^L_{a_{n/2},b_{n/2}} \end{aligned} \tag{3.90}$$

where $(-q)^*$ represents an appropriate power of $(-q)$ determined by $(a_1b_1)(a_2b_2)\cdots(a_{n/2}b_{n/2})$.

The right action of e_k can now be calculated. In the first case, we have the index that contains $(k, k+1)$ and we have

$$
\begin{aligned}
q^* z^L_{a_1,b_1} \cdots z^L_{k,k+1} \cdots z^L_{a_{n/2},b_{n/2}} . e_k \\
&= (z^L_{a_1,b_1} . e_k) \cdots (z^L_{k,k+1} . q^{-\alpha_k/2}) \cdots (z^L_{a_{n/2},b_{n/2}} . q^{-\alpha_k/2}) \\
&\quad + \ (z^L_{a_1,b_1} . q^{\alpha_k/2}) \cdots (z^L_{k,k+1} . e_k) \cdots (z^L_{a_{n/2},b_{n/2}} . q^{-\alpha_k/2}) \\
&\quad + \ (z^L_{a_1,b_1} . q^{\alpha_k/2}) \cdots (z^L_{k,k+1} . q^{\alpha_k/2}) \cdots (z^L_{a_{n/2},b_{n/2}} . e_k) \\
&= (0) \cdots (z^L_{k,k+1} . q^{-\alpha_k/2}) \cdots (z^L_{a_{n/2},b_{n/2}} . q^{-\alpha_k/2}) \\
&\quad + \ (z^L_{a_1,b_1} . q^{\alpha_k/2}) \cdots (0) \cdots (z^L_{a_{n/2},b_{n/2}} . q^{-\alpha_k/2}) \\
&\quad + \ (z^L_{a_1,b_1} . q^{\alpha_k/2}) \cdots (z^L_{k,k+1} . q^{\alpha_k/2}) \cdots (0) \\
&= 0
\end{aligned} \tag{3.91}
$$

In the next case, with the indexes of the paired 2-partitions containing $(k, u)(k+1, v)$ and $(k, v)(k+1, u)$, we calculate the right action of e_k as

$$
(-q)^* \begin{pmatrix} z_{a_1 b_1} \cdots z_{k,u} z_{k+1,v} \cdots z_{a_{n/2-2} b_{n/2}} \\ -q z_{a_1 b_1} \cdots z_{k,v} z_{k+1,u} \cdots z_{a_{n/2-2} b_{n/2}} \end{pmatrix} . e_k
$$

$$= (-q)^* \left(\begin{array}{c} (z_{a_1,b_1}.e_k)\cdots(z_{k,u}.q^{-\alpha_k/2})(z_{k+1,v}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{k,u}.e_k)(z_{k+1,v}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ +(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{k,u}.q^{\alpha_k/2})(z_{k+1,v}.e_k)\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{k,u}.q^{\alpha_k/2})(z_{k+1,v}.q^{\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.e_k) \\ -q(z_{a_1,b_1}.e_k)\cdots(z_{k,v}.q^{-\alpha_k/2})(z_{k+1,u}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{k,v}.e_k)(z_{k+1,u}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ -q(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{k,v}.q^{\alpha_k/2})(z_{k+1,u}.e_k)\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{k,v}.q^{\alpha_k/2})(z_{k+1,u}.q^{\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.e_k) \end{array} \right)$$

From Equations 3.68, 3.70, and 3.71, this expression simplifies to the following,

$$= (-q)^* \begin{pmatrix} (0)\cdots(q^{-1/2}z_{k,u})(q^{1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ +(z_{a_1,b_1})\cdots(q^{-1/2}z_{k+1,u})(q^{1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ +(z_{a_1,b_1})\cdots(q^{1/2}z_{k,u})(0)\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ +(z_{a_1,b_1})\cdots(q^{1/2}z_{k,u})(q^{-1/2}z_{k+1,v})\cdots(0) \\ -q(0)\cdots(q^{-1/2}z_{k,v})(q^{1/2}z_{k+1,u})\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ -q(z_{a_1,b_1})\cdots(q^{-1/2}z_{k+1,v})(q^{1/2}z_{k+1,u})\cdots(z_{a_{n/2}b_{n/2}}) \\ -q(z_{a_1,b_1})\cdots(q^{1/2}z_{k,v})(0)\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ -q(z_{a_1,b_1})\cdots(q^{1/2}z_{k,v})(q^{-1/2}z_{k+1,u})\cdots(0) \end{pmatrix}$$

$$= (-q)^* \begin{pmatrix} (z_{a_1,b_1})\cdots(q^{-1/2}z_{k+1,u})(q^{1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ -q(z_{a_1,b_1})\cdots(q^{-1/2}z_{k+1,v})(q^{1/2}z_{k+1,u})\cdots(z_{a_{n/2}b_{n/2}}) \end{pmatrix}$$

Then by Equation 3.37

$$= 0 \tag{3.92}$$

In the next case, with the indices of the paired 2-partitions containing $(u, k)\cdots(k+$

$1, v)$ and $(u, k+1) \cdots (k, v)$, we calculate the right action of e_k as

$$(-q)^* \begin{pmatrix} z_{a_1 b_1} \cdots z_{u,k} \cdots z_{k+1,v} \cdots z_{a_{n/2-2} b_{n/2}} \\ \\ -q z_{a_1 b_1} \cdots z_{u,k+1} \cdots z_{k,v} \cdots z_{a_{n/2-2} b_{n/2}} \end{pmatrix} .e_k$$

$$= (-q)^* \begin{pmatrix} (z_{a_1,b_1}.e_k) \cdots (z_{u,k}.q^{-\alpha_k/2}) \cdots (z_{k+1,v}.q^{-\alpha_k/2}) \cdots (z_{a_{n/2} b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2}) \cdots (z_{u,k}.e_k) \cdots (z_{k+1,v}.q^{-\alpha_k/2}) \cdots (z_{a_{n/2} b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2}) \cdots (z_{u,k}.q^{\alpha_k/2}) \cdots (z_{k+1,v}.e_k) \cdots (z_{a_{n/2} b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2}) \cdots (z_{u,k}.q^{\alpha_k/2}) \cdots (z_{k+1,v}.q^{\alpha_k/2}) \cdots (z_{a_{n/2} b_{n/2}}.e_k) \\ -q(z_{a_1,b_1}.e_k) \cdots (z_{u,k+1}.q^{-\alpha_k/2}) \cdots (z_{k,v}.q^{-\alpha_k/2}) \cdots (z_{a_{n/2} b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2}) \cdots (z_{u,k+1}.e_k) \cdots (z_{k,v}.q^{-\alpha_k/2}) \cdots (z_{a_{n/2} b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2}) \cdots (z_{u,k+1}.q^{\alpha_k/2}) \cdots (z_{k,v}.e_k) \cdots (z_{a_{n/2} b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2}) \cdots (z_{u,k+1}.q^{\alpha_k/2}) \cdots (z_{k,v}.q^{\alpha_k/2}) \cdots (z_{a_{n/2} b_{n/2}}.e_k) \end{pmatrix}$$

From Equations 3.68, 3.70, and 3.71, this expression simplifies to the following,

$$= (-q)^* \begin{pmatrix} (0)\cdots(q^{-1/2}z_{u,k})\cdots(q^{1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ +(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ +(z_{a_1,b_1})\cdots(q^{1/2}z_{u,k})\cdots(0)\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ +(z_{a_1,b_1})\cdots(q^{1/2}z_{u,k})\cdots(q^{-1/2}z_{k+1,v})\cdots(0) \\ -q(0)\cdots(q^{1/2}z_{u,k+1})\cdots(q^{-1/2}z_{k,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ -q(z_{a_1,b_1})\cdots(0)\cdots(q^{-1/2}z_{k,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ -q(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{-1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ \vdots \\ -q(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{1/2}z_{k,v})\cdots(0) \end{pmatrix}$$

$$= (-q)^* \begin{pmatrix} (z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \\ -q(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{-1/2}z_{k+1,v})\cdots(z_{a_{n/2}b_{n/2}}) \end{pmatrix}$$

$$= 0 \tag{3.93}$$

In the next case, with the indices of the paired 2-partitions containing $(u,k)\cdots(v,k+$

1) and $(u,k+1)\cdots(v,k)$, we calculate the right action of e_k as

$$(-q)^*\begin{pmatrix} z_{a_1b_1}\cdots z_{u,k}\cdots z_{v,k+1}\cdots z_{a_{n/2-2}b_{n/2}} \\ -qz_{a_1b_1}\cdots z_{u,k+1}\cdots z_{v,k}\cdots z_{a_{n/2-2}b_{n/2}} \end{pmatrix}.e_k$$

$$= (-q)^*\begin{pmatrix} (z_{a_1,b_1}.e_k)\cdots(z_{u,k}.q^{-\alpha_k/2})\cdots(z_{v,k+1}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{u,k}.e_k)\cdots(z_{v,k+1}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{u,k}.q^{\alpha_k/2})\cdots(z_{v,k+1}.e_k)\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ +(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{u,k}.q^{\alpha_k/2})\cdots(z_{v,k+1}.q^{\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.e_k) \\ -q(z_{a_1,b_1}.e_k)\cdots(z_{u,k+1}.q^{-\alpha_k/2})\cdots(z_{v,k}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{u,k+1}.e_k)\cdots(z_{v,k}.q^{-\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{u,k+1}.q^{\alpha_k/2})\cdots(z_{v,k}.e_k)\cdots(z_{a_{n/2}b_{n/2}}.q^{-\alpha_k/2}) \\ \vdots \\ -q(z_{a_1,b_1}.q^{\alpha_k/2})\cdots(z_{u,k+1}.q^{\alpha_k/2})\cdots(z_{v,k}.q^{\alpha_k/2})\cdots(z_{a_{n/2}b_{n/2}}.e_k) \end{pmatrix}$$

From Equations 3.68, 3.70, and 3.71, this expression simplifies to the following,

$$
\begin{aligned}
&= (-q)^* \left(\begin{array}{c}
(0)\cdots(q^{-1/2}z_{u,k})\cdots(q^{1/2}z_{v,k+1})\cdots(z_{a_{n/2}b_{n/2}}) \\
\vdots \\
+(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{1/2}z_{v,k+1})\cdots(z_{a_{n/2}b_{n/2}}) \\
\vdots \\
+(z_{a_1,b_1})\cdots(q^{1/2}z_{u,k})\cdots(0)\cdots(z_{a_{n/2}b_{n/2}}) \\
\vdots \\
+(z_{a_1,b_1})\cdots(q^{1/2}z_{u,k})\cdots(q^{-1/2}z_{v,k+1})\cdots(0) \\
-q(0)\cdots(q^{1/2}z_{u,k+1})\cdots(q^{-1/2}z_{k,v})\cdots(z_{a_{n/2}b_{n/2}}) \\
\vdots \\
-q(z_{a_1,b_1})\cdots(0)\cdots(q^{-1/2}z_{v,k})\cdots(z_{a_{n/2}b_{n/2}}) \\
\vdots \\
-q(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{-1/2}z_{v,k+1})\cdots(z_{a_{n/2}b_{n/2}}) \\
\vdots \\
-q(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{1/2}z_{v,k})\cdots(0)
\end{array} \right) \\
&= (-q)^* \left(\begin{array}{c}
(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{1/2}z_{v,k+1})\cdots(z_{a_{n/2}b_{n/2}}) \\
-q(z_{a_1,b_1})\cdots(q^{-1/2}z_{u,k+1})\cdots(q^{-1/2}z_{v,k+1})\cdots(z_{a_{n/2}b_{n/2}})
\end{array} \right) \\
&= 0 \qquad (3.94)
\end{aligned}
$$

Thus,

$$Pf_q.e_k = 0 \qquad (3.95)$$

A similar argument shows

$$Pf_q.f_k = 0 \tag{3.96}$$

Since Pf_q is an element of $A_q(X)$ annihilated by the right action of all e_k and f_k, $1 \leq k < n$, Pf_q must be generated by det_q. By comparing degree and coefficients, we see $Pf_q(Z) = det_q(X)$. □

Here we note the element ${Pf_q}^{\{1,2,\ldots,r\}}$ (where r must be even) provides a realization of an element in $A_q(X)^K \cap A(B_-\backslash G; z^{\Lambda_r})$. This follows from the fact that $Pf_q(Z) = det_q(X)$ for a given n (n is even). If we increase the number of variables from n^2 to $(n+2)^2$, the contents of this same pfaffian is increased by only two columns indexed by $n+1$ and $n+2$. This is from the fact that each z_{ij}^L factor appearing in the construction of ${Pf_q}^{\{1,\ldots,n\}}$ has an additional term added. This term is a quantum 2-minor determinant in rows i and j and in columns $n+1$ and $n+2$. With this additional term in each of the z_{ij}^L we can write the following

$$\begin{aligned} {Pf_q}^{\{1,\ldots,n\}} &= \xi^{1,\ldots,n}_{1,\ldots,n} + \frac{1}{q^2}\xi^{1,\ldots,n}_{1,\ldots,n-2,n+1,n+2} \\ &+ \frac{1}{q^4}\xi^{1,\ldots,n}_{1,\ldots,n-4,n-1,n,n+1,n+2} \\ &\vdots \\ &+ \frac{1}{q^{n-2}}\xi^{1,\ldots,n}_{3,4,\ldots,n,n+1,n+2} \end{aligned} \tag{3.97}$$

which is also an element in $A(B_-\backslash G; z^{\Lambda_n})$ and because it is constructed as a polynomial of the generators of $A^L(\mathcal{A})$, it is also a left q-symplectic invariant. In this manner we can inductively show that ${Pf_q}^{\{1,\ldots,r\}} \in A(B_-\backslash G; z^{\Lambda_r})$ for $r \leq n$ for r even.

3.0.11 Decomposition of ${}^K A_q(X))$ and $A_q(X)^K$

We show the decomposition of ${}^K A_q(X))$ as a right $A_q(G)$-comodule (resp. left $U_q(\mathfrak{g})$-module) and the decomposition of $A_q(X))^K$ as a left $A_q(G)$-comodule (resp. right $U_q(\mathfrak{g})$-module. To perform this decomposition, several preliminary propositions are presented, along with the introduction of some notational conventions. First some notation:

We define the map ϕ from the power set of $\{1, 2, 3, \ldots, n/2\}$ into the power set of $\{1, 2, 3, \ldots, n\}$ by

$$\mathcal{P}\{1, 2, 3, \ldots, n/2\} \xrightarrow{\phi} \mathcal{P}\{1, 2, 3, \ldots, n\}$$

$$\phi(A) = \bigcup_{\alpha \in A} \{2\alpha - 1, 2\alpha\} \tag{3.98}$$

Example 3.6.

$$\phi(\{1, 3\}) = \{1, 2, 5, 6\}$$

$$\phi(\{2\}) = \{3, 4\}$$

$$\phi(\{1, 3, 4, 5\}) = \{1, 2, 5, 6, 7, 8, 9, 10\}$$

We will use the map ϕ to construct indices for the rows and columns of quantum minor determinants used in q-symplectic invariants. Our q-symplectic invariants will be constructed of quantum minor determinants whose row and column indices appear in pairs of consecutive positive integers, in which the first of each pair is odd. As such, the elements in the range of ϕ provide us the indexing sets for these rows and columns. Also, in the discussion of q-symplectic invariants we use a specific set of dominant weights. We define these as

$$P_n^{\mathcal{A}} = \left\{ \lambda \in P_n \ ; \ \lambda = (\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_{n/2}, \mu_{n/2}), \ \mu \in P_{n/2} \right\} \tag{3.99}$$

Example 3.7.

$$(2, 2, 1, 1) \in P_4^{\mathcal{A}}$$

$$(4, 4, 4, 4, 3, 3, 2, 2, 2, 2, 1, 1) \in P_{12}^{\mathcal{A}}$$

One of the key ideas used in the decomposition of ${}^K A_q(X)$ and $KA_q(X)^K$ is presented in the following proposition by Jing and Yamada [7].

Proposition 3.2. *(by Jing and Yamada [7]) Let $\mu \in P_n$ be a dominant integral weight and $V_q^R(\mu)$ be the irreducible left $U_q(\mathfrak{g})$ submodule with highest weight μ. Then the space of the q-symplectic invariants in V_q^R has the dimension equal to the multiplicity of V in ${}^K A_q(X)$).*

Proof. To decompose the algebra ${}^K A_q(X)$) as a right $A_q(G)$-comodule (or left $U_q(\mathfrak{g})$ module), it suffices to find the singular weight vectors, i.e. the weight vectors $\phi \in {}^K A_q(X)$ such that $e_k.\phi = 0$ for $k = 1, \ldots, n-1$. Since such a singular vector φ is contained in the space ${}^K A_q(X) \cap A(X/B^+; z^\lambda)$ for some dominant integral weigh $\lambda \in L_n$, and generates an irreducible right $A_q(G)$-comodule with highest weight λ. Thus if there are m_λ singular weight vectors of weight λ in ${}^K A_q(X)$, then the irreducible right $A_q(G)$-comodule isomorphic to $V_q^R(\lambda)$ occurs m_λ times in the decomposition of ${}^K A_q(X)$. On the other hand, a singular vector φ in ${}^K A_q(X) \cap A(X/B^+; z^\lambda)$ is regarded as a left q-orthogonal invariant in V_q^L (i.e. annihilated on left). Since $V_q^L(\lambda)$ and $V_q^R(\lambda)$ are dual to each other, the dimension of the space of q-symplectic invariants coincides. □

Next, we show by construction, the existence of a left invariant in the left $U_q(\mathfrak{g})$-module $A(B_-\backslash G; z^\lambda)$. Note: although we have already shown the existence of such an element in 3.0.10, the construction presented here gives us a form that lets us more easily construct a bi-invariant in a later section.

We build this left invariant from elements of the following form

$$a_r^R = \sum_J q^{-2|J|} \xi_{\phi(J)}^{1,\ldots,2r} \tag{3.100}$$

where the sum is over all J such that $\#J = r$ and $J \subseteq \{1, 2, \ldots, n/2\}$. $|J|$ represents the sum of the elements of J. We first notice that when $n = 2$ there is only the one element $a_1^R = \xi_{1,2}^{1,2}$ to consider, and it is obviously in $A_q(X)^K$. So for the rest of this discussion we will assume $n \geq 4$. Next we notice that the columns of each component of a_r^R are indexed by $\phi(J)$. This causes the columns of each quantum minor determinant to appear in pairs of adjacent positive integers, the first of each pair is odd. As a result, the action of e_k and f_k for any odd k, $1 \leq k < n$, annihilates a_r^R. In other words

$$e_k.a_r^R = 0, \quad f_k.a_r^R = 0 \qquad \text{for } 1 \leq k < n, k \text{ odd} \tag{3.101}$$

As such,

$$sp_e(i,i).a_r^R = 0, \quad sp_f(i,i).a_r^R = 0 \qquad 1 \leq i \leq n/2 \tag{3.102}$$

Now we need to show that a_r^R is annihilated by elements of the form $sp_e(i, i+1)$ and $sp_f(i, i+2)$. Let us consider any four consecutive integers $k, k+1, k+2, k+3 \in \{1, \ldots, n\}$ with k being odd. With these restrictions on k we have

$$sp_e\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right) = e_k e_{k+1} e_{k+2} - e_k e_{k+2} e_{k+1}$$

$$- e_{k+1} e_{k+2} e_k + e_{k+2} e_{k+1} e_k + \frac{1}{q^2} f_{k+1} \tag{3.103}$$

$$\tag{3.104}$$

$$sp_f\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right) = \frac{1}{q^2} f_{k+2} f_{k+1} f_k - \frac{1}{q^2} f_{k+2} f_k f_{k+1}$$

$$- \frac{1}{q^2} f_{k+1} f_k f_{k+2} + \frac{1}{q^2} f_k f_{k+1} f_{k+2} + e_{k+1} \tag{3.105}$$

Relative to $k, k+1, k+2, k+3$, the components $q^{-2|J|}\xi_{\phi(J)}^{1,\ldots,2r}$ of a_r^R fall into one of the following three cases:

Case 1 These four consecutive integers are contained in the column indexing of $q^{-2|J|}\xi_{\phi(J)}^{1,\ldots,2r}$.

In other words,

$$\{k, k+1, k+2, k+3\} \subseteq \phi(J) \tag{3.106}$$

As a result of the action of e_{k+1} shifting the $k+2$ column to the $k+1$ column and the action of f_{k+1} shifting the $k+1$ column to the $k+2$ column, we have

$$e_{k+1}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \quad and \quad f_{k+1}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \tag{3.107}$$

We also already have

$$e_{k}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \quad and \quad f_{k}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \tag{3.108}$$

$$e_{k+2}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \quad and \quad f_{k+2}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \tag{3.109}$$

Thus $sp_e\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right).\xi_{\phi(J)}^{1,\ldots,2r} = 0$ and $sp_f\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right).\xi_{\phi(J)}^{1,\ldots,2r} = 0$ for all odd $k \leq n-3$.

Case 2 None of these four integers is in the column index of $q^{-2|J|}\xi_{\phi(J)}^{1,\ldots,2r}$. In other words,

$$\{k, k+1, k+2, k+3\} \subseteq \{1, \ldots, n\} \backslash \phi(J) \tag{3.110}$$

As in the previous case,

$$e_{k}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \quad and \quad f_{k}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \tag{3.111}$$

$$e_{k+1}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \quad and \quad f_{k+1}.\xi_{\phi(J)}^{1,\ldots,2r} = 0 \tag{3.112}$$

$$e_{k+2}.\xi^{1,\ldots,2r}_{\phi(J)} = 0 \quad and \quad f_{k+2}.\xi^{1,\ldots,2r}_{\phi(J)} = 0 \tag{3.113}$$

So, again we have $sp_e\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right).\xi^{1,\ldots,2r}_{\phi(J)} = 0$ and $sp_f\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right).\xi^{1,\ldots,2r}_{\phi(J)} = 0$ for all odd $k \leq n-3$.

Case 3 These elements appear in pairs

$$q^{-2|J_1|}\xi^I_{J_1} \text{ and } q^{-2|J_2|}\xi^I_{J_2} \tag{3.114}$$

where

$$\{k, k+1\} \subseteq J_1, \quad \{k+2, k+3\} \subseteq J_2 \tag{3.115}$$

and

$$J_1 \backslash \{k, k+1\} = J_2 \backslash \{k+2, k+3\} \tag{3.116}$$

In this case we have the following actions:

$$f_{k+1}.\xi^I_{J_1} = e_k e_{k+2} e_{k+1}.\xi^I_{J_2} \tag{3.117}$$

$$f_{k+1}.\xi^I_{J_2} = 0 \tag{3.118}$$

$$e_{k+1}.\xi^I_{J_2} = f_{k+2} f_k f_{k+1}.\xi^I_{J_1} \tag{3.119}$$

$$e_{k+1}.\xi^I_{J_1} = 0 \tag{3.120}$$

$$f_k.\xi^I_{J_1} = 0 \tag{3.121}$$

$$f_k.\xi^I_{J_2} = 0 \tag{3.122}$$

$$e_k.\xi^I_{J_1} = 0 \tag{3.123}$$

$$e_k.\xi^I_{J_2} = 0 \tag{3.124}$$

$$f_{k+2}.\xi^I_{J_1} = 0 \tag{3.125}$$

$$f_{k+2}.\xi^I_{J_2} = 0 \tag{3.126}$$

$$e_{k+2}.\xi^I_{J_1} = 0 \tag{3.127}$$

$$e_{k+2}.\xi^I_{J_2} = 0 \tag{3.128}$$

These actions then imply

$$sp_e\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right).\left(q^{-2|J_1|}\xi^I_{J_1} + q^{-2|J_2|}\xi^I_{J_2}\right) = 0 \tag{3.129}$$

$$sp_f\left(\frac{k+1}{2}, \frac{k+1}{2}+1\right).\left(q^{-2|J_1|}\xi^I_{J_1} + q^{-2|J_2|}\xi^I_{J_2}\right) = 0 \tag{3.130}$$

for all odd $k \leq n-3$. Thus, in all three cases we have

$$sp_e(i, i+1).a^R_r = 0, \qquad 1 \leq i < n/2 \tag{3.131}$$

$$sp_f(i, i+1).a^R_r = 0, \qquad 1 \leq i < n/2 \tag{3.132}$$

As such $a^R_r \in A_q(X)^K$. □

Lemma 3.2. *(Existence) For* $\lambda = \sum_{r=1}^{n/2} m_{2r}\Lambda_{2r}$*, i.e,* $\lambda \in P^{\mathcal{A}}_n$*,* $A(B_-\backslash G; z^\lambda)$ *contains a left q-symplectic invariant.*

Proof. Suppose $\lambda = \sum_{r=1}^{n/2} m_{2r}\Lambda_{2r}$, then we define

$$a_\lambda^R = \prod_{r=1}^{n/2} \left(a_r^R\right)^{m_{2r}} \tag{3.133}$$

where a_r^R is defined by Equation 3.100. We see by its construction, $a_\lambda^R \in A_q(X)^K \cap A(B_-\backslash G; z^\lambda)$. As such, each right $A_q(G)$-comodule $V_q^R(\lambda)$ has a q-symplectic invariant.

□

Lemma 3.3. *(Nonexistence) There does not exist a left q-symplectic invariant in the irreducible right $U_q(\mathfrak{g})$-submodule $V_q^L(\lambda)$ if $\lambda \notin P_n^{\mathcal{A}}$.*

Proof. $A_q(X)^K$ is a right $U_q(\mathfrak{g})$-submodule of $A_q(X)$. As such, it has its own decomposition into irreducible right $U_q(\mathfrak{g})$-submodules indexed by $\lambda \in P_n$, where λ is a dominant integral weight

$$A_q(X)^K = \bigoplus_\lambda V_q^L(\lambda) \tag{3.134}$$

Each $V_q^L(\lambda)$ is a highest weight module (Noumi, Yamada, and Mimachi [4]). Each of these highest weight modules has a realization of as $A(G/B^+; z^\lambda)$ with highest weight vector of the form

$$v_\lambda = \left(\xi_{1,\ldots,s}^{1,\ldots,s}\right)^{m_s} \left(\xi_{1,\ldots,s-1}^{1,\ldots,s-1}\right)^{m_{s-1}} \cdots \left(\xi_1^1\right)^{m_1} \tag{3.135}$$

However, because, the elements of $A_q(X)^K$ are annihilated on the left by all e_k and f_k where k is odd, then for the highest weight vector v_λ, it must be true that $\lambda = \sum_{r=1}^{n} m_r\Lambda_r$

where $m_r = 0$ when r is odd. Thus,

$$A_q(X)^K = \bigoplus_{\lambda \in P_n^A} V_q^L(\lambda) \tag{3.136}$$

□

Lemma 3.4. *(Uniqueness) The multiplicity of $V_q^R(\lambda)$ an irreducible right $U_q(\mathfrak{g})$-module, in the decomposition of $A_q(X)^K$ is exactly one.*

Proof. As mentioned earlier, by Proposition 3.2, the multiplicity of $V_q^R(\lambda)$ in the decomposition of $A_q(X)^K$ is equal to the number of left q-symplectic invariants in $A(B_-\backslash G; z^\lambda)$. Let v^K be a non zero left invariant in $A(B_-\backslash G; z^\lambda)$, as such, it can be written as a linear combination of weight vectors from the standard basis of $A(B_-\backslash G; z^\lambda)$ (Noumi, Yamada, and Mimachi [4]).

However, since $A(B_-\backslash G; z^\lambda)$ is a highest weight vector space, there must be at least one basis (weight) vector, η, in the composition of v^K for which there are no higher weight vectors in v^K. In other words

$$v^K = \eta \oplus v_1 \oplus \cdots v_j \tag{3.137}$$

where the weights of $v_1, \ldots, v_j$ are less than or equal to that of η. As such, η must be annihilated by all e_k, where $k < n$ and k is odd. Additionally, the elements of the form

$$sp_f(i,i+1) = e_{2i} + \frac{1}{q^2}\left(f_{2i-1}f_{2i}f_{2i+1} - f_{2i}f_{2i-1}f_{2i+1}\right.$$
$$\left. - f_{2i+1}f_{2i-1}f_{2i} + f_{2i+1}f_{2i}f_{2i-1}\right) \tag{3.138}$$

where $1 \leq i < n/2$, also annihilate v^K, and this in turn requires that η also be annihilated by all e_k, where $k < n$ and k is even. Therefore η must be a highest weight vector of $A(B_-\backslash G; z^\lambda)$, but this vector is unique up to constant multiple, because $A(B_-\backslash G; z^\lambda)$ is a highest weight module. So

$$\eta = cv_\lambda, \quad c \in \mathbb{C} \tag{3.139}$$

where v_λ is defined by Equation 3.135. This tells us that any non-zero left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$ must be written as

$$cv_\lambda \oplus w_1 \oplus \cdots w_j, \quad c \in \mathbb{C}, c \neq 0 \tag{3.140}$$

where $w_1, \ldots, w_j$ are lower weight vectors of $A(B_-\backslash G; z^\lambda)$.

Now assume there is more than one left quantum q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$, say v^K and w^K. Each of these may be written as a sum of standard basis elements, each including a non-zero term for the highest weight vector v_λ. In other words, they may be written as

$$v^K = c_0 v_\lambda + c_1 v_1 + c_2 v_2 + \cdots + c_i v_i, \quad c_0 \neq 0 \tag{3.141}$$

$$w^K = k_0 v_\lambda + d_1 v_1 + d_2 v_2 + \cdots + d_j v_j, \quad k_0 \neq 0 \tag{3.142}$$

Since the linear combination of any left q-symplectic invariant is also a left q-symplectic invariant then it must be true that $k_0 v^K - c_0 w^K$ is also a left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$. If $k_0 v^K - c_0 w^K \neq 0$ then we have a contradiction to the requirement that any left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$ has a nonzero v_λ component. On the other hand, if $k_0 v^K - c_0 w^K = 0$ then w^K is a constant multiple of v^K. Therefore, any left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$ is unique up to a constant multiple.

□

The following proposition summarizes the Lemmas 3.2, 3.3, and 3.4.

Proposition 3.3. *The space of q-symplectic invariants in the right $A_q(G)$-comodule $V_q^R(\mu)$ is one dimensional if and only if $\mu = \sum_{r=1}^{n/2} m_{2r}\Lambda_{2r}$, in other words, $\mu \in P_n^{\mathcal{A}}$. Otherwise there are no q-symplectic invariants in V_q^R*

By Proposition 3.2 we may then summarize our results with the following theorem

Theorem 3.2. *The irreducible decomposition of $A_q(X)^K$ as a right $U_q(\mathfrak{g})$-module is given by*

$$A_q(X)^K = \bigoplus_{\lambda \in P_n^{\mathcal{A}}} V_q^L(\lambda) \tag{3.143}$$

similarly ${}^{K}A_q(X)$*, as a left* $U_q(\mathfrak{g})$*-module has the irreducible decomposition*

$$ {}^{K}A_q(X) = \bigoplus_{\lambda \in P_n^{\mathcal{A}}} V_q^R(\lambda) \tag{3.144}$$

Where $P_n^{\mathcal{A}}$ *is defined by 3.99.*

Proposition 3.4. *The space* $A_q^L(\mathcal{A}) = A_q(X)^K$*, (resp.* ${}^{K}A_q(X) = A_q^R(\mathcal{A})$*). As such,* $A_q^L(\mathcal{A})$ *(resp.* $A_q^R(\mathcal{A})$*) also have the decompositions as a right (resp. left)* $U_q(\mathfrak{g})$*-modules,*

$$ A_q^L(\mathcal{A}) = \bigoplus_{\lambda \in P_{n/2}^{\mathcal{A}}} V_q^L(\lambda) \tag{3.145}$$

$$ A_q^R(\mathcal{A}) = \bigoplus_{\lambda \in P_{n/2}^{\mathcal{A}}} V_q^R(\lambda) \tag{3.146}$$

Proof. From its definition, we already have $A_q^L(\mathcal{A}) \subseteq A_q(X)^K$. The elements, ${Pf_q}^I$ described by Equation 3.63 provide a formula for explicitly constructing a left $U_q(\mathfrak{sp}(n,\mathbb{C}))$ invariant in $A(B_-\backslash G; z^\lambda)$ for any λ. As such, $A_q(X)^K \subseteq A_q^L(\mathcal{A})$, and we have $A_q^L(\mathcal{A}) = A_q(X)^K$.

3.0.12 Bi-invariants

In this section we define a subalgebra of $A_q(X)$ by the intersection of $A_q^R(\mathcal{A})$ and $A_q^L(\mathcal{A})$. Defined in this way, this space is annihilated on the left and right by $U_q(\mathfrak{sp}(n,\mathbb{C}))$. We then proceed to show that this algebra is really $\mathbb{C}[s_1,\ldots,s_{n/2}]^{\mathcal{S}_{n/2}}$, the symmetric algebra of $n/2$ variables. To start, we define A_{ZP}, as

$$A_{ZP} = A_q^R(\mathcal{A}) \cap A_q^L(\mathcal{A}) = \bigoplus_{m=0}^{\infty} A_{ZP,2m}, \tag{3.147}$$

Recall, the polynomials of $A_q^R(\mathcal{A})$ and $A_q^L(\mathcal{A})$ have even degree so it has the natural grading into the subspaces $A_{ZP,2m}$.

Now we define

$$E_r = \sum_{I,J} q^{2(|I|-|J|)} \xi_{\phi(J)}^{\phi(I)}, \quad 1 \leq r \leq n/2 \tag{3.148}$$

where the summation runs over all subsets I and J of $\{1, \ldots, \frac{n}{2}\}$ and $\#I = \#J = r$. Here, $|I|$ and $|J|$ are the sums of the elements of I and J respectively.

Lemma 3.5. $E_r \in A_{ZP,2r}$

Proof. If we examine the component of E_r that is obtained by holding I fixed at $I = \{1, 2, \ldots, r\}$, we see that this component is precisely a_r^R, defined in Equation 3.100. As such, this component is invariant under the left action of $U_q(\mathfrak{sp}(n, \mathbb{C}))$. The remaining components of E_r (the components obtained by fixing I at other values) can be obtained by the right action of $U_q(\mathfrak{g})$ on a_r^R. Since $A_q(X)^K$ is a right submodule of $A_q(X)$ these other components of E_r must also be left invariant. Thus, $E_r \in A_q(X)^K$. Similarly, we see that the component of E_r associated with the fixed $J = \{1, 2, \ldots, n/2\}$ is in ${}^K A_q(X)$ and likewise the other components of E_r can be obtained by the left action of $U_q(\mathfrak{g})$. Thus, $E_r \in {}^K A_q(X)$. Since E_r has degree $2r$ (by its construction) and $E_r \in A_q^R(\mathcal{A}) \cap A_q^L(\mathcal{A})$, it follows that $E_r \in A_{ZP,2r}$. $\square$

Theorem 3.3. *The algebra A_{ZP} is generated by $E_r(1 \leq r \leq n/2)$ and the algebra A_{ZP} is isomorphic to the algebra of symmetric polynomials of $n/2$ variables;*

$$\pi : A_{ZP} \xrightarrow{\sim} \mathbb{C}[s_1, \ldots, s_{n/2}]^{\mathcal{S}_{n/2}} \tag{3.149}$$

Proof. Because of the decomposition given in Proposition 3.4, the dimension of the bi-invariant space associated with each $\lambda \in P_n^{\mathcal{A}}$ must be exactly one. Since the degree of the polynomial in each of these bi-invariant spaces is $\sum_{k=1}^{n} \lambda_k$, the dimension of $A_{ZP,2m}$ can then be calculated as the number of partitions in $P_n^{\mathcal{A}}$ of $2m$. As these partitions are in $P_n^{\mathcal{A}}$ we may also consider this as the number of partitions of m whose number of parts is less than or equal to $n/2$. Adopting the notation of Jing and Yamada [7] we denote this by $p_{n/2}(m)$.

Consider the restriction of the projection map π to A_{ZP}

$$\pi'_H : A_{ZP} \longrightarrow A_+(H), \tag{3.150}$$

where $A_+(H) = \mathbb{C}[t_1, \ldots, t_n]$. Then $Ker(\pi'_H) = \bigoplus_{r=0}^{\infty} Ker(\pi'_{H,2r})$, where

$$\pi'_{H,2r} : A_{ZP,2r} \longrightarrow A_{2r}(H). \tag{3.151}$$

Similar to the proof by Jing and Yamada [7], the monomials $E_{r_1} E_{r_2} \ldots E_{r_k}$ $(r_1 \leq r_2 \leq \ldots \leq r_k)$ have the degree $2(r_1 + r_2 + \ldots + r_k)$ and are linearly independent over $\mathbb{C}$.

As such the space of degree $2m$ spanned by these monomials has dimension $p_{n/2}(m)$. This shows that the space A_{ZP} is generated by E_r $(1 \leq r \leq n/2)$.

Additionally, the map $\pi'_{H,2r}$ acts on the generators of A_{ZP} in the following manner

$$\pi'(E_r) = \pi'\left(\sum_I \xi^{\phi(I)}_{\phi(I)}\right) \tag{3.152}$$

$$= \sum_I (t_{2i_1-1}t_{2i_1})(t_{2i_2-1}t_{2i_2})\cdots(t_{2i_r-1}t_{2i_r}) \tag{3.153}$$

$$\neq 0 \tag{3.154}$$

where the sum runs over all subsets I of $\{1, 2, \ldots, \frac{n}{2}\}$ and $\#I = r$, thus $Ker(\pi'_{H,2r}) = (0)$. Another way of viewing this is that each of these E_r has monomials which are products of diagonal elements. As such, $\pi(E_r) \neq 0$ for $1 \leq r \leq n$. Thus we have the isomorphism

$$A_{ZP} \cong \mathbb{C}[(t_1t_2), (t_3t_4), \ldots, (t_{n-1}t_n)]^{\mathcal{S}_{n/2}} \tag{3.155}$$

$$\cong \mathbb{C}[s_1, s_2, \ldots, s_{n/2}]^{\mathcal{S}_{n/2}} \tag{3.156}$$

where we let $s_i = t_{2i-1}t_{2i}$. □

Appendix A

Proofs for the Left Action of $U_q(\mathfrak{sp}(n, \mathbb{C}))$ on $A_q^L(\mathcal{A})$ Generators

Lemma A.1. $e_k.\xi^{i,j}_{r,s} = \delta_{k+1,r}\xi^{i,j}_{k,s} + \delta_{k+1,s}\xi^{i,j}_{r,k}$

Proof.

$$
\begin{aligned}
e_k.\xi^{i,j}_{r,s} = {} & e_k.(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}) \\
= {} & (e_k \otimes q^{-\alpha_k/2} + q^{\alpha_k/2} \otimes e_k).(x_{i,r} \otimes x_{j,s} - qx_{i,s} \otimes x_{j,r}) \\
= {} & ((id \otimes e_k)(x_{i,1} \otimes x_{1,r} + x_{i,2} \otimes x_{2,r} + \ldots + x_{i,n} \otimes x_{n,r}) \\
& \otimes (id \otimes q^{-\alpha_k/2})(x_{j,1} \otimes x_{1,s} + x_{j,2} \otimes x_{2,s} + \ldots + x_{j,n} \otimes x_{n,s})) \\
& + ((id \otimes q^{\alpha_k/2})(x_{i,1} \otimes x_{1,r} + x_{i,2} \otimes x_{2,r} + \ldots + x_{i,n} \otimes x_{n,r}) \\
& \otimes (id \otimes e_k)(x_{j,1} \otimes x_{1,s} + x_{j,2} \otimes x_{2,s} + \ldots + x_{j,n} \otimes x_{n,s})) \\
& - q((id \otimes e_k)(x_{i,1} \otimes x_{1,s} + x_{i,2} \otimes x_{2,s} + \ldots + x_{i,n} \otimes x_{n,s})
\end{aligned}
$$

$$\otimes (id \otimes q^{-\alpha_k/2})(x_{j,1} \otimes x_{1,r} + x_{j,2} \otimes x_{2,r} + \ldots + x_{j,n} \otimes x_{n,r}))$$

$$+ ((id \otimes q^{\alpha_k/2})(x_{i,1} \otimes x_{1,s} + x_{i,2} \otimes x_{2,s} + \ldots + x_{i,n} \otimes x_{n,s})$$

$$\otimes (id \otimes e_k)(x_{j,1} \otimes x_{1,r} + x_{j,2} \otimes x_{2,r} + \ldots + x_{j,n} \otimes x_{n,r})))$$

$$= \delta_{r,k+1} x_{i,k} \otimes q^{\frac{-\delta_{k,s}+\delta_{k+1,s}}{2}} x_{j,s} + q^{\frac{\delta_{k,r}-\delta_{k+1,r}}{2}} \otimes \delta_{s,k+1} x_{j,k}$$

$$+ q \left(\delta_{s,k+1} x_{i,k} \otimes q^{\frac{-\delta_{k,r}+\delta_{k+1,r}}{2}} x_{j,r} + q^{\frac{\delta_{k,s}-\delta_{k+1,s}}{2}} x_{i,s} \otimes \delta_{r,k+1} x_{j,k} \right)$$

$$= \delta_{r,k+1} \left(x_{i,k} x_{js} q^{\frac{-\delta_{k,s}+\delta_{k+1,s}}{2}} - q x_{i,s} x_{j,k} q^{\frac{\delta_{k,s}-\delta_{k+1,s}}{2}} \right)$$

$$+ \delta_{s,k+1} \left(x_{i,r} x_{j,k} q^{\frac{\delta_{k,r}-\delta_{k+1,r}}{2}} - q x_{i,k} x_{j,r} q^{\frac{-\delta_{k,r}+\delta_{k+1,r}}{2}} \right)$$

$$= \delta_{k+1,r} \xi^{i,j}_{k,s} + \delta_{k+1,s} \xi^{i,j}_{r,k}$$

□

Lemma A.2. $f_k.\xi^{i,j}_{r,s} = \delta_{k,r}\xi^{i,j}_{k+1,s} + \delta_{k,s}\xi^{i,j}_{r,k+1}$

Proof.

$$\begin{aligned}
f_k.\xi^{i,j}_{r,s} &= f_k.(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}) \\
&= (f_k \otimes q^{-\alpha_k/2} + q^{\alpha_k/2} \otimes e_k).(x_{i,r} \otimes x_{j,s} - qx_{i,s} \otimes x_{j,r}) \\
&= ((id \otimes f_k)(x_{i,1} \otimes x_{1,r} + x_{i,2} \otimes x_{2,r} + \ldots + x_{i,n} \otimes x_{n,r}) \\
&\qquad \otimes (id \otimes q^{-\alpha_k/2})(x_{j,1} \otimes x_{1,s} + x_{j,2} \otimes x_{2,s} + \ldots + x_{j,n} \otimes x_{n,s})) \\
&\quad + ((id \otimes q^{\alpha_k/2})(x_{i,1} \otimes x_{1,r} + x_{i,2} \otimes x_{2,r} + \ldots + x_{i,n} \otimes x_{n,r}) \\
&\qquad \otimes (id \otimes f_k)(x_{j,1} \otimes x_{1,s} + x_{j,2} \otimes x_{2,s} + \ldots + x_{j,n} \otimes x_{n,s})) \\
&\quad - q((id \otimes f_k)(x_{i,1} \otimes x_{1,s} + x_{i,2} \otimes x_{2,s} + \ldots + x_{i,n} \otimes x_{n,s}) \\
&\qquad \otimes (id \otimes q^{-\alpha_k/2})(x_{j,1} \otimes x_{1,r} + x_{j,2} \otimes x_{2,r} + \ldots + x_{j,n} \otimes x_{n,r})) \\
&\quad + ((id \otimes q^{\alpha_k/2})(x_{i,1} \otimes x_{1,s} + x_{i,2} \otimes x_{2,s} + \ldots + x_{i,n} \otimes x_{n,s}) \\
&\qquad \otimes (id \otimes f_k)(x_{j,1} \otimes x_{1,r} + x_{j,2} \otimes x_{2,r} + \ldots + x_{j,n} \otimes x_{n,r}))) \\
&= \delta_{r,k}x_{i,k+1} \otimes q^{\frac{-\delta_{k,s}+\delta_{k+1,s}}{2}}x_{j,s} + q^{\frac{\delta_{k,r}-\delta_{k+1,r}}{2}} \otimes \delta_{s,k}x_{j,k+1} \\
&\quad - q\left(\delta_{s,k}x_{i,k+1} \otimes q^{\frac{-\delta_{k,r}+\delta_{k+1,r}}{2}}x_{j,r} + q^{\frac{\delta_{k,s}-\delta_{k+1,s}}{2}}x_{i,s} \otimes \delta_{r,k}x_{j,k+1}\right) \\
&= \delta_{r,k}\left(x_{i,k+1}x_{js}q^{\frac{-\delta_{k,s}+\delta_{k+1,s}}{2}} - qx_{i,s}x_{j,k+1}q^{\frac{\delta_{k,s}-\delta_{k+1,s}}{2}}\right) \\
&\quad + \delta_{s,k}\left(x_{i,r}x_{j,k+1}q^{\frac{\delta_{k,r}-\delta_{k+1,r}}{2}} - qx_{i,k+1}x_{j,r}q^{\frac{-\delta_{k,r}+\delta_{k+1,r}}{2}}\right) \\
&= \delta_{k,r}\xi^{i,j}_{k+1,s} + \delta_{k,s}\xi^{i,j}_{r,k+1}
\end{aligned}$$

□

Lemma A.3. $e_k.z^L_{i,j} = \begin{cases} q^{(i+j-2k-3)/2}\xi^{i,j}_{k,k+2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$

Proof.

$$\begin{aligned}
e_k.z^L_{i,j} &= \sum_{r=1}^{n/2} q^{(i+j+1-4r)/2} e_k.\xi^{i,j}_{2r-1,2r} \\
&= \sum_{r=1}^{n/2} q^{(i+j+1-4r)/2} \left(\delta_{k+1,2r-1}\xi^{i,j}_{k,2r} + \delta_{k+1,2r}\xi^{i,j}_{2r-1,k}\right) \\
&= \begin{cases} q^{(i+j+1-(2k+4))/2}\xi^{i,j}_{k,2r}, & k = 2r-2 \\ q^{(i+j+1-(2k+2))/2}\xi^{i,j}_{2r-1,k}, & k = 2r-1 \end{cases} \\
&= \begin{cases} q^{(i+j+1-(2k+4))/2}\xi^{i,j}_{k,k+2}, & k \text{ even} \\ q^{(i+j+1-(2k+2))/2}\xi^{i,j}_{k,k}, & k \text{ odd} \end{cases} \\
&= \begin{cases} q^{(i+j-2k-3)/2}\xi^{i,j}_{k,k+2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}
\end{aligned}$$

□

Lemma A.4. $f_k.z^L_{i,j} = \begin{cases} 0, & k \text{ odd} \\ q^{(i+j+1-2k)/2}\xi^{i,j}_{k-1,k+1}, & k \text{ even} \end{cases}$

Proof.

$$\begin{aligned} f_k.z^L_{i,j} &= \sum_{r=1}^{n/2} q^{(i+j+1-4r)/2} f_k.\xi^{i,j}_{2r-1,2r} \\ &= \sum_{r=1}^{n/2} q^{(i+j+1-4r)/2} \left(\delta_{k,2r-1}\xi^{i,j}_{k+1,2r} + \delta_{k,2r}\xi^{i,j}_{2r-1,k+1}\right) \\ &= \begin{cases} q^{(i+j+1-(2k+2))/2}\xi^{i,j}_{k+1,2r}, & k = 2r-1 \\ q^{(i+j+1-2k)/2}\xi^{i,j}_{2r-1,k+1}, & k = 2r \end{cases} \\ &= \begin{cases} q^{(i+j+1-(2k+2))/2}\xi^{i,j}_{k+1,k+1}, & k \text{ odd} \\ q^{(i+j+1-2k)/2}\xi^{i,j}_{k-1,k+1}, & k \text{ even} \end{cases} \\ &= \begin{cases} 0, & k \text{ odd} \\ q^{(i+j+1-2k)/2}\xi^{i,j}_{k-1,k+1}, & k \text{ even} \end{cases} \end{aligned}$$

□

Lemma A.5. $sp_e(k,k+1).z^L_{i,j} = 0$

Proof

$$\begin{aligned}
sp_e(k,k+1).z^L_{i,j} =& [e_{2k-1}e_{2k}e_{2k+1} - e_{2k-1}e_{2k+1}e_{2k} \\
& -e_{2k}e_{2k+1}e_{2k-1} + e_{2k+1}e_{2k}e_{2k-1} + q^{-2}f_{2k}\big] .z^L_{i,j} \\
=& \left[-e_{2k-1}e_{2k+1}e_{2k} + q^{-2}f_{2k}\right] .z^L_{i,j} \\
=& - q^{(i+j-4k-3)/2}e_{2k-1}e_{2k+1}.\xi^{i,j}_{2k,2k+2} + q^{-2}q^{(i+j+1-4k)/2}\xi^{i,j}_{2k-1,2k+1} \\
=& q^{(i+j-4k-3)/2}\left(-e_{2k-1}e_{2k+1}.\xi^{i,j}_{2k,2k+2} + \xi^{i,j}_{2k-1,2k+1}\right) \\
=& q^{(i+j-4k-3)/2}\left(-\xi^{i,j}_{2k-1,2k+1} + \xi^{i,j}_{2k-1,2k+1}\right) \\
=& 0
\end{aligned}$$

□

Lemma A.6. $sp_f(k,k+1).z^L_{i,j} = 0$

Proof.

$$\begin{aligned} sp_f(k,k+1).z^L_{i,j} &= \big[e_{2k} + q^{-2}(f_{2k+1}f_{2k}f_{2k-1} - f_{2k+1}f_{2k-1}f_{2k} \\ &\quad -f_{2k}f_{2k-1}f_{2k+1} + f_{2k-1}f_{2k}f_{2k+1})\big].z^L_{i,j} \\ &= \big[e_{2k} - q^{-2}f_{2k+1}f_{2k-1}f_{2k}\big].z^L_{i,j} \\ &= q^{(i+j-4k-3)/2}\xi^{i,j}_{2k,2k+2} - q^{-2}q^{(i+j+1-4k)/2}f_{2k+1}f_{2k-1}.\xi^{i,j}_{2k-1,2k+1} \\ &= q^{(i+j-4k-3)/2}\left(\xi^{i,j}_{2k,2k+2} - f_{2k+1}f_{2k-1}.\xi^{i,j}_{2k-1,2k+1}\right) \\ &= q^{(i+j-4k-3)/2}\left(\xi^{i,j}_{2k,2k+2} - \xi^{i,j}_{2k,2k+2}\right) \\ &= 0 \end{aligned}$$

□

Appendix B

Proofs for Relations of Quantum Antisymmetric Generators

Proposition B.1. $z_{i,j} = -\frac{1}{q} z_{j,i}$ *where* $i < j$

Proof.

$$\begin{aligned}
z_{i,j} &= \sum_{k=1}^{m} q^{2-2*k} \left(x_{i,k}x_{j,m+k} - qx_{i,m+k}x_{j,k}\right) \\
&= \sum_{k=1}^{m} q^{2-2*k} \left((q - \frac{1}{q})x_{i,m+k}x_{j,k} + x_{j,m+k}x_{i,k} - qx_{i,m+k}x_{j,k}\right) \\
&= \sum_{k=1}^{m} q^{2-2*k} \left(-\frac{1}{q}x_{i,m+k}x_{j,k} + x_{j,m+k}x_{i,k}\right) \\
&= -\frac{1}{q} \sum_{k=1}^{m} q^{2-2*k} \left(x_{j,k}x_{i,m+k} - qx_{j,m+k}x_{i,k}\right) \\
&= -\frac{1}{q} z_{j,i}
\end{aligned}$$

□

Proposition B.2. $z_{i,i} = 0$

Proof.

$$\begin{aligned} z_{ii} &= \sum_{k=1}^{m} q^{2-2k} \left(x_{i,2k-1}x_{i,2k} - qx_{i,2k}x_{i,2k-1}\right) \\ &= \sum_{k=1}^{m} q^{2-2k} \left(qx_{i,2k}x_{i,2k-1} - qx_{i,2k}x_{i,2k-1}\right) \\ &= 0 \end{aligned}$$

□

Proposition B.3. $z_{i,j}z_{j,i} = z_{j,i}z_{i,j}$ *where* $i < j$

Proof. Here we will use the previously established relation, B.1 , (and without loss of generality the assumption is made that $i < j$).

$$\begin{aligned} z_{i,j}z_{j,i} &= -\frac{1}{q}z_{j,i}z_{j,i} \\ &= z_{j,i}\left(-\frac{1}{q}\right)z_{j,i} \\ &= z_{j,i}z_{i,j} \end{aligned}$$

□

Proposition B.4. $z_{i,l}z_{j,k} = z_{j,k}z_{i,l}$ *where* $i < j < k < l$

Proof.

$$\begin{aligned} z_{i,l}z_{j,k} &= \left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(\sum_{t=1}^{m} q^{2-2t}\xi^{j,k}_{2t-1,2t}\right) \\ &= \sum_{1\le s,t\le m}\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right) \end{aligned}$$

$$\begin{aligned} &= \sum_{1\le s=t\le m}\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right) + \sum_{1\le s<t\le m}\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right) \\ &\quad + \sum_{1\le t<s\le m}\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right) \end{aligned}$$

$$\begin{aligned} &= \sum_{1\le s=t\le m}\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right) + \sum_{1\le s<t\le m}\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right) \\ &\quad + \sum_{1\le s<t\le m}\left(q^{2-2t}\xi^{i,l}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{j,k}_{2s-1,2s}\right) \end{aligned}$$

Then by using C.1, C.2 and C.3

$$\begin{aligned} &= \sum_{1\le s=t\le m}\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right) \\ &\quad + \sum_{1\le s<t\le m}\left(q^{2-2t}\xi^{j,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,l}_{2s-1,2s}\right) \\ &\quad + \left(q-\frac{1}{q}\right)\sum_{1\le s<t\le m}\left(q^{2-2t}\xi^{j,l}_{2s-1.2s}\right)\left(q^{2-2s}\xi^{i,k}_{2t-1,2t}\right) \\ &\quad - q\left(q-\frac{1}{q}\right)\sum_{1\le s<t\le m}\left(q^{2-2t}\xi^{i,j}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{k,l}_{2s-1,2s}\right) \end{aligned}$$

$$+ \sum_{1 \le s < t \le m} \left(q^{2-2s} \xi^{j,k}_{2s-1,2s}\right) \left(q^{2-2t} \xi^{i,l}_{2t-1,2t}\right)$$

$$- \left(q - \frac{1}{q}\right) \sum_{1 \le s < t \le m} \left(q^{2-2s} \xi^{j,l}_{2s-2,2s}\right) \left(q^{2-2t} \xi^{i,k}_{2t-1,2t}\right)$$

$$+ q \left(q - \frac{1}{q}\right) \sum_{1 \le s < t \le m} \left(q^{2-2s} \xi^{i,j}_{2t-1,2t}\right) \left(q^{2-2t} \xi^{k,l}_{2s-1,2s}\right)$$

$$= \sum_{1 \le s = t \le m} \left(q^{2-2t} \xi^{j,k}_{2t-1,2t}\right) \left(q^{2-2s} \xi^{i,l}_{2s-1,2s}\right)$$

$$+ \sum_{1 \le s < t \le m} \left(q^{2-2t} \xi^{j,k}_{2t-1,2t}\right) \left(q^{2-2s} \xi^{i,l}_{2s-1,2s}\right)$$

$$+ \sum_{1 \le t < s \le m} \left(q^{2-2t} \xi^{j,k}_{2t-1,2t}\right) \left(q^{2-2s} \xi^{i,l}_{2s-1,2s}\right)$$

$$= \sum_{1 \le s,t \le m} \left(q^{2-2t} \xi^{j,k}_{2t-1,2t}\right) \left(q^{2-2s} \xi^{i,l}_{2s-1,2s}\right)$$

$$= \left(\sum_{t=1}^{m} q^{2-2t} \xi^{j,k}_{2t-1,2t}\right) \left(\sum_{s=1}^{m} q^{2-2s} \xi^{i,l}_{2s-1,2s}\right)$$

$$= z_{j,k} z_{i,l}$$

□

Proposition B.5. $z_{i,j}z_{i,k} = qz_{i,k}z_{i,j}$ *where* $i < j < k$

Proof.

$$z_{i,j}z_{i,k} = \left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(\sum_{t=1}^{m} q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)$$

$$= \sum_{1\leq s,t\leq m} \left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)$$

$$= \sum_{1\leq s=t\leq m} \left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right) + \sum_{1\leq s<t\leq m} \left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)$$

$$+ \sum_{1\leq t<s\leq m} \left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)$$

$$= \sum_{1\leq s=t\leq m} \left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right) + \sum_{1\leq s<t\leq m} \left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)$$

$$+ \sum_{1\leq s<t\leq m} \left(q^{2-2t}\xi^{i,j}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,k}_{2s-1,2s}\right)$$

(by C.4, C.5 and C.6)

$$= q\sum_{1\leq s=t\leq m} \left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$+ q\sum_{1\leq s<t\leq m} \left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$+ \left(q-\frac{1}{q}\right)\sum_{1\leq s<t\leq m} \left(q^{2-2t}\xi^{i,k}_{2s-1,2s}\right)\left(q^{2-2s}\xi^{i,j}_{2t-1,2t}\right)$$

$$+ \frac{1}{q}\sum_{1\leq s<t\leq m} \left(q^{2-2s}\xi^{i,k}_{2s-1,2s}\right)\left(q^{2-2t}\xi^{i,j}_{2t-1,2t}\right)$$

$$= q \sum_{1\le s=t\le m} \left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right) + q \sum_{1\le s<t\le m} \left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$+ q \sum_{1\le s<t\le m} \left(q^{2-2t}\xi^{i,k}_{2s-1,2s}\right)\left(q^{2-2s}\xi^{i,j}_{2t-1,2t}\right)$$

$$= \sum_{1\le s=t\le m} q\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right) + \sum_{1\le s<t\le m} q\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$+ \sum_{1\le t<s\le m} q\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$= \sum_{1\le s,t\le m} q\left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$= q\sum_{1\le s,t\le m} \left(q^{2-2t}\xi^{i,k}_{2t-1,2t}\right)\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$= q\sum_{t=1}^{m} \left(q^{2-2t}x^{i,k}_{2t-1,2t}\right)\sum_{s=1}^{m}\left(q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$= qz_{i,k}z_{i,j}$$

□

Proposition B.6. $z_{i,k}z_{j,l} = z_{j,l}z_{i,k} + \left(q - \frac{1}{q}\right) z_{i,l}z_{j,k}$ *where* $i < j < k < l$

Proof

$$z_{i,k}z_{j,l} = \left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,k}_{2s-1,2s}\right)\left(\sum_{t=1}^{m} q^{2-2t}\xi^{j,l}_{2t-1,2t}\right)$$

$$= \sum_{1\le s,t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$

$$= \sum_{1\le s=t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t} + \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$
$$+ \sum_{1\le t<s\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$

$$= \sum_{1\le s=t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t} + \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$
$$+ \sum_{1\le s<t\le m} q^{2-2t}\xi^{i,k}_{2s-1,2s}q^{2-2s}\xi^{j,l}_{2t-1,2t}$$

Then by C.13, C.15 and C.16 we get

$$= \sum_{1\le s=t\le m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$
$$+ \left(q - \frac{1}{q}\right) \sum_{1\le s=t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$
$$+ \sum_{1\le s<t\le m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$
$$+ \left(q - \frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2t}\xi^{i,l}_{2t-1,2t}q^{2-2s}\xi^{j,k}_{2s-1,2s}$$

$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s<t\leq m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s<t\leq m} q^{2-2t}\xi^{i,j}_{2t-1,2t}q^{2-2s}\xi^{k,l}_{2s-2,2s}$$

$$+\sum_{1\leq s<t\leq m} q^{2-2s}\xi^{j,l}_{2s-1,2s}q^{2-2t}\xi^{i,k}_{2t-1,2t}$$

$$+\left(\frac{1}{q}-q\right)\sum_{1\leq s<t\leq m} q^{2-2t}\xi^{i,j}_{2t-1,2t}q^{2-2s}\xi^{k,l}_{2s-1,2s}$$

$$=\sum_{1\leq s=t\leq m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$

$$+\sum_{1\leq s<t\leq m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$

$$+\sum_{1\leq s<t\leq m} q^{2-2s}\xi^{j,l}_{2s-1,2s}q^{2-2t}\xi^{i,k}_{2t-1,2t}$$

$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s=t\leq m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s<t\leq m} q^{2-2t}\xi^{i,l}_{2t-1,2t}q^{2-2s}\xi^{j,k}_{2s-1,2s}$$

$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s<t\leq m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$=\sum_{1\leq s=t\leq m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$

$$+\sum_{1\leq s<t\leq m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$

$$+\sum_{1\leq t<s\leq m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$

$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s=t\leq m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$
$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s<t\leq m} q^{2-2t}\xi^{i,l}_{2t-1,2t}q^{2-2s}\xi^{j,k}_{2s-1,2s}$$
$$+\left(q-\frac{1}{q}\right)\sum_{1\leq t<s\leq m} q^{2-2t}\xi^{i,l}_{2t-1,2t}q^{2-2s}\xi^{j,k}_{2s-1,2s}$$

$$=\sum_{1\leq s,t\leq m} q^{2-2t}\xi^{j,l}_{2t-1,2t}q^{2-2s}\xi^{i,k}_{2s-1,2s}$$
$$+\left(q-\frac{1}{q}\right)\sum_{1\leq s,t\leq m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$=\left(\sum_{t=1}^{m} q^{2-2t}\xi^{j,l}_{2t-1,2t}\right)\left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,k}_{2s-1,2s}\right)$$
$$+\left(q-\frac{1}{q}\right)\left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(\sum_{t=1}^{m} q^{2-2t}\xi^{j,k}_{2t-1,2t}\right)$$
$$=z_{j,l}z_{i,k}+\left(q-\frac{1}{q}\right)z_{i,l}z_{j,k}$$

□

Proposition B.7. $z_{i,j}z_{k,l} = z_{k,l}z_{i,j} + \left(q - \frac{1}{q}\right) z_{i,k}z_{j,l} - q\left(q - \frac{1}{q}\right) z_{i,l}z_{j,k}$ *where* $i < j < k < l$

Proof.

$$\begin{aligned}
z_{i,j}z_{k,l} &= \left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)\left(\sum_{t=1}^{m} q^{2-2t}\xi^{k,l}_{2t-1,2t}\right) \\
&= \sum_{1\le s,t\le m} q^{2-2s}\xi^{i,j}_{2s-1,2s}q^{2-2t}\xi^{k,l}_{2t-1,2t}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{1\le s=t\le m} q^{2-2s}\xi^{i,j}_{2s-1,2s}q^{2-2t}\xi^{k,l}_{2t-1,2t} \\
&\quad + \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,j}_{2s-1,2s}q^{2-2t}\xi^{k,l}_{2t-1,2t} \\
&\quad + \sum_{1\le t<s\le m} q^{2-2s}\xi^{i,j}_{2s-1,2s}q^{2-2t}\xi^{k,l}_{2t-1,2t} \\
&= \sum_{1\le s=t\le m} q^{2-2s}\xi^{i,j}_{2s-1,2s}q^{2-2t}\xi^{k,l}_{2t-1,2t} \\
&\quad + \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,j}_{2s-1,2s}q^{2-2t}\xi^{k,l}_{2t-1,2t} \\
&\quad + \sum_{1\le s<t\le m} q^{2-2t}\xi^{i,j}_{2t-1,2t}q^{2-2s}\xi^{k,l}_{2s-1,2s} \\
&= \sum_{1\le s=t\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t}q^{2-2s}\xi^{i,j}_{2s-1,2s} \\
&\quad + \left(q - \frac{1}{q}\right)\sum_{1\le s=t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t} \\
&\quad - q\left(q - \frac{1}{q}\right)\sum_{1\le s=t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}
\end{aligned}$$

$$+ \sum_{1\le s<t\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t} q^{2-2s}\xi^{i,j}_{2s-1,2s}$$
$$+ \left(q-\frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s} q^{2-2t}\xi^{j,l}_{2t-1,2t}$$
$$+ \left(q-\frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2t}\xi^{i,k}_{2t-1,2t} q^{2-2s}\xi^{j,l}_{2s-1,2s}$$
$$- q\left(q-\frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2t}\xi^{i,l}_{2t-1,2t} q^{2-2s}\xi^{j,k}_{2s-1,2s}$$
$$- q\left(q-\frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s} q^{2-2t}\xi^{j,k}_{2t-1,2t}$$
$$+ \sum_{1\le s<t\le m} q^{2-2s}\xi^{k,l}_{2s-1,2s} q^{2-2t}\xi^{i,j}_{2t-1,2t}$$

$$= \sum_{1\le s=t\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t} q^{2-2s}\xi^{i,j}_{2s-1,2s}$$
$$+ \sum_{1\le s<t\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t} q^{2-2s}\xi^{i,j}_{2s-1,2s}$$
$$+ \sum_{1\le s<t\le m} q^{2-2s}\xi^{k,l}_{2s-1,2s} q^{2-2t}\xi^{i,j}_{2t-1,2t}$$
$$+ \left(q-\frac{1}{q}\right) \sum_{1\le s=t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s} q^{2-2t}\xi^{j,l}_{2t-1,2t}$$
$$+ \left(q-\frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s} q^{2-2t}\xi^{j,l}_{2t-1,2t}$$
$$+ \left(q-\frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2t}\xi^{i,k}_{2t-1,2t} q^{2-2s}\xi^{j,l}_{2s-1,2s}$$
$$- q\left(q-\frac{1}{q}\right) \sum_{1\le s=t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s} q^{2-2t}\xi^{j,k}_{2t-1,2t}$$
$$- q\left(q-\frac{1}{q}\right) \sum_{1\le s<t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s} q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$- q\left(q-\frac{1}{q}\right)\sum_{1\le s<t\le m} q^{2-2t}\xi^{i,l}_{2t-1,2t}q^{2-2s}\xi^{j,k}_{2s-1,2s}$$

$$= \sum_{1\le s=t\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t}q^{2-2s}\xi^{i,j}_{2s-1,2s}$$

$$+ \sum_{1\le s<t\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t}q^{2-2s}\xi^{i,j}_{2s-1,2s}$$

$$+ \sum_{1\le t<s\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t}q^{2-2s}\xi^{i,j}_{2s-1,2s}$$

$$+ \left(q-\frac{1}{q}\right)\sum_{1\le s=t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$

$$+ \left(q-\frac{1}{q}\right)\sum_{1\le s<t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$

$$+ \left(q-\frac{1}{q}\right)\sum_{1\le t<s\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$

$$- q\left(q-\frac{1}{q}\right)\sum_{1\le s=t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$- q\left(q-\frac{1}{q}\right)\sum_{1\le s<t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$- q\left(q-\frac{1}{q}\right)\sum_{1\le t<s\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$= \sum_{1\le s,t\le m} q^{2-2t}\xi^{k,l}_{2t-1,2t}q^{2-2s}\xi^{i,j}_{2s-1,2s}$$

$$+ \left(q-\frac{1}{q}\right)\sum_{1\le s,t\le m} q^{2-2s}\xi^{i,k}_{2s-1,2s}q^{2-2t}\xi^{j,l}_{2t-1,2t}$$

$$- q\left(q-\frac{1}{q}\right)\sum_{1\le s,t\le m} q^{2-2s}\xi^{i,l}_{2s-1,2s}q^{2-2t}\xi^{j,k}_{2t-1,2t}$$

$$= \left(\sum_{t=1}^{m} q^{2-2t}\xi^{k,l}_{2t-1,2t}\right)\left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,j}_{2s-1,2s}\right)$$

$$+ \left(q - \frac{1}{q}\right)\left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,k}_{2s-1,2s}\right)\left(\sum_{t=1}^{m} q^{2-2t}\xi^{j,l}_{2t-1,2t}\right)$$

$$- q\left(q - \frac{1}{q}\right)\left(\sum_{s=1}^{m} q^{2-2s}\xi^{i,l}_{2s-1,2s}\right)\left(\sum_{t=1}^{m} q^{2-2t}\xi^{j,k}_{2t-1,2t}\right)$$

$$= z_{k,l}z_{i,j} + \left(q - \frac{1}{q}\right) z_{i,k}z_{j,l} - q\left(q - \frac{1}{q}\right) z_{i,l}z_{j,k}$$

or by B.1

$$= z_{k,l}z_{i,j} + \left(q - \frac{1}{q}\right) z_{i,k}z_{j,l} + \left(q - \frac{1}{q}\right) z_{i,l}z_{k,j}$$

□

Appendix C

Relations of Quantum 2-Minor Determinants

The quantum antisymmetric generators $z_{1,1}, z_{1,2}, \ldots, z_{n,n}$ are defined in terms of certain quantum 2-minor determinants. Therefore, to calculate the relations for the quantum antisymmetric generators, the relations for these quantum 2-minor determinants had to be calculated. This part of the appendix is a catalog of the relations of these particular quantum 2-minor determinants and the associated calculations.

In the following, each relation is stated in a general manner, in other words, using generic indices on the variables. Then to give a visual interpretation of the determinant, a portion of the matrix $X = (x_{i,j})$ is shown. In this picture, the circled variables are the variables of the first quantum 2-minor in the product. The variables of the second quantum 2-minor are displayed in squares. Shown here are the relations that are relevant to the algebra of quantum antisymmetric polynomials defined in Definition 3.1. No claims about other possible relations between these quantum minor determinants are made here. Additionally, given two quantum determinants, there may be many equivalent ways of stating how they multiply, for example, see C.13 and C.14.

Regarding the following calculations, no claim is made that they are the shortest most efficient calculations. There are many (and probably better) ways to prove these relations.

C.1 Quantum minor determinant relations associated with $z_{i,l}z_{j,k}$ where $i < j < k < l$

Proposition C.1. *Given the quantum minor determinants $\xi^{i,l}_{r,s}$ and $\xi^{j,k}_{r,s}$ where $i < j < k < l$ and $r < s$*

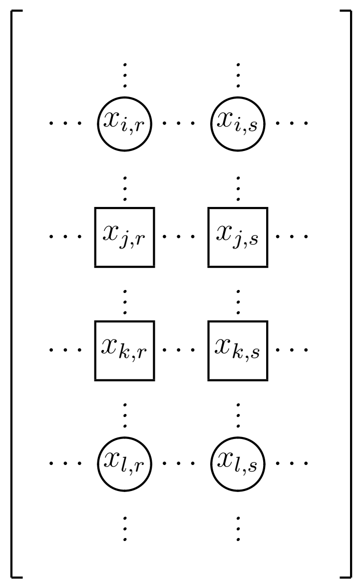

Figure C.1: Relative positions of $\xi^{i,l}_{r,s}$ and $\xi^{j,k}_{r,s}$

then

$$\xi^{i,l}_{r,s}\xi^{j,k}_{r,s} = \xi^{j,k}_{r,s}\xi^{i,l}_{r,s} \tag{C.1}$$

Proof.

$$\begin{aligned}\xi^{i,l}_{r,s}\xi^{j,k}_{r,s} &= (x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}) \\ &= x_{i,r}x_{l,s}x_{j,r}x_{k,s}\end{aligned}$$

$$- qx_{i,r}x_{l,s}x_{j,s}x_{k,r}$$

$$- qx_{i,s}x_{l,r}x_{j,r}x_{k,s}$$

$$+ q^2x_{i,s}x_{l,r}x_{j,s}x_{k,r}$$

$$=x_{i,r}\left[\left(\frac{1}{q} - q\right)x_{j,s}x_{l,r} + x_{j,r}x_{l,s}\right]x_{k,s}$$

$$- x_{i,r}x_{j,s}x_{l,s}x_{k,r}$$

$$- x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$

$$+ q^2x_{i,s}x_{j,s}x_{l,r}x_{k,r}$$

$$= \left(\frac{1}{q} - q\right)x_{i,r}x_{j,s}x_{l,r}x_{k,s}$$

$$+ x_{i,r}x_{j,r}x_{l,s}x_{k,s}$$

$$- x_{i,r}x_{j,s}x_{l,s}x_{k,r}$$

$$- x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$

$$+ q^2x_{i,s}x_{j,s}x_{l,r}x_{k,r}$$

$$= \left(\frac{1}{q} - q\right)\left[\left(q - \frac{1}{q}\right)x_{j,r}x_{i,s} + x_{j,s}x_{i,r}\right]x_{k,s}x_{l,r}$$

$$+ x_{j,r}x_{i,r}x_{k,s}x_{l,s}$$

$$- \left[\left(q - \frac{1}{q}\right)x_{j,r}x_{i,s} + x_{j,s}x_{i,r}\right]\left[\left(\frac{1}{q} - q\right)x_{k,s}x_{l,r} + x_{k,r}x_{l,s}\right]$$

$$- x_{j,r}x_{i,s}x_{k,s}x_{l,r}$$

$$+ q^2 x_{j,s}x_{i,s}x_{k,r}x_{l,r}$$

$$\begin{aligned}
&= \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{k,s}x_{l,r} \\
&\quad + \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{k,s}x_{l,r} \\
&\quad + x_{j,r}x_{i,r}x_{k,s}x_{l,s} \\
&\quad - \left(q - \frac{1}{q}\right)\left(\frac{1}{q} - q\right) x_{j,r}x_{i,s}x_{k,s}x_{l,r} \\
&\quad - \left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{k,r}x_{l,s} \\
&\quad - \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{k,s}x_{l,r} \\
&\quad - x_{j,s}x_{i,r}x_{k,r}x_{l,s} \\
&\quad - x_{j,r}x_{i,s}x_{k,s}x_{l,r} \\
&\quad + q^2 x_{j,s}x_{i,s}x_{k,r}x_{l,r}
\end{aligned}$$

$$\begin{aligned}
&= x_{j,r}x_{i,r}x_{k,s}x_{l,s} \\
&\quad - \left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{k,r}x_{l,s} \\
&\quad - x_{j,s}x_{i,r}x_{k,r}x_{l,s} \\
&\quad - x_{j,r}x_{i,s}x_{k,s}x_{l,r} \\
&\quad + q^2 x_{j,s}x_{i,s}x_{k,r}x_{l,r}
\end{aligned}$$

$$= x_{j,r}\left[\left(q-\frac{1}{q}\right)x_{k,r}x_{i,s}+x_{k,s}x_{i,r}\right]x_{l,s}$$
$$-\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{k,r}x_{l,s}$$
$$-x_{j,s}x_{i,r}x_{k,r}x_{l,s}$$
$$-x_{j,r}x_{i,s}x_{k,s}x_{l,r}$$
$$+q^2x_{j,s}x_{i,s}x_{k,r}x_{l,r}$$

$$= x_{j,r}x_{k,s}x_{i,r}x_{l,s}$$
$$+\left(q-\frac{1}{q}\right)x_{j,r}x_{k,r}x_{i,s}x_{l,s}$$
$$-\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{k,r}x_{l,s}$$
$$-qx_{j,r}x_{k,s}x_{i,s}x_{l,r}$$
$$-qx_{j,s}x_{k,r}x_{i,r}x_{l,s}$$
$$+q^2x_{j,s}x_{i,s}x_{k,r}x_{l,r}$$

$$= x_{j,r}x_{k,s}x_{i,r}x_{l,s}$$
$$-qx_{j,r}x_{k,s}x_{i,s}x_{l,r}$$
$$-qx_{j,s}x_{k,r}x_{i,r}x_{l,s}$$
$$+q^2x_{j,s}x_{i,s}x_{k,r}x_{l,r}$$

$$= (x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})(x_{i,r}x_{l,s}-qx_{i,s}x_{l,r})$$

$$= \xi^{j,k}_{r,s}\xi^{i,l}_{r,s}$$

□

Proposition C.2. *Given the quantum minor determinants* $\xi_{r,s}^{i,l}$ *and* $\xi_{t,u}^{j,k}$ *where* $i < j < k < l$ *and* $r < s < t < u$

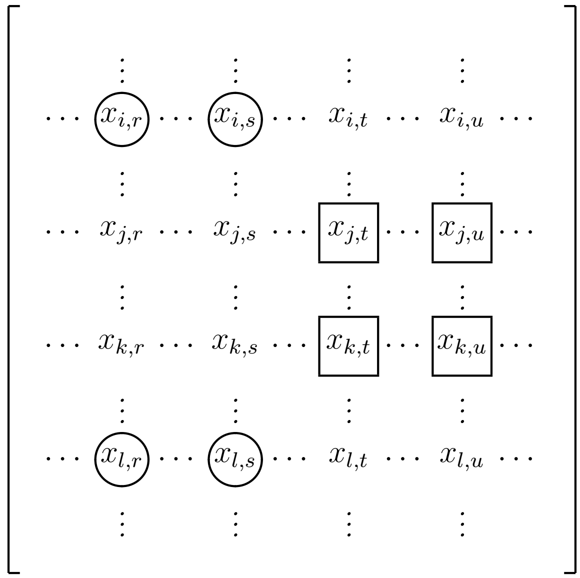

Figure C.2: Relative positions of $\xi_{r,s}^{i,l}$ *and* $\xi_{t,u}^{j,k}$

then

$$\xi_{r,s}^{i,l}\xi_{t,u}^{j,k} = \xi_{t,u}^{j,k}\xi_{r,s}^{i,l} + \left(q - \frac{1}{q}\right)\xi_{r,s}^{j,l}\xi_{t,u}^{i,k} - q\left(q - \frac{1}{q}\right)\xi_{t,u}^{i,j}\xi_{r,s}^{k,l} \tag{C.2}$$

Proof

$$\begin{aligned}
\xi_{r,s}^{i,l}\xi_{t,u}^{j,k} &= (x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})(x_{j,t}x_{k,u} - qx_{j,u}x_{k,t}) \\
&= x_{i,r}x_{l,s}x_{j,t}x_{k,u} \\
&\quad - qx_{i,r}x_{l,s}x_{j,u}x_{k,t} \\
&\quad - qx_{i,s}x_{l,r}x_{j,t}x_{k,u} \\
&\quad + q^2x_{i,s}x_{l,r}x_{j,u}x_{k,t}
\end{aligned}$$

$$= x_{i,r}x_{j,t}x_{l,s}x_{k,u}$$

$$- qx_{i,r}x_{j,u}x_{l,s}x_{k,t}$$

$$- qx_{i,s}x_{j,t}x_{l,r}x_{k,u}$$

$$+ q^2x_{i,s}x_{j,u}x_{l,r}x_{k,t}$$

$$= \left[\left(q - \frac{1}{q}\right)x_{j,r}x_{i,t} + x_{j,t}x_{i,r}\right]x_{k,u}x_{l,s}$$
$$- q\left[\left(q - \frac{1}{q}\right)x_{j,r}x_{i,u} + x_{j,u}x_{i,r}\right]x_{k,t}x_{l,s}$$
$$- q\left[\left(q - \frac{1}{q}\right)x_{j,s}x_{i,t} + x_{j,t}x_{i,s}\right]x_{k,u}x_{l,r}$$
$$+ q^2\left[\left(q - \frac{1}{q}\right)x_{j,s}x_{i,u} + x_{j,u}x_{i,s}\right]x_{k,t}x_{l,r}$$

$$= x_{j,t}x_{i,r}x_{k,u}x_{l,s}$$

$$- qx_{j,u}x_{i,r}x_{k,t}x_{l,s}$$

$$- qx_{j,t}x_{i,s}x_{k,u}x_{l,r}$$

$$+ q^2x_{j,u}x_{i,s}x_{k,t}x_{l,r}$$

$$+ \left(q - \frac{1}{q}\right)x_{j,r}x_{i,t}x_{k,u}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right)x_{j,r}x_{i,u}x_{k,t}x_{l,s}$$

$$- q\left(q-\frac{1}{q}\right)x_{j,s}x_{i,t}x_{k,u}x_{l,r}$$
$$+ q^2\left(q-\frac{1}{q}\right)x_{j,s}x_{i,u}x_{k,t}x_{l,r}$$

$$= x_{j,t}\left[\left(q-\frac{1}{q}\right)x_{i,u}x_{k,r} + x_{k,u}x_{i,r}\right]x_{l,s}$$
$$- qx_{j,u}\left[\left(q-\frac{1}{q}\right)x_{i,t}x_{k,r} + x_{k,t}x_{i,r}\right]x_{l,s}$$
$$- qx_{j,t}\left[\left(q-\frac{1}{q}\right)x_{i,u}x_{k,s} + x_{k,u}x_{i,s}\right]x_{l,r}$$
$$+ q^2x_{j,u}\left[\left(q-\frac{1}{q}\right)x_{i,t}x_{k,s} + x_{k,t}x_{i,s}\right]x_{l,r}$$

$$+ \left(q-\frac{1}{q}\right)x_{j,r}x_{l,s}x_{i,t}x_{k,u}$$
$$- q\left(q-\frac{1}{q}\right)x_{j,r}x_{l,s}x_{i,u}x_{k,t}$$
$$- q\left(q-\frac{1}{q}\right)x_{j,s}x_{l,r}x_{i,t}x_{k,u}$$
$$+ q^2\left(q-\frac{1}{q}\right)x_{j,s}x_{l,r}x_{i,u}x_{k,t}$$

$$= (x_{j,t}x_{k,u} - qx_{j,u}x_{k,t})(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})$$
$$+ \left(q-\frac{1}{q}\right)(x_{j,t}x_{i,u} - qx_{j,u}x_{i,t})(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r})$$
$$+ \left(q-\frac{1}{q}\right)(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})$$

$$= (x_{j,t}x_{k,u} - qx_{j,u}x_{k,t})(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})$$

$$- q\left(q - \frac{1}{q}\right)(x_{i,t}x_{j,u} - qx_{i,u}x_{j,t})(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r})$$
$$+ \left(q - \frac{1}{q}\right)(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})$$

$$= \xi^{j,k}_{t,u}\xi^{i,l}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{j,l}_{r,s}\xi^{i,k}_{t,u} - q\left(q - \frac{1}{q}\right)\xi^{i,j}_{t,u}\xi^{k,l}_{r,s}$$

$\square$

Proposition C.3. *Given the quantum minor determinants* $\xi^{i,l}_{t,u}$ *and* $\xi^{j,k}_{r,s}$ *where* $i < j < k < l$ *and* $r < s < t < u$

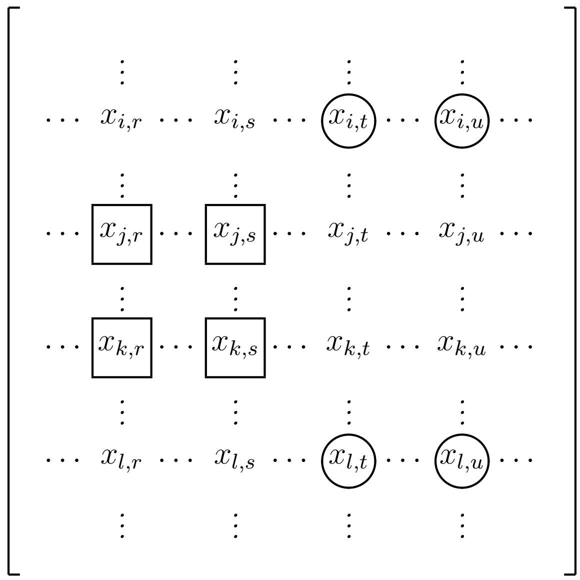

Figure C.3: Relative positions of $\xi^{i,l}_{t,u}$ *and* $\xi^{j,k}_{r,s}$

then

$$\xi^{i,l}_{t,u}\xi^{j,k}_{r,s} = \xi^{j,k}_{r,s}\xi^{i,l}_{t,u} - \left(q - \frac{1}{q}\right)\xi^{j,l}_{r,s}\xi^{i,k}_{t,u} + q\left(q - \frac{1}{q}\right)\xi^{i,j}_{t,u}\xi^{k,l}_{r,s} \tag{C.3}$$

Proof.

$$\begin{aligned}(x_{i,t}x_{l,u} - qx_{i,u}x_{l,t})(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}) &= x_{i,t}x_{l,u}x_{j,r}x_{k,s} \\ &- qx_{i,t}x_{l,u}x_{j,s}x_{k,r} \\ &- qx_{i,u}x_{l,t}x_{j,r}x_{k,s} \\ &+ q^2x_{i,u}x_{l,t}x_{j,s}x_{k,r}\end{aligned}$$

$$
\begin{aligned}
&= x_{i,t}\left[\left(\frac{1}{q}-q\right)x_{l,r}x_{j,u}+x_{j,r}x_{l,u}\right]x_{k,s}\\
&\quad -qx_{i,t}\left[\left(\frac{1}{q}-q\right)x_{l,s}x_{j,u}+x_{j,s}x_{l,u}\right]x_{k,r}\\
&\quad -qx_{i,u}\left[\left(\frac{1}{q}-q\right)x_{l,r}x_{j,t}+x_{j,r}x_{l,t}\right]x_{k,s}\\
&\quad +q^2x_{i,u}\left[\left(\frac{1}{q}-q\right)x_{l,s}x_{j,t}+x_{j,s}x_{l,t}\right]x_{k,r}
\end{aligned}
$$

$$
\begin{aligned}
&=x_{i,t}x_{j,r}x_{l,u}x_{k,s}\\
&\quad -qx_{i,t}x_{j,s}x_{l,u}x_{k,r}\\
&\quad -qx_{i,u}x_{j,r}x_{l,t}x_{k,s}\\
&\quad +q^2x_{i,u}x_{j,s}x_{l,t}x_{k,r}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\left(\frac{1}{q}-q\right)x_{i,t}x_{j,u}x_{l,r}x_{k,s}\\
&\quad -q\left(\frac{1}{q}-q\right)x_{i,t}x_{j,u}x_{l,s}x_{k,r}\\
&\quad -q\left(\frac{1}{q}-q\right)x_{i,u}x_{j,t}x_{l,r}x_{k,s}\\
&\quad +q^2\left(\frac{1}{q}-q\right)x_{i,u}x_{j,t}x_{l,s}x_{k,r}
\end{aligned}
$$

$$
\begin{aligned}
&= x_{j,r}x_{i,t}\left[\left(\frac{1}{q}-q\right)x_{l,s}x_{k,u}+x_{k,s}x_{l,u}\right]\\
&\quad -qx_{j,s}x_{i,t}\left[\left(\frac{1}{q}-q\right)x_{l,r}x_{k,u}+x_{k,r}x_{l,u}\right]\\
&\quad -qx_{j,r}x_{i,u}\left[\left(\frac{1}{q}-q\right)x_{l,s}x_{k,t}+x_{k,s}x_{l,t}\right]
\end{aligned}
$$

$$+ q^2 x_{j,s} x_{i,u} \left[\left(\frac{1}{q} - q \right) x_{l,r} x_{k,t} + x_{k,r} x_{l,t} \right]$$

$$+ \left(\frac{1}{q} - q \right) (x_{i,t} x_{j,u} - q x_{i,u} x_{j,t})(x_{l,r} x_{k,s} - q x_{l,s} x_{k,r})$$

$$= x_{j,r} x_{i,t} x_{k,s} x_{l,u}$$

$$- q x_{j,s} x_{i,t} x_{k,r} x_{l,u}$$

$$- q x_{j,r} x_{i,u} x_{k,s} x_{l,t}$$

$$+ q^2 x_{j,s} x_{i,u} x_{k,r} x_{l,t}$$

$$+ \left(\frac{1}{q} - q \right) x_{j,r} x_{i,t} x_{l,s} x_{k,u}$$

$$- q \left(\frac{1}{q} - q \right) x_{j,s} x_{i,t} x_{l,r} x_{k,u}$$

$$- q \left(\frac{1}{q} - q \right) x_{j,r} x_{i,u} x_{l,s} x_{k,t}$$

$$+ q^2 \left(\frac{1}{q} - q \right) x_{j,s} x_{i,u} x_{l,r} x_{k,t}$$

$$- \left(q - \frac{1}{q} \right) (x_{i,t} x_{j,u} - q x_{i,u} x_{j,t})(x_{l,r} x_{k,s} - q x_{l,s} x_{k,r})$$

$$= x_{j,r} x_{k,s} x_{i,t} x_{l,u}$$

$$- q x_{j,s} x_{k,r} x_{i,t} x_{l,u}$$

$$- qx_{j,r}x_{k,s}x_{i,u}x_{l,t}$$
$$+ q^2x_{j,s}x_{k,r}x_{i,u}x_{l,t}$$

$$+ \left(\frac{1}{q} - q\right) x_{j,r}x_{l,s}x_{i,t}x_{k,u}$$
$$- q\left(\frac{1}{q} - q\right) x_{j,s}x_{l,r}x_{i,t}x_{k,u}$$
$$- q\left(\frac{1}{q} - q\right) x_{j,r}x_{l,s}x_{i,u}x_{k,t}$$
$$+ q^2\left(\frac{1}{q} - q\right) x_{j,s}x_{l,r}x_{i,u}x_{k,t}$$

$$- \left(q - \frac{1}{q}\right)(x_{i,t}x_{j,u} - qx_{i,u}x_{j,t})(x_{l,r}x_{k,s} - qx_{l,s}x_{k,r})$$

$$=(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r})(x_{i,t}x_{l,u} - qx_{i,u}x_{l,t})$$
$$+ \left(\frac{1}{q} - q\right)(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})$$
$$- \left(q - \frac{1}{q}\right)(x_{i,t}x_{j,u} - qx_{i,u}x_{j,t})(x_{l,r}x_{k,s} - qx_{l,s}x_{k,r})$$

$$=(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r})(x_{i,t}x_{l,u} - qx_{i,u}x_{l,t})$$
$$- \left(q - \frac{1}{q}\right)(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})$$
$$+ q\left(q - \frac{1}{q}\right)(x_{i,t}x_{j,u} - qx_{i,u}x_{j,t})(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r})$$

$$=\xi^{j,k}_{r,s}\xi^{i,l}_{t,u} - \left(q - \frac{1}{q}\right)\xi^{j,l}_{r,s}\xi^{i,k}_{t,u} + q\left(q - \frac{1}{q}\right)\xi^{i,j}_{t,u}\xi^{k,l}_{r,s}$$

□

C.2 Quantum minor determinant relations associated with $z_{i,j}z_{i,k} = qz_{i,k}z_{i,j}$ where $i < j < k$

Proposition C.4. *Given the quantum minor determinants $\xi^{i,j}_{r,s}$ and $\xi^{i,k}_{r,s}$ where $i < j < k$ and $r < s$*

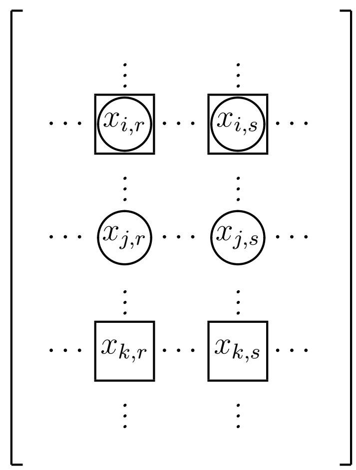

Figure C.4: Relative positions of $\xi^{i,j}_{r,s}$ and $\xi^{i,k}_{r,s}$

then

$$\xi^{i,j}_{r,s}\xi^{i,k}_{r,s} = q\xi^{i,k}_{r,s}\xi^{i,j}_{r,s} \tag{C.4}$$

Proof.

$$\begin{aligned}
\xi^{i,j}_{r,s}\xi^{i,k}_{r,s} &= \left(x_{i,r}x_{j,s} - qx_{j,r}x_{i,s}\right)\left(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s}\right) \\
&= x_{i,r}x_{j,s}x_{i,r}x_{k,s} \\
&\quad - qx_{i,r}x_{j,s}x_{i,s}x_{k,r}
\end{aligned}$$

$$- qx_{i,s}x_{j,r}x_{i,r}x_{k,s}$$

$$+ q^2x_{i,s}x_{j,r}x_{i,s}x_{k,r}$$

$$= x_{i,r}\left[\left(\frac{1}{q} - q\right)x_{i,s}x_{j,r} + x_{i,r}x_{j,s}\right]x_{k,s}$$

$$- x_{i,r}x_{i,s}x_{j,s}x_{k,r}$$

$$- x_{i,s}x_{i,r}x_{j,r}x_{k,s}$$

$$+ q^2x_{i,s}x_{i,s}x_{j,r}x_{k,r}$$

$$= x_{i,r}x_{i,r}x_{j,s}x_{k,s}$$

$$+ \left(\frac{1}{q} - q\right)x_{i,r}x_{i,s}x_{j,r}x_{k,s}$$

$$- x_{i,r}x_{i,s}x_{j,s}x_{k,r}$$

$$- x_{i,s}x_{i,r}x_{j,r}x_{k,s}$$

$$+ q^2x_{i,s}x_{i,s}x_{j,r}x_{k,r}$$

$$= qx_{i,r}x_{i,r}x_{k,s}x_{j,s}$$

$$+ \left(\frac{1}{q} - q\right)x_{i,r}x_{i,s}x_{j,r}x_{k,s}$$

$$- qx_{i,s}x_{i,r}x_{k,r}x_{j,s}$$

$$- \frac{1}{q}x_{i,r}x_{i,s}\left[\left(q - \frac{1}{q}\right)x_{k,r}x_{j,s} + x_{k,s}x_{j,r}\right]$$

$$+ q^3x_{i,s}x_{i,s}x_{k,r}x_{j,r}$$

$$= qx_{i,r}x_{i,r}x_{k,s}x_{j,s}$$
$$+ \left(\frac{1}{q} - q\right) x_{i,r}x_{i,s}x_{j,r}x_{k,s}$$
$$- qx_{i,s}x_{i,r}x_{k,r}x_{j,s}$$
$$- \frac{1}{q}\left(q - \frac{1}{q}\right) x_{i,r}x_{i,s}x_{k,r}x_{j,s}$$
$$- \frac{1}{q}x_{i,r}x_{i,s}x_{k,s}x_{j,r}$$
$$+ q^3x_{i,s}x_{i,s}x_{k,r}x_{j,r}$$

$$= qx_{i,r}\left[\left(q - \frac{1}{q}\right) x_{i,s}x_{k,r} + x_{k,s}x_{i,r}\right] x_{j,s}$$
$$+ \left(\frac{1}{q} - q\right) x_{i,r}x_{i,s}x_{j,r}x_{k,s}$$
$$- q^2x_{i,s}x_{k,r}x_{i,r}x_{j,s}$$
$$- \frac{1}{q}\left(q - \frac{1}{q}\right) x_{i,r}x_{i,s}x_{k,r}x_{j,s}$$
$$- \frac{1}{q}x_{i,r}x_{i,s}x_{k,s}x_{j,r}$$
$$+ q^3x_{i,s}x_{k,r}x_{i,s}x_{j,r}$$

$$= qx_{i,r}x_{k,s}x_{i,r}x_{j,s}$$
$$q\left(q - \frac{1}{q}\right) x_{i,r}x_{i,s}x_{k,r}x_{j,s}$$
$$+ \left(\frac{1}{q} - q\right) x_{i,r}x_{i,s}x_{j,r}x_{k,s}$$

$$- q^2 x_{i,s}x_{k,r}x_{i,r}x_{j,s}$$

$$- \frac{1}{q}\left(q - \frac{1}{q}\right) x_{i,r}x_{i,s}x_{k,r}x_{j,s}$$

$$- \frac{1}{q} x_{i,r}x_{i,s}x_{k,s}x_{j,r}$$

$$+ q^3 x_{i,s}x_{k,r}x_{i,s}x_{j,r}$$

$$= q x_{i,r}x_{k,s}x_{i,r}x_{j,s}$$

$$- q^2 x_{i,s}x_{k,r}x_{i,r}x_{j,s}$$

$$+ q^3 x_{i,s}x_{k,r}x_{i,s}x_{j,r}$$

$$\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,r}x_{i,s}x_{k,r}x_{j,s}$$

$$+ \left(\frac{1}{q} - q\right) x_{i,r}x_{i,s}x_{j,r}x_{k,s}$$

$$- x_{i,r}x_{k,s}x_{i,s}x_{j,r}$$

$$= q x_{i,r}x_{k,s}x_{i,r}x_{j,s}$$

$$- q^2 x_{i,s}x_{k,r}x_{i,r}x_{j,s}$$

$$+ q^3 x_{i,s}x_{k,r}x_{i,s}x_{j,r}$$

$$\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,r}x_{i,s}x_{k,r}x_{j,s}$$

$$+ \left(\frac{1}{q} - q\right) x_{i,r}x_{i,s}\left[\left(q - \frac{1}{q}\right) x_{k,r}x_{j,s} + x_{k,s}x_{j,r}\right]$$

$$- x_{i,r}x_{k,s}x_{i,s}x_{j,r}$$

$$
\begin{aligned}
&= q x_{i,r} x_{k,s} x_{i,r} x_{j,s} \\
&\quad - q^2 x_{i,s} x_{k,r} x_{i,r} x_{j,s} \\
&\quad + q^3 x_{i,s} x_{k,r} x_{i,s} x_{j,r} \\
&\quad \left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,r} x_{i,s} x_{k,r} x_{j,s} \\
&\quad + \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{i,r} x_{i,s} x_{k,r} x_{j,s} \\
&\quad + \left(\frac{1}{q} - q\right) x_{i,r} x_{i,s} x_{k,s} x_{j,r} \\
&\quad - x_{i,r} x_{k,s} x_{i,s} x_{j,r}
\end{aligned}
$$

$$
\begin{aligned}
&= q x_{i,r} x_{k,s} x_{i,r} x_{j,s} \\
&\quad - q^2 x_{i,s} x_{k,r} x_{i,r} x_{j,s} \\
&\quad + q^3 x_{i,s} x_{k,r} x_{i,s} x_{j,r} \\
&\quad + q \left(\frac{1}{q} - q\right) x_{i,r} x_{k,s} x_{i,s} x_{j,r} \\
&\quad - x_{i,r} x_{k,s} x_{i,s} x_{j,r}
\end{aligned}
$$

$$
\begin{aligned}
&= q x_{i,r} x_{k,s} x_{i,r} x_{j,s} \\
&\quad - q^2 x_{i,r} x_{k,s} x_{i,s} x_{j,r} \\
&\quad - q^2 x_{i,s} x_{k,r} x_{i,r} x_{j,s} \\
&\quad + q^3 x_{i,s} x_{k,r} x_{i,s} x_{j,r}
\end{aligned}
$$

$$= q\left(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s}\right)\left(x_{i,r}x_{j,s} - qx_{j,r}x_{i,s}\right)$$

$$= q\xi^{i,k}_{r,s}\xi^{i,j}_{r,s}$$

□

Proposition C.5. *Given the quantum minor determinants* $\xi^{i,j}_{r,s}$ *and* $\xi^{i,k}_{t,u}$ *where* $i < j < k$ *and* $r < s < t < u$

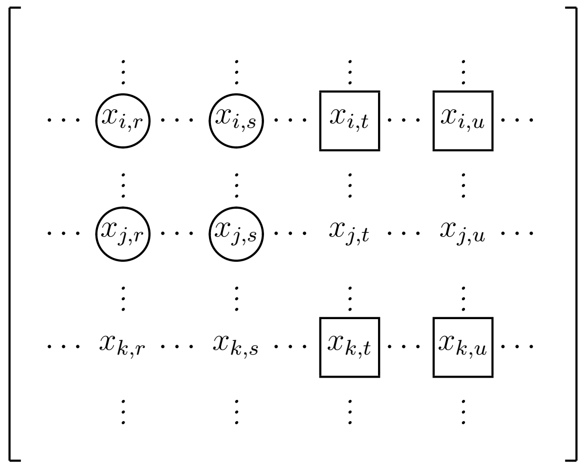

Figure C.5: Relative positions of $\xi^{i,j}_{r,s}$ *and* $\xi^{i,k}_{t,u}$

then

$$\xi^{i,j}_{r,s}\xi^{i,k}_{t,u} = q\xi^{i,k}_{t,u}\xi^{i,j}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{i,j}_{t,u} \tag{C.5}$$

Proof.

$$\begin{aligned}
\xi^{i,j}_{r,s}\xi^{i,k}_{t,u} &= \left(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}\right)\left(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t}\right) \\
&= x_{i,r}x_{j,s}x_{i,t}x_{k,u} \\
&\quad - qx_{i,r}x_{j,s}x_{i,u}x_{k,t} \\
&\quad - qx_{i,s}x_{j,r}x_{i,t}x_{k,u} \\
&\quad + q^2x_{i,s}x_{j,r}x_{i,u}x_{k,t}
\end{aligned}$$

$$= x_{i,r}x_{i,t}x_{j,s}x_{k,u}$$
$$- qx_{i,r}x_{i,u}x_{j,s}x_{k,t}$$
$$- qx_{i,s}x_{i,t}x_{j,r}x_{k,u}$$
$$+ q^2x_{i,s}x_{i,u}x_{j,r}x_{k,t}$$

$$= qx_{i,t}x_{i,r}\left[\left(q - \frac{1}{q}\right)x_{k,s}x_{j,u} + x_{k,u}x_{j,s}\right]$$
$$- q^2x_{i,u}x_{i,r}\left[\left(q - \frac{1}{q}\right)x_{k,s}x_{j,t} + x_{k,t}x_{j,s}\right]$$
$$- q^2x_{i,t}x_{i,s}\left[\left(q - \frac{1}{q}\right)x_{k,r}x_{j,u} + x_{k,u}x_{j,r}\right]$$
$$+ q^3x_{i,u}x_{i,s}\left[\left(q - \frac{1}{q}\right)x_{k,r}x_{j,t} + x_{k,t}x_{j,r}\right]$$

$$= qx_{i,t}x_{i,r}x_{k,u}x_{j,s}$$
$$- q^2x_{i,u}x_{i,r}x_{k,t}x_{j,s}$$
$$- q^2x_{i,t}x_{i,s}x_{k,u}x_{j,r}$$
$$+ q^3x_{i,u}x_{i,s}x_{k,t}x_{j,r}$$

$$+ q\left(q - \frac{1}{q}\right)x_{i,t}x_{i,r}x_{k,s}x_{j,u}$$
$$- q^2\left(q - \frac{1}{q}\right)x_{i,u}x_{i,r}x_{k,s}x_{j,t}$$
$$- q^2\left(q - \frac{1}{q}\right)x_{i,t}x_{i,s}x_{k,r}x_{j,u}$$

$$+ q^3 \left(q - \frac{1}{q}\right) x_{i,u} x_{i,s} x_{k,r} x_{j,t}$$

$$= q x_{i,t} \left[\left(q - \frac{1}{q}\right) x_{i,u} x_{k,r} + x_{k,u} x_{i,r}\right] x_{j,s}$$

$$- q^2 x_{i,u} \left[\left(q - \frac{1}{q}\right) x_{i,t} x_{k,r} + x_{k,t} x_{i,r}\right] x_{j,s}$$

$$- q^2 x_{i,t} \left[\left(q - \frac{1}{q}\right) x_{i,u} x_{k,s} + x_{k,u} x_{i,s}\right] x_{j,r}$$

$$+ q^3 x_{i,u} \left[\left(q - \frac{1}{q}\right) x_{i,t} x_{k,s} + x_{k,t} x_{i,s}\right] x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r} x_{i,t} x_{k,s} x_{j,u}$$

$$- q \left(q - \frac{1}{q}\right) x_{i,r} x_{i,u} x_{k,s} x_{j,t}$$

$$- q \left(q - \frac{1}{q}\right) x_{i,s} x_{i,t} x_{k,r} x_{j,u}$$

$$+ q^2 \left(q - \frac{1}{q}\right) x_{i,s} x_{i,u} x_{k,r} x_{j,t}$$

$$= q x_{i,t} x_{k,u} x_{i,r} x_{j,s}$$

$$- q^2 x_{i,u} x_{k,t} x_{i,r} x_{j,s}$$

$$- q^2 x_{i,t} x_{k,u} x_{i,s} x_{j,r}$$

$$+ q^3 x_{i,u} x_{k,t} x_{i,s} x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,t} x_{i,u} x_{k,r} x_{j,s}$$

$$- q^2\left(q - \frac{1}{q}\right) x_{i,u}x_{i,t}x_{k,r}x_{j,s}$$

$$- q^2\left(q - \frac{1}{q}\right) x_{i,t}x_{i,u}x_{k,s}x_{j,r}$$

$$+ q^3\left(q - \frac{1}{q}\right) x_{i,u}x_{i,t}x_{k,s}x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r}x_{k,s}x_{i,t}x_{j,u}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,r}x_{k,s}x_{i,u}x_{j,t}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,s}x_{k,r}x_{i,t}x_{j,u}$$

$$+ q^2\left(q - \frac{1}{q}\right) x_{i,s}x_{k,r}x_{i,u}x_{j,t}$$

$$= qx_{i,t}x_{k,u}x_{i,r}x_{j,s}$$

$$- q^2x_{i,t}x_{k,u}x_{i,s}x_{j,r}$$

$$- q^2x_{i,u}x_{k,t}x_{i,r}x_{j,s}$$

$$+ q^3x_{i,u}x_{k,t}x_{i,s}x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r}x_{k,s}x_{i,t}x_{j,u}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,r}x_{k,s}x_{i,u}x_{j,t}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,s}x_{k,r}x_{i,t}x_{j,u}$$

$$+ q^2\left(q - \frac{1}{q}\right) x_{i,s}x_{k,r}x_{i,u}x_{j,t}$$

$$
\begin{aligned}
&= q\left(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t}\right)\left(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}\right) \\
&\quad + \left(q - \frac{1}{q}\right)\left(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}\right)\left(x_{i,t}x_{j,u} - qx_{i,u}x_{j,t}\right) \\
&= q\xi^{i,k}_{t,u}\xi^{i,j}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{i,j}_{t,u}
\end{aligned}
$$

□

Proposition C.6. *Given the quantum minor determinants $\xi^{i,j}_{t,u}$ and $\xi^{i,k}_{r,s}$ where $i < j < k$ and $r < s < t < u$ then*

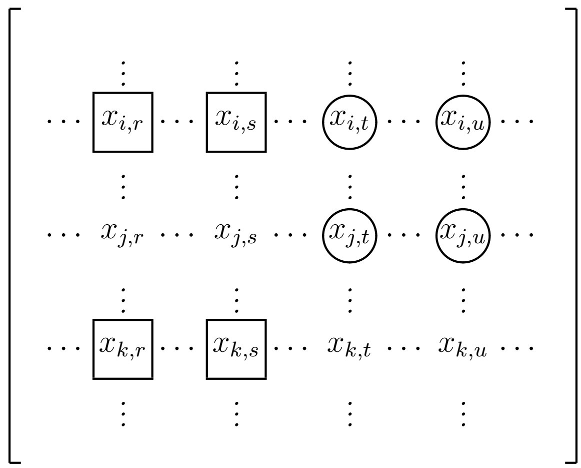

Figure C.6: Relative positions of $\xi^{i,j}_{t,u}$ and $\xi^{i,k}_{r,s}$

$$\xi^{i,j}_{t,u}\xi^{i,k}_{r,s} = \frac{1}{q}\xi^{i,k}_{r,s}\xi^{i,j}_{t,u} \tag{C.6}$$

Proof.

$$\begin{aligned}
\xi^{i,j}_{t,u}\xi^{i,k}_{r,s} &= (x_{i,t}x_{j,u} - qx_{j,t}x_{i,u})(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s}) \\
&= x_{i,t}x_{j,u}x_{i,r}x_{k,s} \\
&\quad - qx_{i,t}x_{j,u}x_{i,s}x_{k,r} \\
&\quad - qx_{i,u}x_{j,t}x_{i,r}x_{k,s} \\
&\quad + q^2x_{i,u}x_{j,t}x_{i,s}x_{k,r}
\end{aligned}$$

$$\begin{aligned}
&= x_{i,t}\left[\left(\frac{1}{q}-q\right)x_{i,u}x_{j,r}+x_{i,r}x_{j,u}\right]x_{k,s}\\
&\quad - qx_{i,t}\left[\left(\frac{1}{q}-q\right)x_{i,u}x_{j,s}+x_{i,s}x_{j,u}\right]x_{k,r}\\
&\quad - qx_{i,u}\left[\left(\frac{1}{q}-q\right)x_{i,t}x_{j,r}+x_{i,r}x_{j,t}\right]x_{k,s}\\
&\quad + q^2x_{i,u}\left[\left(\frac{1}{q}-q\right)x_{i,t}x_{j,s}+x_{i,s}x_{j,t}\right]x_{k,r}
\end{aligned}$$

$$\begin{aligned}
&= x_{i,t}x_{i,r}x_{j,u}x_{k,s}\\
&\quad - qx_{i,t}x_{i,s}x_{j,u}x_{k,r}\\
&\quad - qx_{i,u}x_{i,r}x_{j,t}x_{k,s}\\
&\quad + q^2x_{i,u}x_{i,s}x_{j,t}x_{k,r}
\end{aligned}$$

$$\begin{aligned}
&\quad + \left(\frac{1}{q}-q\right)x_{i,t}x_{i,u}x_{j,r}x_{k,s}\\
&\quad - q\left(\frac{1}{q}-q\right)x_{i,t}x_{i,u}x_{j,s}x_{k,r}\\
&\quad - q\left(\frac{1}{q}-q\right)x_{i,u}x_{i,t}x_{j,r}x_{k,s}\\
&\quad + q^2\left(\frac{1}{q}-q\right)x_{i,u}x_{i,t}x_{j,s}
\end{aligned}$$

$$\begin{aligned}
&= x_{i,t}x_{i,r}x_{j,u}x_{k,s}\\
&\quad - qx_{i,t}x_{i,s}x_{j,u}x_{k,r}\\
&\quad - qx_{i,u}x_{i,r}x_{j,t}x_{k,s}
\end{aligned}$$

$$+ q^2 x_{i,u} x_{i,s} x_{j,t} x_{k,r}$$

$$= \frac{1}{q} x_{i,r} x_{i,t} x_{j,u} x_{k,s}$$
$$- x_{i,s} x_{i,t} x_{j,u} x_{k,r}$$
$$- x_{i,r} x_{i,u} x_{j,t} x_{k,s}$$
$$+ q x_{i,s} x_{i,u} x_{j,t} x_{k,r}$$

$$= \frac{1}{q} x_{i,r} x_{k,s} x_{i,t} x_{j,u}$$
$$- x_{i,s} x_{k,r} x_{i,t} x_{j,u}$$
$$- x_{i,r} x_{k,s} x_{i,u} x_{j,t}$$
$$+ q x_{i,s} x_{k,r} x_{i,u} x_{j,t}$$

$$= \frac{1}{q} \left(x_{i,r} x_{k,s} - q x_{i,s} x_{k,r} \right) \left(x_{i,t} x_{j,u} - q x_{i,u} x_{j,t} \right)$$
$$= \frac{1}{q} \xi^{i,k}_{r,s} \xi^{i,j}_{t,u}$$

□

C.3 Quantum determinant relations associated with $z_{i,j}z_{j,k} = qz_{j,k}z_{i,j}$ where $i < j < k$

Proposition C.7. *Given the quantum minor determinants $\xi^{i,j}_{r,s}$ and $\xi^{j,k}_{r,s}$ where $i < j < k$ and $r < s$*

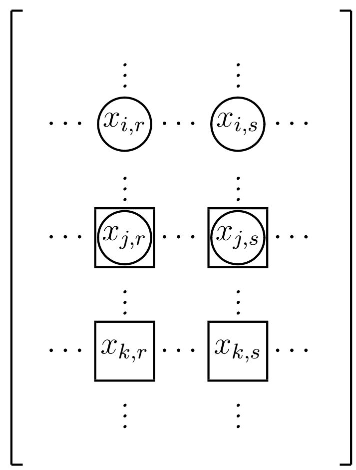

Figure C.7: Relative positions of $\xi^{i,j}_{r,s}$ and $\xi^{j,k}_{r,s}$

then

$$\xi^{i,j}_{r,s}\xi^{j,k}_{r,s} = q\xi^{j,k}_{r,s}\xi^{i,j}_{r,s} \tag{C.7}$$

Proof.

$$\begin{aligned}\xi^{i,j}_{r,s}\xi^{j,k}_{r,s} = &\ (x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}) \\ &= x_{i,r}x_{j,s}x_{j,r}x_{k,s} \\ &\quad - qx_{i,r}x_{j,s}x_{j,s}x_{k,r}\end{aligned}$$

$$- q x_{i,s} x_{j,r} x_{j,r} x_{k,s}$$

$$+ q^2 x_{i,s} x_{j,r} x_{j,s} x_{k,r}$$

$$= \frac{1}{q} x_{i,r} x_{j,r} x_{j,s} x_{k,s}$$

$$- q x_{i,r} x_{j,s} x_{j,s} x_{k,r}$$

$$- q x_{i,s} x_{j,r} x_{j,r} x_{k,s}$$

$$+ q^3 x_{i,s} x_{j,s} x_{j,r} x_{k,r}$$

$$= q x_{j,r} x_{i,r} x_{k,s} x_{j,s}$$

$$- q \left[\left(q - \frac{1}{q} \right) x_{j,r} x_{i,s} + x_{j,s} x_{i,r} \right] x_{k,r} x_{j,s}$$

$$- q x_{j,r} x_{i,s} \left[\left(q - \frac{1}{q} \right) x_{k,r} x_{j,s} + x_{k,s} x_{j,r} \right]$$

$$+ q^5 x_{j,s} x_{i,s} x_{k,r} x_{j,r}$$

$$= q x_{j,r} x_{i,r} x_{k,s} x_{j,s}$$

$$- q \left(q - \frac{1}{q} \right) x_{j,r} x_{i,s} x_{k,r} x_{j,s}$$

$$- q x_{j,s} x_{i,r} x_{k,r} x_{j,s}$$

$$- q \left(q - \frac{1}{q} \right) x_{j,r} x_{i,s} x_{k,r} x_{j,s}$$

$$- q x_{j,r} x_{i,s} x_{k,s} x_{j,r}$$

$$+ q^5 x_{j,s} x_{i,s} x_{k,r} x_{j,r}$$

$$= q x_{j,r} \left[\left(q - \frac{1}{q} \right) x_{k,r} x_{i,s} + x_{k,s} x_{i,r} \right] x_{j,s}$$

$$- q \left(q - \frac{1}{q} \right) x_{j,r} x_{k,r} x_{i,s} x_{j,s}$$

$$- q^2 x_{j,s} x_{k,r} x_{i,r} x_{j,s}$$

$$- q \left(q - \frac{1}{q} \right) x_{j,r} x_{k,r} x_{i,s} x_{j,s}$$

$$- q^2 x_{j,r} x_{k,s} x_{i,s} x_{j,r}$$

$$+ q^5 x_{j,s} x_{k,r} x_{i,s} x_{j,r}$$

$$= q x_{j,r} x_{k,s} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,s} x_{k,r} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,r} x_{k,s} x_{i,s} x_{j,r}$$

$$+ q^5 x_{j,s} x_{k,r} x_{i,s} x_{j,r}$$

$$- q \left(q - \frac{1}{q} \right) x_{j,r} x_{k,r} x_{i,s} x_{j,s}$$

$$= q x_{j,r} x_{k,s} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,s} x_{k,r} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,r} x_{k,s} x_{i,s} x_{j,r}$$

$$+ q^5 x_{j,s} x_{k,r} x_{i,s} x_{j,r}$$

$$- q^3 \left(q - \frac{1}{q} \right) x_{k,r} x_{j,r} x_{j,s} x_{i,s}$$

$$= q x_{j,r} x_{k,s} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,s} x_{k,r} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,r} x_{k,s} x_{i,s} x_{j,r}$$

$$+ q^5 x_{j,s} x_{k,r} x_{i,s} x_{j,r}$$

$$- q^4 \left(q - \frac{1}{q} \right) x_{j,s} x_{k,r} x_{i,s} x_{j,r}$$

$$= q x_{j,r} x_{k,s} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,s} x_{k,r} x_{i,r} x_{j,s}$$

$$- q^2 x_{j,r} x_{k,s} x_{i,s} x_{j,r}$$

$$+ q^3 x_{j,s} x_{k,r} x_{i,s} x_{j,r}$$

$$= q \left(x_{j,r} x_{k,s} - q x_{j,s} x_{k,r} \right) \left(x_{i,r} x_{j,s} - q x_{i,s} x_{j,r} \right)$$

$$= q \xi^{j,k}_{r,s} \xi^{i,j}_{r,s}$$

□

Proposition C.8. *Given the quantum minor determinants $\xi^{i,j}_{r,s}$ and $\xi^{j,k}_{t,u}$ where $i < j < k$ and $r < s < t < u$ then*

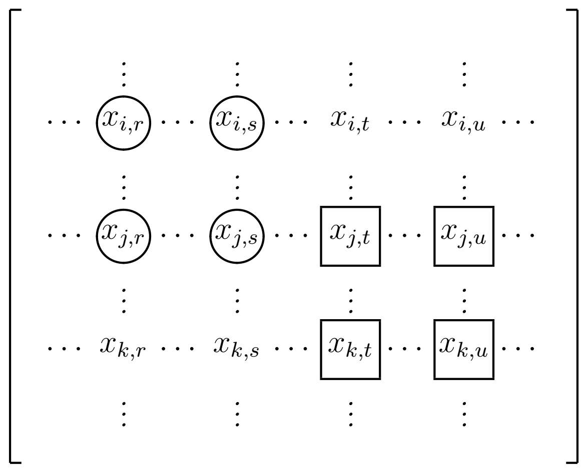

Figure C.8: Relative positions of $\xi^{i,j}_{r,s}$ and $\xi^{j,k}_{t,u}$

$$\xi^{i,j}_{r,s}\xi^{j,k}_{t,u} = q\xi^{j,k}_{t,u}\xi^{i,j}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{j,k}_{r,s}\xi^{i,j}_{t,y} \tag{C.8}$$

Proof.

$$\begin{aligned}
\xi^{i,j}_{r,s}\xi^{j,k}_{t,u} &= (x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})\,(x_{j,t}x_{k,u} - qx_{j,u}x_{k,t}) \\
&= x_{i,r}x_{j,s}x_{j,t}x_{k,u} \\
&\quad - qx_{i,r}x_{j,s}x_{j,u}x_{k,t} \\
&\quad - qx_{i,s}x_{j,r}x_{j,t}x_{k,u} \\
&\quad + q^2x_{i,s}x_{j,r}x_{j,u}x_{k,t}
\end{aligned}$$

$$= qx_{i,r}x_{j,t}x_{j,s}x_{k,u}$$
$$- q^2x_{i,r}x_{j,u}x_{j,s}x_{k,t}$$
$$- q^2x_{i,s}x_{j,t}x_{j,r}x_{k,u}$$
$$+ q^3x_{i,s}x_{j,u}x_{j,r}x_{k,t}$$

$$= q\left[\left(q-\frac{1}{q}\right)x_{j,r}x_{i,t}+x_{j,t}x_{i,r}\right]\left[\left(q-\frac{1}{q}\right)x_{k,s}x_{j,u}+x_{k,u}x_{j,s}\right]$$
$$- q^2\left[\left(q-\frac{1}{q}\right)x_{j,r}x_{i,u}+x_{j,u}x_{i,r}\right]\left[\left(q-\frac{1}{q}\right)x_{k,s}x_{j,t}+x_{k,t}x_{j,s}\right]$$
$$- q^2\left[\left(q-\frac{1}{q}\right)x_{j,s}x_{i,t}+x_{j,t}x_{i,s}\right]\left[\left(q-\frac{1}{q}\right)x_{k,r}x_{j,u}+x_{k,u}x_{j,r}\right]$$
$$+ q^3\left[\left(q-\frac{1}{q}\right)x_{j,s}x_{i,u}+x_{j,u}x_{i,s}\right]\left[\left(q-\frac{1}{q}\right)x_{k,r}x_{j,t}+x_{k,t}x_{j,r}\right]$$

$$= q\left(q-\frac{1}{q}\right)^2 x_{j,r}x_{i,t}x_{k,s}x_{j,u}$$
$$- q^2\left(q-\frac{1}{q}\right)^2 x_{j,r}x_{i,u}x_{k,s}x_{j,t}$$
$$- q^2\left(q-\frac{1}{q}\right)^2 x_{j,s}x_{i,t}x_{k,r}x_{j,u}$$
$$+ q^3\left(q-\frac{1}{q}\right)^2 x_{j,s}x_{i,u}x_{k,r}x_{j,t}$$

$$+ q\left(q-\frac{1}{q}\right)x_{j,r}x_{i,t}x_{k,u}x_{j,s}$$
$$- q^2\left(q-\frac{1}{q}\right)x_{j,r}x_{i,u}x_{k,t}x_{j,s}$$
$$- q^2\left(q-\frac{1}{q}\right)x_{j,s}x_{i,t}x_{k,u}x_{j,r}$$

$$+ q^3 \left(q - \frac{1}{q}\right) x_{j,s}x_{i,u}x_{k,t}x_{j,r}$$

$$+ q \left(q - \frac{1}{q}\right) x_{j,t}x_{i,r}x_{k,s}x_{j,u}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{j,u}x_{i,r}x_{k,s}x_{j,t}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{j,t}x_{i,s}x_{k,r}x_{j,u}$$
$$+ q^3 \left(q - \frac{1}{q}\right) x_{j,u}x_{i,s}x_{k,r}x_{j,t}$$

$$+ qx_{j,t}x_{i,r}x_{k,u}x_{j,s}$$
$$- q^2x_{j,u}x_{i,r}x_{k,t}x_{j,s}$$
$$- q^2x_{j,t}x_{i,s}x_{k,u}x_{j,r}$$
$$+ q^3x_{j,u}x_{i,s}x_{k,t}x_{j,r}$$

$$= q \left(q - \frac{1}{q}\right)^2 x_{j,r}x_{k,s}x_{i,t}x_{j,u}$$
$$- q^2 \left(q - \frac{1}{q}\right)^2 x_{j,r}x_{k,s}x_{i,u}x_{j,t}$$
$$- q^2 \left(q - \frac{1}{q}\right)^2 x_{j,s}x_{k,r}x_{i,t}x_{j,u}$$
$$+ q^3 \left(q - \frac{1}{q}\right)^2 x_{j,s}x_{k,r}x_{i,u}x_{j,t}$$

$$+ q \left(q - \frac{1}{q}\right) x_{i,t}x_{j,r}x_{k,u}x_{j,s}$$

$$- q^2 \left(q - \frac{1}{q} \right) x_{i,u} x_{j,r} x_{k,t} x_{j,s}$$
$$- q^2 \left(q - \frac{1}{q} \right) x_{i,t} x_{j,s} x_{k,u} x_{j,r}$$
$$+ q^3 \left(q - \frac{1}{q} \right) x_{i,u} x_{j,s} x_{k,t} x_{j,r}$$

$$+ q \left(q - \frac{1}{q} \right) x_{j,t} x_{i,r} x_{j,u} x_{k,s}$$
$$- q^2 \left(q - \frac{1}{q} \right) x_{j,u} x_{i,r} x_{j,t} x_{k,s}$$
$$- q^2 \left(q - \frac{1}{q} \right) x_{j,t} x_{i,s} x_{j,u} x_{k,r}$$
$$+ q^3 \left(q - \frac{1}{q} \right) x_{j,u} x_{i,s} x_{j,t} x_{k,r}$$

$$+ q x_{j,t} \left[\left(q - \frac{1}{q} \right) x_{i,u} x_{k,r} + x_{k,u} x_{i,r} \right] x_{j,s}$$
$$- q^2 x_{j,u} \left[\left(q - \frac{1}{q} \right) x_{i,t} x_{k,r} + x_{k,t} x_{i,r} \right] x_{j,s}$$
$$- q^2 x_{j,t} \left[\left(q - \frac{1}{q} \right) x_{i,u} x_{k,s} + x_{k,u} x_{i,s} \right] x_{j,r}$$
$$+ q^3 x_{j,u} \left[\left(q - \frac{1}{q} \right) x_{i,u} x_{k,s} + x_{k,t} x_{i,s} \right] x_{j,r}$$

$$= q \left(q - \frac{1}{q} \right)^2 (x_{j,r} x_{k,s} - q x_{j,s} x_{k,r}) (x_{i,t} x_{j,u} - q x_{i,u} x_{j,t})$$

$$+ q \left(q - \frac{1}{q} \right) x_{i,t} x_{j,r} x_{k,u} x_{j,s}$$

$$- q^2 \left(q - \frac{1}{q}\right) x_{i,u} x_{j,r} x_{k,t} x_{j,s}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{i,t} x_{j,s} x_{k,u} x_{j,r}$$
$$+ q^3 \left(q - \frac{1}{q}\right) x_{i,u} x_{j,s} x_{k,t} x_{j,r}$$

$$+ q \left(q - \frac{1}{q}\right) x_{j,t} x_{i,r} x_{j,u} x_{k,s}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{j,u} x_{i,r} x_{j,t} x_{k,s}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{j,t} x_{i,s} x_{j,u} x_{k,r}$$
$$+ q^3 \left(q - \frac{1}{q}\right) x_{j,u} x_{i,s} x_{j,t} x_{k,r}$$

$$+ q \left(q - \frac{1}{q}\right) \left(x_{j,t} x_{i,u} - q x_{j,u} x_{i,t}\right) \left(x_{k,r} x_{j,s} - q x_{k,s} x_{j,r}\right)$$
$$+ q \left(x_{j,t} x_{k,u} - q x_{j,u} x_{k,t}\right) \left(x_{i,r} x_{j,s} - q x_{i,s} x_{j,r}\right)$$
$$= q \left(q - \frac{1}{q}\right)^2 \left(x_{j,r} x_{k,s} - q x_{j,s} x_{k,r}\right) \left(x_{i,t} x_{j,u} - q x_{i,u} x_{j,t}\right)$$

$$+ q \left(q - \frac{1}{q}\right) x_{i,t} \left[\left(q - \frac{1}{q}\right) x_{j,u} x_{k,r} + x_{k,u} x_{j,r}\right] x_{j,s}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{i,u} \left[\left(q - \frac{1}{q}\right) x_{j,t} x_{k,r} + x_{k,t} x_{j,r}\right] x_{j,s}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{i,t} \left[\left(q - \frac{1}{q}\right) x_{j,u} x_{k,s} + x_{k,u} x_{j,s}\right] x_{j,r}$$
$$+ q^3 \left(q - \frac{1}{q}\right) x_{i,u} \left[\left(q - \frac{1}{q}\right) x_{j,t} x_{k,s} + x_{k,t} x_{j,s}\right] x_{j,r}$$

$$+ q\left(q-\frac{1}{q}\right)x_{j,t}\left[\left(q-\frac{1}{q}\right)x_{i,u}x_{j,r}+x_{j,u}x_{i,r}\right]x_{k,s}$$
$$- q^2\left(q-\frac{1}{q}\right)x_{j,u}\left[\left(q-\frac{1}{q}\right)x_{i,t}x_{j,r}+x_{j,t}x_{i,r}\right]x_{k,s}$$
$$- q^2\left(q-\frac{1}{q}\right)x_{j,t}\left[\left(q-\frac{1}{q}\right)x_{i,u}x_{j,s}+x_{j,u}x_{i,s}\right]x_{k,r}$$
$$+ q^3\left(q-\frac{1}{q}\right)x_{j,u}\left[\left(q-\frac{1}{q}\right)x_{i,t}x_{j,s}+x_{j,t}x_{i,s}\right]x_{k,r}$$

$$+ q\left(q-\frac{1}{q}\right)\left(x_{j,t}x_{i,u}-qx_{j,u}x_{i,t}\right)\left(x_{k,r}x_{j,s}-qx_{k,s}x_{j,r}\right)$$
$$+ q\left(x_{j,t}x_{k,u}-qx_{j,u}x_{k,t}\right)\left(x_{i,r}x_{j,s}-qx_{i,s}x_{j,r}\right)$$
$$=q\left(q-\frac{1}{q}\right)^2\left(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r}\right)\left(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t}\right)$$

$$+ q\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{i,t}x_{j,u}x_{k,r}x_{j,s}$$
$$- q^2\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{i,u}x_{j,t}x_{k,r}x_{j,s}$$
$$- q^2\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{i,t}x_{j,u}x_{k,s}x_{j,r}$$
$$+ q^3\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{i,u}x_{j,t}x_{k,s}x_{j,r}$$

$$+ q\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,t}x_{i,u}x_{j,r}x_{k,s}$$
$$- q^2\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,u}x_{i,t}x_{j,r}x_{k,s}$$
$$- q^2\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,t}x_{i,u}x_{j,s}x_{k,r}$$

$$+ q^3\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,u}x_{i,t}x_{j,s}x_{k,r}$$

$$+ q\left(q-\frac{1}{q}\right)(x_{j,t}x_{i,u}-qx_{j,u}x_{i,t})(x_{k,r}x_{j,s}-qx_{k,s}x_{j,r})$$

$$+ q(x_{j,t}x_{k,u}-qx_{j,u}x_{k,t})(x_{i,r}x_{j,s}-qx_{i,s}x_{j,r})$$

$$= q\left(q-\frac{1}{q}\right)^2(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})$$

$$+ q\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})(x_{k,r}x_{j,s}-qx_{k,s}x_{j,r})$$

$$+ q\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)(x_{j,t}x_{i,u}-qx_{j,u}x_{i,t})(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})$$

$$+ q\left(q-\frac{1}{q}\right)(x_{j,t}x_{i,u}-qx_{j,u}x_{i,t})(x_{k,r}x_{j,s}-qx_{k,s}x_{j,r})$$

$$+ q(x_{j,t}x_{k,u}-qx_{j,u}x_{k,t})(x_{i,r}x_{j,s}-qx_{i,s}x_{j,r})$$

$$= q\left(q-\frac{1}{q}\right)^2(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})$$

$$- q^2\left(q-\frac{1}{q}\right)^2(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})$$

$$- q^2\left(q-\frac{1}{q}\right)^2(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})$$

$$+ q^3\left(q-\frac{1}{q}\right)(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})$$

$$+ q(x_{j,t}x_{k,u}-qx_{j,u}x_{k,t})(x_{i,r}x_{j,s}-qx_{i,s}x_{j,r})$$

$$= q\left(q-\frac{1}{q}\right)^2(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})$$

$$- q\left(q-\frac{1}{q}\right)^2(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})$$

$$- q\left(q-\frac{1}{q}\right)^2(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t})$$

$$+ q^2 \left(q - \frac{1}{q}\right) (x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}) (x_{i,t}x_{j,u} - qx_{i,u}x_{j,t})$$

$$+ q (x_{j,t}x_{k,u} - qx_{j,u}x_{k,t}) (x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})$$

$$= q (x_{j,t}x_{k,u} - qx_{j,u}x_{k,t}) (x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})$$

$$+ \left(q - \frac{1}{q}\right) (x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}) (x_{i,t}x_{j,u} - qx_{i,u}x_{j,t})$$

$$= q\xi^{j,k}_{t,u}\xi^{i,j}_{r,s} + \left(q - \frac{1}{q}\right) \xi^{j,k}_{r,s}\xi^{i,j}_{t,u}$$

□

Proposition C.9. *Given the quantum minor determinants* $\xi^{i,j}_{t,u}$ *and* $\xi^{j,k}_{r,s}$ *where* $i < j < t$ *and* $r < s < t < u$

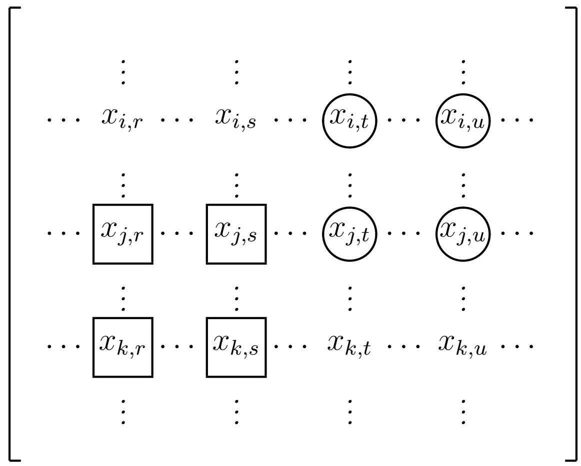

Figure C.9: Relative positions of $\xi^{i,j}_{t,u}$ *and* $\xi^{i,k}_{r,s}$

then

$$\xi^{i,j}_{t,u}\xi^{j,k}_{r,s} = \frac{1}{q}\xi^{j,k}_{r,s}\xi^{i,j}_{t,u} \tag{C.9}$$

Proof.

$$\begin{aligned}
\xi^{i,j}_{t,u}\xi^{j,k}_{r,s} &= \left(x_{i,t}x_{j,u} - qx_{i,u}x_{j,t}\right)\left(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}\right) \\
&= x_{i,t}x_{j,u}x_{j,r}x_{k,s} \\
&\quad - qx_{i,t}x_{j,u}x_{j,s}x_{k,r} \\
&\quad - qx_{i,u}x_{j,t}x_{j,r}x_{k,s} \\
&\quad + q^2x_{i,u}x_{j,t}x_{j,s}x_{k,r}
\end{aligned}$$

$$= \frac{1}{q} x_{i,t} x_{j,r} x_{j,u} x_{k,s}$$

$$- x_{i,t} x_{j,s} x_{j,u} x_{k,r}$$

$$- x_{i,u} x_{j,r} x_{j,t} x_{k,s}$$

$$+ q x_{i,u} x_{j,s} x_{j,t} x_{k,r}$$

$$= \frac{1}{q} x_{j,r} x_{i,t} x_{k,s} x_{j,u}$$

$$- x_{j,s} x_{i,t} x_{k,r} x_{j,u}$$

$$- x_{j,r} x_{i,u} x_{k,s} x_{j,t}$$

$$+ q x_{j,s} x_{i,u} x_{k,r} x_{j,t}$$

$$= \frac{1}{q} x_{j,r} x_{k,s} x_{i,t} x_{j,u}$$

$$- x_{j,s} x_{k,r} x_{i,t} x_{j,u}$$

$$- x_{j,r} x_{k,s} x_{i,u} x_{j,t}$$

$$+ q x_{j,s} x_{k,r} x_{i,u} x_{j,t}$$

$$= \frac{1}{q} \left(x_{j,r} x_{k,s} - q x_{j,s} x_{k,r} \right) \left(x_{i,t} x_{j,u} - q x_{i,u} x_{j,t} \right)$$

$$= \frac{1}{q} \xi^{j,k}_{r,s} \xi^{i,j}_{t,u}$$

□

C.4 Quantum minor determinant relations associated with $z_{i,k}z_{j,k} = qz_{j,k}z_{i,k}$ where $i < j < k$

Proposition C.10. *Given the quantum minor determinants $\xi_{r,s}^{i,k}$ and $\xi_{r,s}^{j,k}$ where $i < j < k$ and $r < s$*

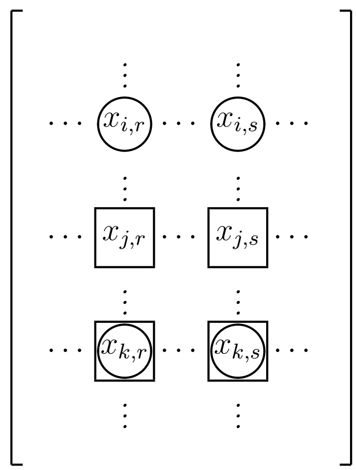

Figure C.10: Relative positions of $\xi_{r,s}^{i,k}$ and $\xi_{r,s}^{j,k}$

then

$$\xi_{r,s}^{i,k}\xi_{r,s}^{j,k} = q\xi_{r,s}^{j,k}\xi_{r,s}^{i,k} \tag{C.10}$$

Proof.

$$\begin{aligned}
\xi_{r,s}^{i,k}\xi_{r,s}^{j,k} &= (x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}) \\
&= x_{i,r}x_{k,s}x_{j,r}x_{k,s} \\
&\quad - qx_{i,r}x_{k,s}x_{j,s}x_{k,r}
\end{aligned}$$

$$- q x_{i,s} x_{k,r} x_{j,r} x_{k,s}$$

$$+ q^2 x_{i,s} x_{k,r} x_{j,s} x_{k,r}$$

$$= x_{i,r} \left[\left(\frac{1}{q} - q \right) x_{j,s} x_{k,r} + x_{j,r} x_{k,s} \right] x_{k,s}$$

$$- x_{i,r} x_{j,s} x_{k,s} x_{k,r}$$

$$- x_{i,s} x_{j,r} x_{k,r} x_{k,s}$$

$$+ q^2 x_{i,s} x_{j,s} x_{k,r} x_{k,r}$$

$$= \left(\frac{1}{q} - q \right) x_{i,r} x_{j,s} x_{k,r} x_{k,s}$$

$$+ x_{i,r} x_{j,r} x_{k,s} x_{k,s}$$

$$- x_{i,r} x_{j,s} x_{k,s} x_{k,r}$$

$$- x_{i,s} x_{j,r} x_{k,r} x_{k,s}$$

$$+ q^2 x_{i,s} x_{j,s} x_{k,r} x_{k,r}$$

$$= \left(\frac{1}{q} - q \right) \left[\left(q - \frac{1}{q} \right) x_{j,r} x_{i,s} + x_{j,s} x_{i,r} \right] x_{k,r} x_{k,s}$$

$$+ q x_{j,r} x_{i,r} x_{k,s} x_{k,s}$$

$$- \frac{1}{q} \left[\left(q - \frac{1}{q} \right) x_{j,r} x_{i,s} + x_{j,s} x_{i,r} \right] x_{k,r} x_{k,s}$$

$$- q x_{j,r} x_{i,s} x_{k,s} x_{k,r}$$

$$+ q^3 x_{j,s} x_{i,s} x_{k,r} x_{k,r}$$

$$
\begin{aligned}
= & \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{k,r}x_{k,s} \\
& + \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{k,r}x_{k,s} \\
& + q x_{j,r}x_{i,r}x_{k,s}x_{k,s} \\
& - \frac{1}{q}\left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{k,r}x_{k,s} \\
& - \frac{1}{q} x_{j,s}x_{i,r}x_{k,r}x_{k,s} \\
& - q x_{j,r}x_{i,s}x_{k,s}x_{k,r} \\
& + q^3 x_{j,s}x_{i,s}x_{k,r}x_{k,r}
\end{aligned}
$$

$$
\begin{aligned}
= & \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,r}x_{k,r}x_{i,s}x_{k,s} \\
& + q\left(\frac{1}{q} - q\right) x_{j,s}x_{k,r}x_{i,r}x_{k,s} \\
& + q x_{j,r}\left[\left(q - \frac{1}{q}\right) x_{k,r}x_{i,s} + x_{k,s}x_{i,r}\right] x_{k,s} \\
& - \frac{1}{q}\left(q - \frac{1}{q}\right) x_{j,r}x_{k,r}x_{i,s}x_{k,s} \\
& - x_{j,s}x_{k,r}x_{i,r}x_{k,s} \\
& - q^2 x_{j,r}x_{k,s}x_{i,s}x_{k,r} \\
& + q^3 x_{j,s}x_{k,r}x_{i,s}x_{k,r}
\end{aligned}
$$

$$
= \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,r}x_{k,r}x_{i,s}x_{k,s}
$$

$$
\begin{aligned}
&+ q\left(\frac{1}{q} - q\right) x_{j,s}x_{k,r}x_{i,r}x_{k,s} \\
&+ q\left(q - \frac{1}{q}\right) x_{j,r}x_{k,r}x_{i,s}x_{k,s} \\
&+ qx_{j,r}x_{k,s}x_{i,r}x_{k,s} \\
&- \frac{1}{q}\left(q - \frac{1}{q}\right) x_{j,r}x_{k,r}x_{i,s}x_{k,s} \\
&- x_{j,s}x_{k,r}x_{i,r}x_{k,s} \\
&- q^2x_{j,r}x_{k,s}x_{i,s}x_{k,r} \\
&+ q^3x_{j,s}x_{k,r}x_{i,s}x_{k,r} \\
= {} & qx_{j,r}x_{k,s}x_{i,r}x_{k,s} \\
&- q^2x_{j,r}x_{k,s}x_{i,s}x_{k,r} \\
&- q^2x_{j,s}x_{k,r}x_{i,r}x_{k,s} \\
&+ q^3x_{j,s}x_{k,r}x_{i,s}x_{k,r} \\
= {} & q\left(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}\right)\left(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}\right) \\
= {} & q\xi^{j,k}_{r,s}\xi^{i,k}_{r,s}
\end{aligned}
$$

□

Proposition C.11. *Given the quantum minor determinants* $\xi^{i,k}_{r,s}$ *and* $\xi^{j,k}_{t,u}$ *where* $i < j < k$ *and* $r < s < t < u$

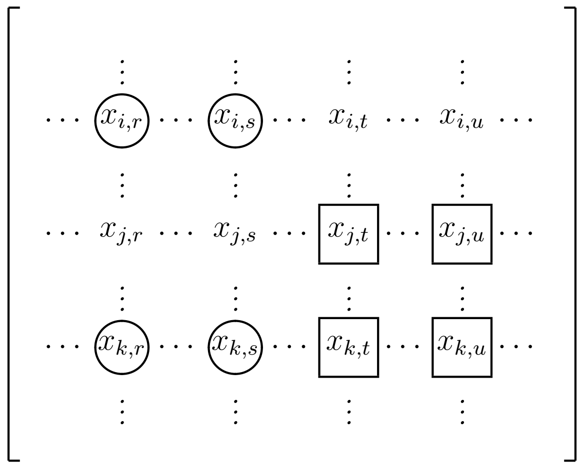

Figure C.11: Relative positions of $\xi^{i,k}_{r,s}$ *and* $\xi^{j,k}_{t,u}$

then

$$\xi^{i,k}_{r,s}\xi^{j,k}_{t,u} = q\xi^{j,k}_{t,u}\xi^{i,k}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{j,k}_{r,s}\xi^{i,k}_{t,u} \tag{C.11}$$

Proof.

$$\begin{aligned}
\xi^{i,k}_{r,s}\xi^{j,k}_{t,u} &= \left(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}\right)\left(x_{j,t}x_{k,u} - qx_{j,u}x_{k,t}\right) \\
&= x_{i,r}x_{k,s}x_{j,t}x_{k,u} \\
&\quad - qx_{i,r}x_{k,s}x_{j,u}x_{k,t} \\
&\quad - qx_{i,s}x_{k,r}x_{j,t}x_{k,u} \\
&\quad + q^2x_{i,s}x_{k,r}x_{j,u}x_{k,t}
\end{aligned}$$

$$
\begin{aligned}
&= x_{i,r}x_{j,t}x_{k,s}x_{k,u} \\
&\quad - qx_{i,r}x_{j,u}x_{k,s}x_{k,t} \\
&\quad - qx_{i,s}x_{j,t}x_{k,r}x_{k,u} \\
&\quad + q^2x_{i,s}x_{j,u}x_{k,r}x_{k,t}
\end{aligned}
$$

$$
\begin{aligned}
&= q\left[\left(q-\frac{1}{q}\right)x_{j,r}x_{i,t} + x_{j,t}x_{i,r}\right]x_{k,u}x_{k,s} \\
&\quad - q^2\left[\left(q-\frac{1}{q}\right)x_{j,r}x_{i,u} + x_{j,u}x_{i,r}\right]x_{k,t}x_{k,s} \\
&\quad - q^2\left[\left(q-\frac{1}{q}\right)x_{j,s}x_{i,t} + x_{j,t}x_{i,s}\right]x_{k,u}x_{k,r} \\
&\quad + q^3\left[\left(q-\frac{1}{q}\right)x_{j,s}x_{i,u} + x_{j,u}x_{i,s}\right]x_{k,t}x_{k,r} \\
&= q\left(q-\frac{1}{q}\right)x_{j,r}x_{i,t}x_{k,u}x_{k,s} \\
&\quad - q^2\left(q-\frac{1}{q}\right)x_{j,r}x_{i,u}x_{k,t}x_{k,s} \\
&\quad - q^2\left(q-\frac{1}{q}\right)x_{j,s}x_{i,t}x_{k,u}x_{k,r} \\
&\quad + q^3\left(q-\frac{1}{q}\right)x_{j,s}x_{i,u}x_{k,t}x_{k,r} \\
&\quad + qx_{j,t}x_{i,r}x_{k,u}x_{k,s} \\
&\quad - q^2x_{j,u}x_{i,r}x_{k,t}x_{k,s} \\
&\quad - q^2x_{j,t}x_{i,s}x_{k,u}x_{k,r} \\
&\quad + q^3x_{j,u}x_{i,s}x_{k,t}x_{k,r}
\end{aligned}
$$

$$
\begin{aligned}
= & \left(q-\frac{1}{q}\right) x_{j,r} x_{i,t} x_{k,s} x_{k,u} \\
& -q\left(q-\frac{1}{q}\right) x_{j,r} x_{i,u} x_{k,s} x_{k,t} \\
& -q\left(q-\frac{1}{q}\right) x_{j,s} x_{i,t} x_{k,r} x_{k,u} \\
& +q^2\left(q-\frac{1}{q}\right) x_{j,s} x_{i,u} x_{k,r} x_{k,t}
\end{aligned}
$$

$$
\begin{aligned}
& +q x_{j,t}\left[\left(q-\frac{1}{q}\right) x_{i,u} x_{k,r}+x_{k,u} x_{i,r}\right] x_{k,s} \\
& -q^2 x_{j,u}\left[\left(q-\frac{1}{q}\right) x_{i,t} x_{k,r}+x_{k,t} x_{i,r}\right] x_{k,s} \\
& -q^2 x_{j,t}\left[\left(q-\frac{1}{q}\right) x_{i,u} x_{k,s}+x_{k,u} x_{i,s}\right] x_{k,r} \\
& +q^3 x_{j,u}\left[\left(q-\frac{1}{q}\right) x_{i,t} x_{k,s}+x_{k,t} x_{i,s}\right] x_{k,r}
\end{aligned}
$$

$$
\begin{aligned}
= & \left(q-\frac{1}{q}\right) x_{j,r} x_{k,s} x_{i,t} x_{k,u} \\
& -q\left(q-\frac{1}{q}\right) x_{j,r} x_{k,s} x_{i,u} x_{k,t} \\
& -q\left(q-\frac{1}{q}\right) x_{j,s} x_{k,r} x_{i,t} x_{k,u} \\
& +q^2\left(q-\frac{1}{q}\right) x_{j,s} x_{k,r} x_{i,u} x_{k,t}
\end{aligned}
$$

$$
\begin{aligned}
& +q\left(q-\frac{1}{q}\right) x_{j,t} x_{i,u} x_{k,r} x_{k,s} \\
& -q^2\left(q-\frac{1}{q}\right) x_{j,u} x_{i,t} x_{k,r} x_{k,s}
\end{aligned}
$$

$$- q^2 \left(q - \frac{1}{q}\right) x_{j,t} x_{i,u} x_{k,s} x_{k,r}$$

$$+ q^3 \left(q - \frac{1}{q}\right) x_{j,u} x_{i,t} x_{k,s} x_{k,r}$$

$$+ q x_{j,t} x_{k,u} x_{i,r} x_{k,s}$$

$$- q^2 x_{j,u} x_{k,t} x_{i,r} x_{k,s}$$

$$- q^2 x_{j,t} x_{k,u} x_{i,s} x_{k,r}$$

$$+ q^3 x_{j,u} x_{k,t} x_{i,s} x_{k,r}$$

$$= \left(q - \frac{1}{q}\right) \left(x_{j,r} x_{k,s} - q x_{j,s} x_{k,r}\right) \left(x_{i,t} x_{k,u} - q x_{i,u} x_{k,t}\right)$$

$$+ q \left(x_{j,t} x_{k,u} - q x_{j,u} x_{k,t}\right) \left(x_{i,r} x_{k,s} - q x_{i,s} x_{k,r}\right)$$

$$= q \xi^{j,k}_{t,u} \xi^{i,k}_{r,s} + \left(q - \frac{1}{q}\right) \xi^{j,k}_{r,s} x^{i,k}_{t,u}$$

□

Proposition C.12. *Given the quantum minor determinants* $\xi^{i,k}_{t,u}$ *and* $\xi^{j,k}_{r,s}$ *where* $i < j < k$ *and* $r < s < t < u$ *then*

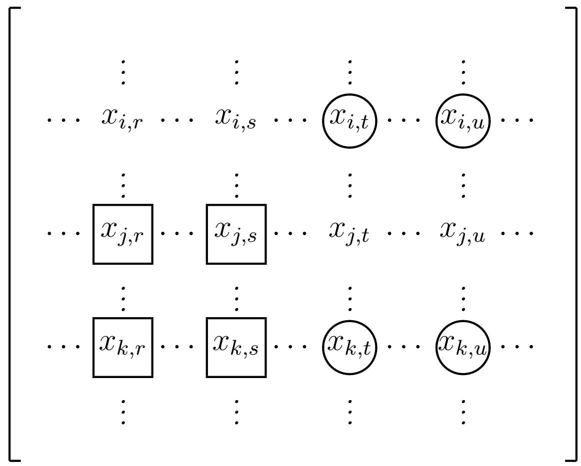

Figure C.12: Relative positions of $\xi^{i,k}_{t,u}$ *and* $\xi^{j,k}_{r,s}$

$$\xi^{i,k}_{t,u}\xi^{j,k}_{r,s} = \frac{1}{q}\xi^{j,k}_{r,s}\xi^{i,k}_{t,u} \tag{C.12}$$

Proof.

$$\begin{aligned}
\xi^{i,k}_{t,u}\xi^{j,k}_{r,s} &= \left(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t}\right)\left(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}\right) \\
&= x_{i,t}x_{k,u}x_{j,r}x_{k,s} \\
&\quad - qx_{i,t}x_{k,u}x_{j,s}x_{k,r} \\
&\quad - qx_{i,u}x_{k,t}x_{j,r}x_{k,s} \\
&\quad + q^2x_{i,u}x_{k,t}x_{j,s}x_{k,r}
\end{aligned}$$

$$= x_{i,t}\left[\left(\frac{1}{q}-q\right)x_{j,u}x_{k,r}+x_{j,r}x_{k,u}\right]x_{k,s}$$
$$-\,qx_{i,t}\left[\left(\frac{1}{q}-q\right)x_{j,u}x_{k,s}+x_{j,s}x_{k,u}\right]x_{k,r}$$
$$-\,qx_{i,u}\left[\left(\frac{1}{q}-q\right)x_{j,t}x_{k,r}+x_{j,r}x_{k,t}\right]x_{k,s}$$
$$+\,q^2x_{i,u}\left[\left(\frac{1}{q}-q\right)x_{j,t}x_{k,s}+x_{k,s}x_{k,t}\right]x_{k,r}$$

$$= x_{i,t}x_{j,r}x_{k,u}x_{k,s}$$
$$-\,qx_{i,t}x_{j,s}x_{k,u}x_{k,r}$$
$$-\,qx_{i,u}x_{j,r}x_{k,t}x_{k,s}$$
$$+\,q^2x_{i,u}x_{j,s}x_{k,t}x_{k,r}$$

$$+\left(\frac{1}{q}-q\right)x_{i,t}x_{j,u}x_{k,r}x_{k,s}$$
$$-\,q\left(\frac{1}{q}-q\right)x_{i,t}x_{j,u}x_{k,s}x_{k,r}$$
$$-\,q\left(\frac{1}{q}-q\right)x_{i,u}x_{j,t}x_{k,r}x_{k,s}$$
$$+\,q^2\left(\frac{1}{q}-q\right)x_{i,u}x_{j,t}x_{k,s}$$

$$= x_{i,t}x_{j,r}x_{k,u}x_{k,s}$$
$$-\,qx_{i,t}x_{j,s}x_{k,u}x_{k,r}$$
$$-\,qx_{i,u}x_{j,r}x_{k,t}x_{k,s}$$

$$+ q^2 x_{i,u} x_{j,s} x_{k,t} x_{k,r}$$

$$= \frac{1}{q} x_{j,r} x_{i,t} x_{k,s} x_{k,u}$$

$$- x_{j,s} x_{i,t} x_{k,r} x_{k,u}$$

$$- x_{j,r} x_{i,u} x_{k,s} x_{k,t}$$

$$+ q x_{j,s} x_{i,u} x_{k,r} x_{k,t}$$

$$= \frac{1}{q} x_{j,r} x_{k,s} x_{i,t} x_{k,u}$$

$$- x_{j,s} x_{k,r} x_{i,t} x_{k,u}$$

$$- x_{j,r} x_{k,s} x_{i,u} x_{k,t}$$

$$+ q x_{j,s} x_{k,r} x_{i,u} x_{k,t}$$

$$= \frac{1}{q} \left(x_{j,r} x_{k,s} - q x_{j,s} x_{k,r} \right) \left(x_{i,t} x_{k,u} - q x_{i,u} x_{,t} \right)$$
$$= \frac{1}{q} \xi^{j,k}_{r,s} \xi^{i,k}_{t,u}$$

□

C.5 Quantum minor determinant relations associated with $z_{i,k}z_{j,l}$ where $i < j < k < l$

Proposition C.13. *Given the quantum minor determinants $\xi^{i,k}_{r,s}$ and $\xi^{j,l}_{r,s}$ where $i < j < k < l$ and $r < s$*

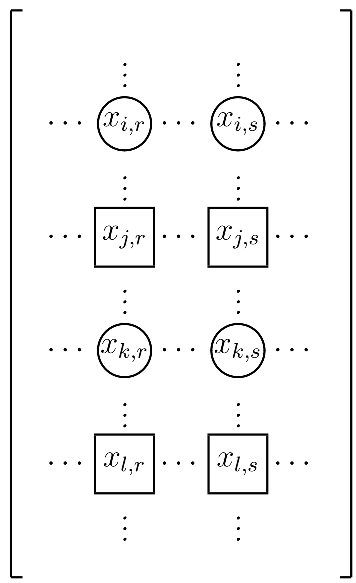

Figure C.13: Relative positions of $\xi^{i,k}_{r,s}$ and $\xi^{j,l}_{r,s}$

then

$$\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} = \xi^{j,l}_{r,s}\xi^{i,k}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{r,s} \tag{C.13}$$

Proof.

$$\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} = (x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})$$

$$= x_{i,r}x_{k,s}x_{j,r}x_{l,s}$$

$$- qx_{i,r}x_{k,s}x_{j,s}x_{l,r}$$

$$- qx_{i,s}x_{k,r}x_{j,r}x_{l,s}$$

$$+ q^2x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$

$$= x_{i,r}\left[\left(\frac{1}{q}-q\right)x_{j,s}x_{k,r}+x_{j,r}x_{k,s}\right]x_{l,s}$$

$$- x_{i,r}x_{j,s}x_{k,s}x_{l,r}$$

$$- x_{i,s}x_{j,r}x_{k,r}x_{l,s}$$

$$+ q^2x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$

$$= \left(\frac{1}{q}-q\right)x_{i,r}x_{j,s}x_{k,r}x_{l,s}$$

$$+ x_{i,r}x_{j,r}x_{k,s}x_{l,s}$$

$$- x_{i,r}x_{j,s}x_{k,s}x_{l,r}$$

$$- x_{i,s}x_{j,r}x_{k,r}x_{l,s}$$

$$+ q^2x_{i,s}x_{j,s}x_{k,r}x_{l,r}$$

$$= \left(\frac{1}{q}-q\right)\left[\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}+x_{j,s}x_{i,r}\right]\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{k,s}+x_{l,s}x_{k,r}\right]$$

$$+ q^2x_{j,r}x_{i,r}x_{l,s}x_{k,s}$$

$$- \left[\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}+x_{j,s}x_{i,r}\right]x_{l,r}x_{k,s}$$

$$- x_{j,r}x_{i,s}\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{k,s}+x_{l,s}x_{k,r}\right]$$

$$+ q^4 x_{j,s}x_{i,s}x_{l,r}x_{k,r}$$

$$= \left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$+ \left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,s}x_{k,r}$$

$$+ \left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{i,r}x_{l,r}x_{k,s}$$

$$+ \left(\frac{1}{q}-q\right)x_{j,s}x_{i,r}x_{l,s}x_{k,r}$$

$$+ q^2 x_{j,r}x_{i,r}x_{l,s}x_{k,s}$$

$$- \left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$- x_{j,s}x_{i,r}x_{l,r}x_{k,s}$$

$$- \left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$- x_{j,r}x_{i,s}x_{l,s}x_{k,r}$$

$$+ q^4 x_{j,s}x_{i,s}x_{l,r}x_{k,r}$$

$$= \left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$+ q\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{l,s}x_{i,s}x_{k,r}$$

$$+ q\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{l,r}x_{i,r}x_{k,s}$$

$$+\left(\frac{1}{q}-q\right)x_{j,s}\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{i,s}+x_{l,s}x_{i,r}\right]x_{k,r}$$

$$+q^2x_{j,r}\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{i,s}+x_{l,s}x_{i,r}\right]x_{k,s}$$
$$-\left(q-\frac{1}{q}\right)x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$
$$-qx_{j,s}x_{l,r}x_{i,r}x_{k,s}$$
$$-\left(q-\frac{1}{q}\right)x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$
$$-qx_{j,r}x_{l,s}x_{i,s}x_{k,r}$$
$$+q^4x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$

$$=\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$
$$+q\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{l,s}x_{i,s}x_{k,r}$$
$$+q\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{l,r}x_{i,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$
$$\left(\frac{1}{q}-q\right)x_{j,s}x_{l,s}x_{i,r}x_{k,r}$$

$$+q^2\left(q-\frac{1}{q}\right)x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$
$$+q^2x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$
$$-\left(q-\frac{1}{q}\right)x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$- qx_{j,s}x_{l,r}x_{i,r}x_{k,s}$$

$$- \left(q - \frac{1}{q}\right) x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$- qx_{j,r}x_{l,s}x_{i,s}x_{k,r}$$

$$+ q^4 x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$

$$= + x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$

$$- qx_{j,r}x_{l,s}x_{i,s}x_{k,r}$$

$$- qx_{j,s}x_{l,r}x_{i,r}x_{k,s}$$

$$+ q^2 x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$

$$+ \left(q^2 - 1\right) x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$

$$+ (q^4 - q^2) x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$

$$\left(\frac{1}{q} - q\right) \left(q - \frac{1}{q}\right) \left(q - \frac{1}{q}\right) x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$+ q^2 \left(q - \frac{1}{q}\right) x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$- \left(q - \frac{1}{q}\right) x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$- \left(q - \frac{1}{q}\right) x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$+ q\left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,r}x_{l,s}x_{i,s}x_{k,r}$$
$$+ q\left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,s}x_{l,r}x_{i,r}x_{k,s}$$
$$+ \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$
$$+ \left(\frac{1}{q} - q\right) x_{j,s}x_{l,s}x_{i,r}x_{k,r}$$
$$= \left(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r}\right)\left(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}\right)$$

$$+ q\left(q - \frac{1}{q}\right) x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$
$$+ q^3\left(q - \frac{1}{q}\right) x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$

$$- \frac{1}{q^2}\left(q - \frac{1}{q}\right) x_{j,r}x_{l,r}x_{i,s}x_{k,s}$$

$$+ q\left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,r}x_{l,s}x_{i,s}x_{k,r}$$
$$+ q\left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,s}x_{l,r}x_{i,r}x_{k,s}$$
$$+ \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{j,s}x_{l,r}x_{i,s}x_{k,r}$$
$$+ \left(\frac{1}{q} - q\right) x_{j,s}x_{l,s}x_{i,r}x_{k,r}$$

$$= \left(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r}\right)\left(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}\right)$$

$$+ q\left(q-\frac{1}{q}\right)x_{j,r}\left[\left(\frac{1}{q}-q\right)x_{i,s}x_{l,r}+x_{i,r}x_{l,s}\right]x_{k,s}$$
$$+ q^3\left(q-\frac{1}{q}\right)x_{j,s}x_{i,s}x_{l,r}x_{k,r}$$

$$-\frac{1}{q^2}\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,s}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{i,r}x_{l,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{i,s}x_{l,r}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)x_{j,s}\left[\left(\frac{1}{q}-q\right)x_{i,s}x_{l,r}+x_{i,r}x_{l,s}\right]x_{k,r}$$

$$= \left(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)$$

$$+ q\left(q-\frac{1}{q}\right)\left(\frac{1}{q}-q\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$
$$+ q\left(q-\frac{1}{q}\right)x_{j,r}x_{i,r}x_{l,s}x_{k,s}$$
$$+ q^3\left(q-\frac{1}{q}\right)x_{j,s}x_{i,s}x_{l,r}x_{k,r}$$

$$-\frac{1}{q^2}\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,s}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{i,r}x_{l,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{i,s}x_{l,r}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)\left(\frac{1}{q}-q\right)x_{j,s}x_{i,s}x_{l,r}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)x_{j,s}x_{i,r}x_{l,s}x_{k,r}$$

$$=\left(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)$$

$$+q\left(q-\frac{1}{q}\right)\left(\frac{1}{q}-q\right)x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$
$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{j,r}x_{l,s}x_{k,s}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{j,s}x_{l,r}x_{k,r}$$

$$-\frac{1}{q^2}\left(q-\frac{1}{q}\right)x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$

$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{i,s}x_{j,r}x_{l,s}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)\left[\left(\frac{1}{q}-q\right)x_{i,s}x_{j,r}+x_{i,r}x_{j,s}\right]x_{l,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)\left[\left(\frac{1}{q}-q\right)x_{i,s}x_{j,r}+x_{i,r}x_{j,s}\right]x_{l,s}x_{k,r}$$

$$= (x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})$$

$$+ q\left(q - \frac{1}{q}\right)\left(\frac{1}{q} - q\right) x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$
$$- \frac{1}{q^2}\left(q - \frac{1}{q}\right) x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$
$$+ \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right)\left(\frac{1}{q} - q\right) x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r}x_{j,r}x_{l,s}x_{k,s}$$
$$+ q^2\left(q - \frac{1}{q}\right) x_{i,s}x_{j,s}x_{l,r}x_{k,r}$$
$$+ \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{i,r}x_{j,s}x_{l,r}x_{k,s}$$
$$+ \left(\frac{1}{q} - q\right) x_{i,r}x_{j,s}x_{l,s}x_{k,r}$$

$$+ \left(\frac{1}{q} - q\right)\left(q - \frac{1}{q}\right) x_{i,s}x_{j,r}x_{l,s}x_{k,r}$$
$$+ \left(\frac{1}{q} - q\right)\left(\frac{1}{q} - q\right) x_{i,s}x_{j,r}x_{l,s}x_{k,r}$$

$$= (x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})$$

$$- \left(q - \frac{1}{q}\right) x_{i,s}x_{j,r}x_{l,r}x_{k,s}$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{j,r}x_{l,s}x_{k,s}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{j,s}x_{l,r}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{i,r}x_{j,s}x_{l,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)x_{i,r}x_{j,s}x_{l,s}x_{k,r}$$

$$=\ (x_{j,r}x_{l,s}-qx_{j,s}x_{l,r})\,(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r})$$

$$-q\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,r}x_{k,s}$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{j,s}+x_{l,s}x_{j,r}\right]x_{k,s}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,s}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{i,r}x_{l,r}x_{j,s}x_{k,s}$$
$$+q\left(\frac{1}{q}-q\right)x_{i,r}x_{l,s}x_{j,s}x_{k,r}$$

$$=\ (x_{j,r}x_{l,s}-qx_{j,s}x_{l,r})\,(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r})$$

$$-q\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,r}x_{k,s}$$

$$+\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{i,r}x_{l,r}x_{j,s}x_{k,s}$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{l,s}x_{j,r}x_{k,s}$$

$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,s}x_{k,r}$$

$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{i,r}x_{l,r}x_{j,s}x_{k,s}$$

$$+q\left(\frac{1}{q}-q\right)x_{i,r}x_{l,s}x_{j,s}x_{k,r}$$

$$=\left(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)$$

$$-q\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,r}x_{k,s}$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{l,s}x_{j,r}x_{k,s}$$

$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,s}x_{k,r}$$

$$+q\left(\frac{1}{q}-q\right)x_{i,r}x_{l,s}x_{j,s}x_{k,r}$$

$$=\left(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{l,s}x_{j,r}x_{k,s}$$

$$-q\left(q-\frac{1}{q}\right)x_{i,r}x_{l,s}x_{j,s}x_{k,r}$$

$$-q\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,r}x_{k,s}$$

$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{l,r}x_{j,s}x_{k,r}$$

$$= (x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}) + \left(q - \frac{1}{q}\right)(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r})$$

$$= \xi^{j,l}_{r,s}\xi^{i,k}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{r,s}$$

$\square$

Additionally, we have the following relation for $\xi^{i,k}_{r,s}$ and $\xi^{j,l}_{r,s}$.

Proposition C.14. *Given the quantum minor determinants* $\xi^{i,k}_{r,s}$ *and* $\xi^{j,l}_{r,s}$ *where* $i < j < k < l$ *and* $r < s$ *then*

$$\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} = q^2\xi^{j,l}_{r,s}\xi^{i,k}_{r,s} + \left(\frac{1}{q} - q\right)\xi^{i,j}_{r,s}\xi^{k,l}_{r,s} \tag{C.14}$$

Proof.

$$\begin{aligned}
\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} = &\ (x_{i,r}x_{k,s} - qx_{k,r}x_{i,s})(x_{j,r}x_{l,s} - qx_{l,r}x_{j,s}) \\
= &\ x_{i,r}x_{k,s}x_{j,r}x_{l,s} \\
&- qx_{i,r}x_{k,s}x_{l,r}x_{j,s} \\
&- qx_{k,r}x_{i,s}x_{j,r}x_{l,s} \\
&+ q^2x_{k,r}x_{i,s}x_{l,r}x_{j,s} \\
= &\ x_{i,r}\left[\left(\frac{1}{q} - q\right)x_{j,s}x_{k,s} + x_{j,r}x_{k,s}\right]x_{l,s} \\
&- qx_{i,r}x_{l,r}x_{k,s}x_{j,s} \\
&- qx_{k,r}x_{j,r}x_{i,s}x_{l,s} \\
&+ q^2x_{k,r}x_{l,r}x_{i,s}x_{j,s} \\
= &\ x_{i,r}x_{j,r}x_{k,s}x_{l,s}
\end{aligned}$$

$$
\begin{aligned}
&- q x_{i,r} x_{l,r} x_{k,s} x_{j,s} \\
&- q x_{k,r} x_{j,r} x_{i,s} x_{l,s} \\
&+ q^2 x_{k,r} x_{l,r} x_{i,s} x_{j,s} \\
&+ \left(\frac{1}{q} - q\right) x_{i,r} x_{j,s} x_{k,r} x_{l,s}
\end{aligned}
$$

$$
\begin{aligned}
= {} & q^2 x_{j,r} x_{i,r} x_{l,s} x_{k,s} \\
&- q x_{l,r} x_{i,r} x_{j,s} x_{k,s} \\
&- q x_{j,r} x_{k,r} x_{l,s} x_{i,s} \\
&+ q^4 x_{l,r} x_{k,r} x_{j,s} x_{i,s} \\
&+ \left(\frac{1}{q} - q\right) x_{i,r} x_{j,s} x_{k,r} x_{l,s}
\end{aligned}
$$

$$
\begin{aligned}
= {} & q^2 x_{j,r} \left[\left(q - \frac{1}{q}\right) x_{l,r} x_{i,s} + x_{l,s} x_{i,r} \right] x_{k,s} \\
&- q x_{l,r} \left[\left(q - \frac{1}{q}\right) x_{j,r} x_{i,s} + x_{j,s} x_{i,r} \right] x_{k,s} \\
&- q x_{j,r} \left[\left(q - \frac{1}{q}\right) x_{l,r} x_{k,s} + x_{l,s} x_{k,r} \right] x_{i,s} \\
&+ q^4 x_{l,r} x_{k,r} x_{j,s} x_{i,s} \\
&+ \left(\frac{1}{q} - q\right) x_{i,r} x_{j,s} x_{k,r} x_{l,s}
\end{aligned}
$$

$$= q^2 x_{j,r} x_{l,s} x_{i,r} x_{k,s}$$

$$- q x_{l,r} x_{j,s} x_{i,r} x_{k,s}$$

$$- q x_{j,r} x_{l,s} x_{k,r} x_{i,s}$$

$$+ q^4 x_{l,r} x_{j,s} x_{k,r} x_{i,s}$$

$$+ q^2 \left(q - \frac{1}{q} \right) x_{j,r} x_{l,r} x_{i,s} x_{k,s}$$

$$- q \left(q - \frac{1}{q} \right) x_{l,r} x_{j,r} x_{i,s} x_{k,s}$$

$$- q \left(q - \frac{1}{q} \right) x_{j,r} x_{l,r} x_{k,s} x_{i,s}$$

$$+ \left(\frac{1}{q} - q \right) x_{i,r} x_{j,s} x_{k,r} x_{l,s}$$

$$= q^2 x_{j,r} x_{l,s} x_{i,r} x_{k,s}$$

$$- q x_{l,r} x_{j,s} x_{i,r} x_{k,s}$$

$$- q x_{j,r} x_{l,s} x_{k,r} x_{i,s}$$

$$+ q^4 x_{l,r} x_{j,s} x_{k,r} x_{i,s}$$

$$+ \left(q^2 - 2 \right) \left(q - \frac{1}{q} \right) x_{i,s} x_{j,r} x_{k,s} x_{l,r}$$

$$+\left(\frac{1}{q}-q\right)x_{i,r}x_{j,s}x_{k,r}x_{l,s}$$

$$= q^2 x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$

$$- q x_{l,r}x_{j,s}x_{i,r}x_{k,s}$$

$$- q x_{j,r}x_{l,s}x_{k,r}x_{i,s}$$

$$+ q^4 x_{l,r}x_{j,s}x_{k,r}x_{i,s}$$

$$+ \left(q^2-2\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$+\left(\frac{1}{q}-q\right)\left[\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}+x_{j,s}x_{i,r}\right]\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{k,s}x_{l,s}x_{k,r}\right]$$

$$= q^2 x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$

$$- q x_{l,r}x_{j,s}x_{i,r}x_{k,s}$$

$$- q x_{j,r}x_{l,s}x_{k,r}x_{i,s}$$

$$+ q^4 x_{l,r}x_{j,s}x_{k,r}x_{i,s}$$

$$+ \left(q^2-2\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,s}x_{k,r}$$
$$+\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,s}x_{i,r}x_{l,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)x_{j,s}x_{i,r}x_{l,s}x_{k,r}$$

$$=q^2x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$
$$-qx_{l,r}x_{j,s}x_{i,r}x_{k,s}$$
$$-qx_{j,r}x_{l,s}x_{k,r}x_{i,s}$$
$$+q^4x_{l,r}x_{j,s}x_{k,r}x_{i,s}$$

$$+\left(q^2-2\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$-\frac{1}{q^2}\left(q-\frac{1}{q}\right)x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$
$$+q\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{j,r}x_{l,s}x_{i,s}x_{k,r}$$
$$+q\left(\frac{1}{q}-q\right)\left(q-\frac{1}{q}\right)x_{l,r}x_{j,s}x_{i,r}x_{k,s}$$
$$+\left(\frac{1}{q}-q\right)x_{j,s}x_{i,r}x_{l,s}x_{k,r}$$

$$=q^2x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$

$$- qx_{l,r}x_{j,s}x_{i,r}x_{k,s}$$

$$- qx_{j,r}x_{l,s}x_{k,r}x_{i,s}$$

$$+ q^4x_{l,r}x_{j,s}x_{k,r}x_{i,s}$$

$$- \frac{1}{q^2}\left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$
$$+ \left(2q - q^3 - \frac{1}{q}\right) x_{j,r}x_{l,s}x_{i,s}x_{k,r}$$
$$+ \left(2q - q^3 - \frac{1}{q}\right) x_{l,r}x_{j,s}x_{i,r}x_{k,s}$$
$$+ \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{l,s}x_{k,r}$$

$$= q^2x_{j,r}x_{l,s}x_{i,r}x_{k,s}$$

$$- q^3x_{l,r}x_{j,s}x_{i,r}x_{k,s}$$

$$- q^3x_{j,r}x_{l,s}x_{k,r}x_{i,s}$$

$$+ q^4x_{l,r}x_{j,s}x_{k,r}x_{i,s}$$

$$- \frac{1}{q^2}\left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$
$$+ \left(2q - q - \frac{1}{q}\right) x_{j,r}x_{l,s}x_{i,s}x_{k,r}$$
$$+ \left(2q - q - \frac{1}{q}\right) x_{l,r}x_{j,s}x_{i,r}x_{k,s}$$
$$+ \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{l,s}x_{k,r}$$

$$= q^2(x_{j,r}x_{l,s} - qx_{l,r}x_{j,s})(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s})$$

$$\begin{aligned}
&- \frac{1}{q^2}\left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{l,r}x_{k,s} \\
&+ \left(2q - q - \frac{1}{q}\right) x_{j,r}x_{l,s}x_{i,s}x_{k,r} \\
&+ \left(2q - q - \frac{1}{q}\right) x_{l,r}x_{j,s}x_{i,r}x_{k,s} \\
&+ \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{l,s}x_{k,r}
\end{aligned}$$

$$= q^2(x_{j,r}x_{l,s} - qx_{l,r}x_{j,s})(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s})$$

$$\begin{aligned}
&- \frac{1}{q^2}\left(q - \frac{1}{q}\right) x_{j,r}x_{i,s}x_{l,r}x_{k,s} \\
&+ \left(q - \frac{1}{q}\right) x_{j,r}x_{l,s}x_{i,s}x_{k,r} \\
&+ \left(q - \frac{1}{q}\right) x_{l,r}x_{j,s}x_{i,r}x_{k,s} \\
&+ \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{l,s}x_{k,r}
\end{aligned}$$

$$= q^2(x_{j,r}x_{l,s} - qx_{l,r}x_{j,s})(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s})$$

$$+ \frac{1}{q^2}\left(\frac{1}{q} - q\right) x_{j,r}x_{i,s}x_{l,r}x_{k,s}$$

$$- \frac{1}{q}\left(\frac{1}{q} - q\right) x_{j,r}x_{i,s}x_{l,s}x_{k,r}$$
$$- \frac{1}{q}\left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{l,r}x_{k,s}$$
$$+ \left(\frac{1}{q} - q\right) x_{j,s}x_{i,r}x_{l,s}x_{k,r}$$

$$= q^2(x_{j,r}x_{l,s} - qx_{l,r}x_{j,s})(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s})$$

$$+ \frac{1}{q^2}\left(\frac{1}{q} - q\right)(x_{j,r}x_{i,s} - qx_{j,s}x_{i,r})(x_{l,r}x_{k,s} - qx_{l,s}x_{k,r})$$

$$= q^2\left(x_{j,r}x_{l,s} - qx_{l,r}x_{j,s}\right)\left(x_{i,r}x_{k,s} - qx_{k,r}x_{i,s}\right)$$
$$+ \left(\frac{1}{q} - q\right)(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r})$$

$$= q^2\xi^{j,l}_{r,s}\xi^{i,k}_{r,s} + \left(\frac{1}{q} - q\right)\xi^{i,j}_{r,s}\xi^{k,l}_{r,s}$$

□

Proposition C.15. *Given the quantum minor determinants* $\xi^{i,k}_{r,s}$ *and* $\xi^{j,l}_{t,u}$ *where* $i < j < k < l$ *and* $r < s < t < u$

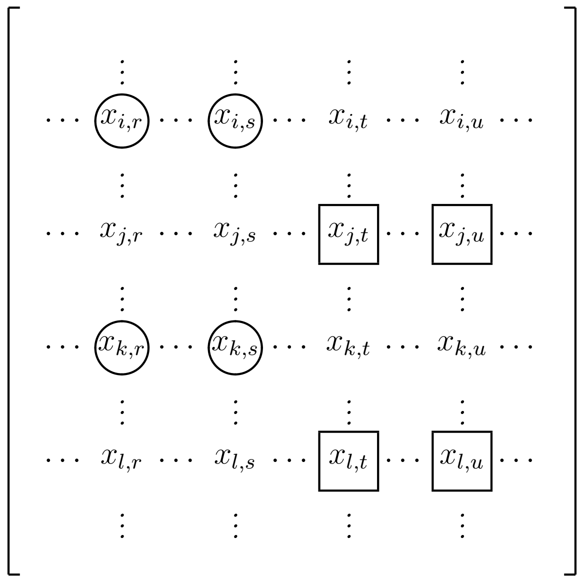

Figure C.14: Relative positions of $\xi^{i,k}_{r,s}$ *and* $\xi^{j,l}_{t,u}$

then

$$\begin{aligned}\xi^{i,k}_{r,s}\xi^{j,l}_{t,u} &= \xi^{j,l}_{t,u}\xi^{i,k}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,l}_{t,u}\xi^{j,k}_{r,s} \\ &\quad + \left(q - \frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{t,u} + \left(q - \frac{1}{q}\right)\xi^{i,j}_{t,u}\xi^{k,l}_{r,s} \qquad \text{(C.15)}\end{aligned}$$

Proof

$$\begin{aligned}\xi^{i,k}_{r,s}\xi^{j,l}_{t,u} &= \left(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}\right)\left(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t}\right) \\ &= x_{i,r}x_{k,s}x_{j,t}x_{l,u}\end{aligned}$$

$$- qx_{i,r}x_{k,s}x_{j,u}x_{l,t}$$

$$- qx_{i,s}x_{k,r}x_{j,t}x_{l,u}$$

$$+ q^2x_{i,s}x_{k,r}x_{j,u}x_{l,t}$$

$$= x_{i,r}x_{j,t}x_{k,s}x_{l,u}$$

$$- qx_{i,r}x_{j,u}x_{k,s}x_{l,t}$$

$$- qx_{i,s}x_{j,t}x_{k,r}x_{l,u}$$

$$+ q^2x_{i,s}x_{j,u}x_{k,r}x_{l,t}$$

$$\begin{aligned}
= &\left[\left(q - \frac{1}{q}\right) x_{i,t}x_{j,r} + x_{j,t}x_{i,r}\right]\left[\left(q - \frac{1}{q}\right) x_{k,u}x_{l,s} + x_{l,u}x_{k,s}\right] \\
&- q\left[\left(q - \frac{1}{q}\right) x_{i,u}x_{j,r} + x_{j,u}x_{i,r}\right]\left[\left(q - \frac{1}{q}\right) x_{k,t}x_{l,s} + x_{l,t}x_{k,s}\right] \\
&- q\left[\left(q - \frac{1}{q}\right) x_{i,t}x_{j,s} + x_{j,t}x_{i,s}\right]\left[\left(q - \frac{1}{q}\right) x_{k,u}x_{l,r} + x_{l,u}x_{k,r}\right] \\
&+ q^2\left[\left(q - \frac{1}{q}\right) x_{i,u}x_{j,s} + x_{j,u}x_{i,s}\right]\left[\left(q - \frac{1}{q}\right) x_{k,t}x_{l,r} + x_{l,t}x_{k,r}\right]
\end{aligned}$$

$$= x_{j,t}x_{i,r}x_{l,u}x_{k,s}$$

$$- qx_{j,u}x_{i,r}x_{l,t}x_{k,s}$$

$$- qx_{j,t}x_{i,s}x_{l,u}x_{k,r}$$

$$+ q^2x_{j,u}x_{i,s}x_{l,t}x_{k,r}$$

$$= \left(q - \frac{1}{q}\right) x_{i,t}x_{j,r}x_{l,u}x_{k,s}$$
$$- q\left(q - \frac{1}{q}\right) x_{i,u}x_{j,r}x_{l,t}x_{k,s}$$
$$- q\left(q - \frac{1}{q}\right) x_{i,t}x_{j,s}x_{l,u}x_{k,r}$$
$$+ q^2\left(q - \frac{1}{q}\right) x_{i,u}x_{j,s}x_{l,t}x_{k,r}$$

$$= \left(q - \frac{1}{q}\right) x_{j,t}x_{i,r}x_{k,u}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right) x_{j,u}x_{i,r}x_{k,t}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right) x_{j,t}x_{i,s}x_{k,u}x_{l,r}$$
$$+ q^2\left(q - \frac{1}{q}\right) x_{j,u}x_{i,s}x_{k,t}x_{l,r}$$

$$= \left(q - \frac{1}{q}\right)^2 x_{i,t}x_{j,r}x_{k,u}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right)^2 x_{i,u}x_{j,r}x_{k,t}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right)^2 x_{i,t}x_{j,s}x_{k,u}x_{l,r}$$
$$+ q^2\left(q - \frac{1}{q}\right)^2 x_{i,u}x_{j,s}x_{k,t}x_{l,r}$$

$$= x_{j,t}\left[\left(q - \frac{1}{q}\right) x_{i,u}x_{l,r} + x_{l,u}x_{i,r}\right] x_{k,s}$$
$$- qx_{j,u}\left[\left(q - \frac{1}{q}\right) x_{i,t}x_{l,r} + x_{l,t}x_{i,r}\right] x_{k,s}$$

$$- qx_{j,t}\left[\left(q-\frac{1}{q}\right)x_{i,u}x_{l,s}+x_{l,u}x_{i,s}\right]x_{k,r}$$
$$+ q^2x_{j,u}\left[\left(q-\frac{1}{q}\right)x_{i,t}x_{l,s}+x_{l,t}x_{i,s}\right]x_{k,r}$$

$$+\left(q-\frac{1}{q}\right)x_{i,t}\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{j,u}+x_{l,u}x_{j,r}\right]x_{k,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{i,u}\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{j,t}+x_{l,t}x_{j,r}\right]x_{k,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{i,t}\left[\left(q-\frac{1}{q}\right)x_{l,s}x_{j,u}+x_{l,u}x_{j,s}\right]x_{k,r}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{i,u}\left[\left(q-\frac{1}{q}\right)x_{l,s}x_{j,t}+x_{l,t}x_{j,s}\right]x_{k,r}$$

$$+\left(q-\frac{1}{q}\right)\left[\left(\frac{1}{q}-q\right)x_{i,t}x_{j,r}+x_{i,r}x_{j,t}\right]x_{k,u}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)\left[\left(\frac{1}{q}-q\right)x_{i,u}x_{j,r}+x_{i,r}x_{j,u}\right]x_{k,t}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)\left[\left(\frac{1}{q}-q\right)x_{i,t}x_{j,s}+x_{i,s}x_{j,t}\right]x_{k,u}x_{l,r}$$
$$+q^2\left(q-\frac{1}{q}\right)\left[\left(\frac{1}{q}-q\right)x_{i,u}x_{j,s}+x_{i,s}x_{j,u}\right]x_{k,t}x_{l,r}$$

$$+\left(q-\frac{1}{q}\right)^2x_{i,t}x_{j,r}x_{k,u}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)^2x_{i,u}x_{j,r}x_{k,t}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)^2x_{i,t}x_{j,s}x_{k,u}x_{l,r}$$
$$+q^2\left(q-\frac{1}{q}\right)^2x_{i,u}x_{j,s}x_{k,t}x_{l,r}$$

$$
\begin{aligned}
&= x_{j,t}x_{l,u}x_{i,r}x_{k,s} \\
&\quad - qx_{j,u}x_{l,t}x_{i,r}x_{k,s} \\
&\quad - qx_{j,t}x_{l,u}x_{i,s}x_{k,r} \\
&\quad + q^2x_{j,u}x_{l,t}x_{i,s}x_{k,r} \\
&\\
&\quad + \left(q - \frac{1}{q}\right) x_{j,t}x_{i,u}x_{l,r}x_{k,s} \\
&\quad - q\left(q - \frac{1}{q}\right) x_{j,u}x_{i,t}x_{l,r}x_{k,s} \\
&\quad - q\left(q - \frac{1}{q}\right) x_{j,t}x_{i,u}x_{l,s}x_{k,r} \\
&\quad + q^2\left(q - \frac{1}{q}\right) x_{j,u}x_{i,t}x_{l,s}x_{k,r} \\
&\\
&\quad + \left(q - \frac{1}{q}\right)^2 x_{i,t}x_{l,r}x_{j,u}x_{k,s} \\
&\quad - q\left(q - \frac{1}{q}\right)^2 x_{i,u}x_{l,r}x_{j,t}x_{k,s} \\
&\quad - q\left(q - \frac{1}{q}\right)^2 x_{i,t}x_{l,s}x_{j,u}x_{k,r} \\
&\quad + q^2\left(q - \frac{1}{q}\right)^2 x_{i,u}x_{l,s}x_{j,t}x_{k,r} \\
&\\
&\quad + \left(q - \frac{1}{q}\right) x_{i,t}x_{l,u}x_{j,r}x_{k,s} \\
&\quad - q\left(q - \frac{1}{q}\right) x_{i,u}x_{l,t}x_{j,r}x_{k,s} \\
&\quad - q\left(q - \frac{1}{q}\right) x_{i,t}x_{l,u}x_{j,s}x_{k,r}
\end{aligned}
$$

$$+ q^2 \left(q - \frac{1}{q} \right) x_{i,u} x_{l,t} x_{j,s} x_{k,r}$$

$$+ \left(q - \frac{1}{q} \right) x_{i,r} x_{j,t} x_{k,u} x_{l,s}$$
$$- q \left(q - \frac{1}{q} \right) x_{i,r} x_{j,u} x_{k,t} x_{l,s}$$
$$- q \left(q - \frac{1}{q} \right) x_{i,s} x_{j,t} x_{k,u} x_{l,r}$$
$$+ q^2 \left(q - \frac{1}{q} \right) x_{i,s} x_{j,u} x_{k,t} x_{l,r}$$

$$= \left(x_{j,t} x_{l,u} - q x_{j,u} x_{l,t} \right) \left(x_{i,r} x_{k,s} - q x_{i,s} x_{k,r} \right)$$

$$+ \left(q - \frac{1}{q} \right) \left(x_{j,t} x_{i,u} - q x_{j,u} x_{i,t} \right) \left(x_{l,r} x_{k,s} - q x_{l,s} x_{k,r} \right)$$

$$+ \left(q - \frac{1}{q} \right)^2 x_{i,t} x_{l,r} x_{j,u} x_{k,s}$$
$$- q \left(q - \frac{1}{q} \right)^2 x_{i,u} x_{l,r} x_{j,t} x_{k,s}$$
$$- q \left(q - \frac{1}{q} \right)^2 x_{i,t} x_{l,s} x_{j,u} x_{k,r}$$
$$+ q^2 \left(q - \frac{1}{q} \right)^2 x_{i,u} x_{l,s} x_{j,t} x_{k,r}$$

$$+ \left(q - \frac{1}{q} \right) \left(x_{i,t} x_{l,u} - q x_{i,u} x_{l,t} \right) \left(x_{j,r} x_{k,s} - q x_{j,s} x_{k,r} \right)$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{j,t}x_{k,u}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{i,r}x_{j,u}x_{k,t}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{i,s}x_{j,t}x_{k,u}x_{l,r}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{j,u}x_{k,t}x_{l,r}$$

$$=\left(x_{j,t}x_{l,u}-qx_{j,u}x_{l,t}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)$$

$$+\left(q-\frac{1}{q}\right)\left(x_{j,t}x_{i,u}-qx_{j,u}x_{i,t}\right)\left(x_{l,r}x_{k,s}-qx_{l,s}x_{k,r}\right)$$

$$+\left(q-\frac{1}{q}\right)^2x_{i,t}x_{j,u}x_{l,r}x_{k,s}$$
$$-q\left(q-\frac{1}{q}\right)^2x_{i,u}x_{j,t}x_{l,r}x_{k,s}$$
$$-q\left(q-\frac{1}{q}\right)^2x_{i,t}x_{j,u}x_{l,s}x_{k,r}$$
$$+q^2\left(q-\frac{1}{q}\right)^2x_{i,u}x_{j,t}x_{l,s}x_{k,r}$$

$$+\left(q-\frac{1}{q}\right)\left(x_{i,t}x_{l,u}-qx_{i,u}x_{l,t}\right)\left(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r}\right)$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{l,s}x_{j,t}x_{k,u}$$
$$-q\left(q-\frac{1}{q}\right)x_{i,r}x_{l,s}x_{j,u}x_{k,t}$$

$$
\begin{aligned}
&- q\left(q-\frac{1}{q}\right) x_{i,s}x_{l,r}x_{j,t}x_{k,u}\\
&+ q^2\left(q-\frac{1}{q}\right) x_{i,s}x_{l,r}x_{j,u}x_{k,t}\\
=\;& \left(x_{j,t}x_{l,u}-qx_{j,u}x_{l,t}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)\\
&+ q^2\left(q-\frac{1}{q}\right)\left(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t}\right)\left(x_{k,r}x_{l,s}-qx_{k,s}x_{l,r}\right)\\
&+ \left(q-\frac{1}{q}\right)^2\left(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t}\right)\left(x_{l,r}x_{k,s}-qx_{l,s}x_{k,r}\right)\\
&+ \left(q-\frac{1}{q}\right)\left(x_{i,t}x_{l,u}-qx_{i,u}x_{l,t}\right)\left(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r}\right)\\
&+ \left(q-\frac{1}{q}\right)\left(x_{i,r}x_{l,s}-qx_{i,s}x_{l,r}\right)\left(x_{j,t}x_{k,u}-qx_{j,u}x_{k,t}\right)
\end{aligned}
$$

$$
\begin{aligned}
=\;& \left(x_{j,t}x_{l,u}-qx_{j,u}x_{l,t}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)\\
&+ \left(q-\frac{1}{q}\right)\left(x_{i,t}x_{j,u}-qx_{i,u}x_{j,t}\right)\left(x_{k,r}x_{l,s}-qx_{k,s}x_{l,r}\right)\\
&+ \left(q-\frac{1}{q}\right)\left(x_{i,t}x_{l,u}-qx_{i,u}x_{l,t}\right)\left(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r}\right)\\
&+ \left(q-\frac{1}{q}\right)\left(x_{i,r}x_{l,s}-qx_{i,s}x_{l,r}\right)\left(x_{j,t}x_{k,u}-qx_{j,u}x_{k,t}\right)\\
=\;& \xi^{j,l}_{t,u}\xi^{i,k}_{r,s} + \left(q-\frac{1}{q}\right)\xi^{i,l}_{t,u}\xi^{j,k}_{r,s}\\
&+ \left(q-\frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{t,u} + \left(q-\frac{1}{q}\right)\xi^{i,j}_{t,u}\xi^{k,l}_{r,s}
\end{aligned}
$$

□

Proposition C.16. *Given the quantum minor determinants* $\xi^{i,k}_{t,u}$ *and* $\xi^{j,l}_{r,s}$ *where* $i < j < k < l$ *and* $r < s < t < u$

$$\begin{bmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & x_{i,r} & \cdots & x_{i,s} & \cdots & \enclose{circle}{x_{i,t}} & \cdots & \enclose{circle}{x_{i,u}} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \boxed{x_{j,r}} & \cdots & \boxed{x_{j,s}} & \cdots & x_{j,t} & \cdots & x_{j,u} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & x_{k,r} & \cdots & x_{k,s} & \cdots & \enclose{circle}{x_{k,t}} & \cdots & \enclose{circle}{x_{k,u}} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \boxed{x_{l,r}} & \cdots & \boxed{x_{l,s}} & \cdots & x_{l,t} & \cdots & x_{l,u} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{bmatrix}$$

Figure C.15: Relative positions of $\xi^{i,k}_{t,u}$ *and* $\xi^{j,l}_{r,s}$

then

$$\xi^{i,k}_{t,u}\xi^{j,l}_{r,s} = \xi^{j,l}_{r,s}\xi^{i,k}_{t,u} + \left(\frac{1}{q} - q\right)\xi^{i,j}_{t,u}\xi^{k,l}_{r,s} \tag{C.16}$$

Proof.

$$\begin{aligned} \xi^{i,k}_{t,u}\xi^{j,l}_{r,s} = &\ (x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r}) \\ = &\ x_{i,t}x_{k,u}x_{j,r}x_{l,s} \\ & - qx_{i,t}x_{k,u}x_{j,s}x_{l,r} \\ & - qx_{i,u}x_{k,t}x_{j,r}x_{l,s} \\ & + q^2x_{i,u}x_{k,t}x_{j,s}x_{l,r} \end{aligned}$$

$$= x_{i,t}\left[\left(\frac{1}{q}-q\right)x_{j,u}x_{k,r}+x_{j,r}x_{k,u}\right]x_{l,s}$$
$$-\,qx_{i,t}\left[\left(\frac{1}{q}-q\right)x_{j,u}x_{k,s}+x_{j,s}x_{k,u}\right]x_{l,r}$$
$$-\,qx_{i,u}\left[\left(\frac{1}{q}-q\right)x_{j,t}x_{k,r}+x_{j,r}x_{k,t}\right]x_{l,s}$$
$$+\,q^2x_{i,u}\left[\left(\frac{1}{q}-q\right)x_{k,t}x_{k,s}+x_{j,s}x_{k,t}\right]x_{l,r}$$

$$= x_{i,t}x_{j,r}x_{k,u}x_{l,s}$$
$$-\,qx_{i,t}x_{j,s}x_{k,u}x_{l,r}$$
$$-\,qx_{i,u}x_{j,r}x_{k,t}x_{l,s}$$
$$+\,q^2x_{i,u}x_{j,s}x_{k,t}x_{l,r}$$

$$+\left(\frac{1}{q}-q\right)x_{i,t}x_{j,u}x_{k,r}x_{l,s}$$
$$-\,q\left(\frac{1}{q}-q\right)x_{i,t}x_{j,u}x_{k,s}x_{l,r}$$
$$-\,q\left(\frac{1}{q}-q\right)x_{i,u}x_{j,t}x_{k,r}x_{l,s}$$
$$+\,q^2\left(\frac{1}{q}-q\right)x_{i,u}x_{k,t}x_{k,s}x_{l,r}$$

$$= x_{j,r}x_{i,t}x_{l,s}x_{k,u}$$
$$-\,qx_{j,s}x_{i,t}x_{l,r}x_{k,u}$$

$$- q x_{j,r} x_{i,u} x_{l,s} x_{k,t}$$

$$+ q^2 x_{j,s} x_{i,u} x_{l,r} x_{k,t}$$

$$+ \left(\frac{1}{q} - q\right) (x_{i,t} x_{j,u} - q x_{i,u} x_{k,t}) (x_{k,r} x_{l,s} - q x_{k,s} x_{l,r})$$

$$= x_{j,r} x_{l,s} x_{i,t} x_{k,u}$$

$$- q x_{j,s} x_{l,r} x_{i,t} x_{k,u}$$

$$- q x_{j,r} x_{l,s} x_{i,u} x_{k,t}$$

$$+ q^2 x_{j,s} x_{l,r} x_{i,u} x_{k,t}$$

$$+ \left(\frac{1}{q} - q\right) (x_{i,t} x_{j,u} - q x_{i,u} x_{k,t}) (x_{k,r} x_{l,s} - q x_{k,s} x_{l,r})$$

$$= (x_{j,r} x_{l,s} - q x_{j,s} x_{l,r})(x_{i,t} x_{k,u} - q x_{i,u} x_{k,t})$$
$$+ \left(\frac{1}{q} - q\right) (x_{i,t} x_{j,u} - q x_{i,u} x_{k,t}) (x_{k,r} x_{l,s} - q x_{k,s} x_{l,r})$$

$$= \xi^{j,l}_{r,s} \xi^{i,k}_{t,u} + \left(\frac{1}{q} - q\right) \xi^{i,j}_{t,u} \xi^{k,l}_{r,s}$$

□

C.6 Quantum minor determinant relations associated with $z_{i,j}z_{k,l}$ where $i < j < k < l$

Proposition C.17. *Given the quantum minor determinants* $\xi^{i,j}_{r,s}$ *and* $\xi^{k,l}_{r,s}$ *where* $i < j < k < l$ *and* $r < s$

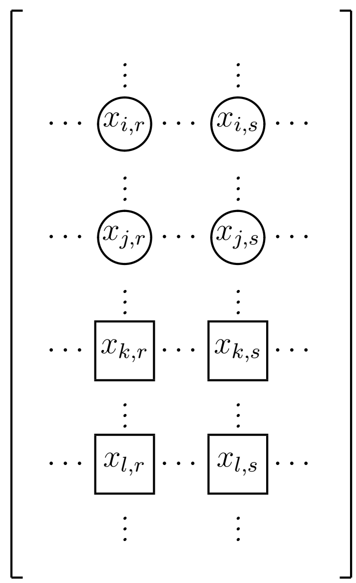

Figure C.16: Relative positions of $\xi^{i,j}_{r,s}$ *and* $\xi^{k,l}_{r,s}$

then

$$\xi^{i,j}_{r,s}\xi^{k,l}_{r,s} = \xi^{k,l}_{r,s}\xi^{i,j}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} - q\left(q - \frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{r,s} \tag{C.17}$$

Proof.

$$\begin{aligned}\xi^{i,j}_{r,s}\xi^{k,l}_{r,s} &= (x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r}) \\ &= x_{i,r}x_{j,s}x_{k,r}x_{l,s}\end{aligned}$$

$$- qx_{i,r}x_{j,s}x_{k,s}x_{l,r}$$

$$- qx_{i,s}x_{j,r}x_{k,r}x_{l,s}$$

$$+ q^2x_{i,s}x_{j,r}x_{k,s}x_{l,r}$$

$$= x_{i,r}x_{k,r}x_{j,s}x_{l,s}$$

$$- q^2x_{i,r}x_{k,s}x_{j,s}x_{l,r}$$

$$- q^2x_{i,s}x_{k,r}x_{j,r}x_{l,s}$$

$$+ q^2x_{i,s}\left[\left(q - \frac{1}{q}\right)x_{k,r}x_{j,s} + x_{k,s}x_{j,r}\right]x_{l,r}$$

$$= x_{i,r}x_{k,r}x_{j,s}x_{l,s}$$

$$- q^2x_{i,r}x_{k,s}x_{j,s}x_{l,r}$$

$$- q^2x_{i,s}x_{k,r}x_{j,r}x_{l,s}$$

$$+ q^2x_{i,s}x_{k,s}x_{j,r}x_{l,r}$$

$$+ q^2\left(q - \frac{1}{q}\right)x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$

$$= q^2x_{k,r}x_{i,r}x_{l,s}x_{j,s}$$

$$- q^2\left[\left(q - \frac{1}{q}\right)x_{k,r}x_{i,s} + x_{k,s}x_{i,r}\right]x_{l,r}x_{j,s}$$

$$- q^2x_{k,r}x_{i,s}\left[\left(q - \frac{1}{q}\right)x_{l,r}x_{j,s} + x_{l,s}x_{j,r}\right]$$

$$+ q^4x_{k,s}x_{i,s}x_{l,r}x_{j,r}$$

$$+ q^2 \left(q - \frac{1}{q}\right) x_{k,r}x_{i,s}x_{l,r}x_{j,s}$$

$$= q^2 x_{k,r}x_{i,r}x_{l,s}x_{j,s}$$
$$- q^2 x_{k,s}x_{i,r}x_{l,r}x_{j,s}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{k,r}x_{i,s}x_{l,r}x_{j,s}$$
$$- q^2 x_{k,r}x_{i,s}x_{l,s}x_{j,r}$$
$$+ q^4 x_{k,s}x_{i,s}x_{l,r}x_{j,r}$$

$$= q^2 x_{k,r} \left[\left(q - \frac{1}{q}\right) x_{l,r}x_{i,s} + x_{l,s}x_{i,r}\right] x_{j,s}$$
$$- q^3 x_{k,s}x_{l,r}x_{i,r}x_{j,s}$$
$$- q^2 \left(q - \frac{1}{q}\right) x_{k,r}x_{l,r}x_{i,s}x_{j,s}$$
$$- q^3 x_{k,r}x_{l,s}x_{i,s}x_{j,r}$$
$$+ q^4 x_{k,s}x_{l,r}x_{i,s}x_{j,r}$$

$$= q^2 x_{k,r}x_{l,s}x_{i,r}x_{j,s}$$
$$- q^3 x_{k,s}x_{l,r}x_{i,r}x_{j,s}$$
$$- q^3 x_{k,r}x_{l,s}x_{i,s}x_{j,r}$$
$$+ q^4 x_{k,s}x_{l,r}x_{i,s}x_{j,r}$$

$$= q^2 \left(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r}\right)\left(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}\right)$$

and by Lemma C.1

$$= \left(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r}\right)\left(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}\right)$$

$$+ \left(q - \frac{1}{q}\right)\left(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}\right)\left(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r}\right)$$

$$- q\left(q - \frac{1}{q}\right)\left(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r}\right)\left(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r}\right)$$

$$= \xi^{k,l}_{r,s}\xi^{i,j}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} - q\left(q - \frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{r,s}$$

□

Lemma C.1.

$$q^2\xi^{k,l}_{r,s}\xi^{i,j}_{r,s} = \xi^{k,l}_{r,s}\xi^{i,j}_{r,s} + \left(q-\frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} - q\left(q-\frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{r,s} \tag{C.18}$$

where $i<j<k<l$ and $r<s$

Proof.

$$\begin{aligned}
\left(q^2-1\right)\xi^{k,l}_{r,s}\xi^{i,j}_{r,s} = &\left(q^2-1\right)\left(x_{k,r}x_{l,s}-qx_{k,s}x_{l,r}\right)\left(x_{i,r}x_{j,s}-qx_{i,s}x_{j,r}\right)\\
= &\left(q^2-1\right)x_{k,r}x_{l,s}x_{i,r}x_{j,s}\\
&-q\left(q^2-1\right)x_{k,r}x_{l,s}x_{i,s}x_{j,r}\\
&-q\left(q^2-1\right)x_{k,s}x_{l,r}x_{i,r}x_{j,s}\\
&+q^2\left(q^2-1\right)x_{k,s}x_{l,r}x_{i,s}x_{j,r}
\end{aligned}$$

$$\begin{aligned}
&= q\left(q-\frac{1}{q}\right)x_{k,r}x_{l,s}x_{i,r}x_{j,s}\\
&\quad -q^2\left(q-\frac{1}{q}\right)x_{k,r}x_{l,s}x_{i,s}x_{j,r}\\
&\quad -q^2\left(q-\frac{1}{q}\right)x_{k,s}x_{l,r}x_{i,r}x_{j,s}\\
&\quad +q^3\left(q-\frac{1}{q}\right)x_{k,s}x_{l,r}x_{i,s}x_{j,r}
\end{aligned}$$

$$\begin{aligned}
&= q\left(q-\frac{1}{q}\right)x_{k,r}\left[\left(\frac{1}{q}-q\right)x_{i,s}x_{l,r}+x_{i,r}x_{l,s}\right]x_{j,s}\\
&\quad -q\left(q-\frac{1}{q}\right)x_{k,r}x_{i,s}x_{l,s}x_{j,r}
\end{aligned}$$

$$- q\left(q-\frac{1}{q}\right) x_{k,s}x_{i,r}x_{l,r}x_{j,s}$$
$$+ q^3\left(q-\frac{1}{q}\right) x_{k,s}x_{i,s}x_{l,r}x_{j,r}$$

$$= q\left(q-\frac{1}{q}\right)\left(\frac{1}{q}-q\right) x_{k,r}x_{i,s}x_{l,r}x_{j,s}$$
$$+ q\left(q-\frac{1}{q}\right) x_{k,r}x_{i,r}x_{l,s}x_{j,s}$$
$$- q\left(q-\frac{1}{q}\right) x_{k,r}x_{i,s}x_{l,s}x_{j,r}$$
$$- q\left(q-\frac{1}{q}\right) x_{k,s}x_{i,r}x_{l,r}x_{j,s}$$
$$+ q^3\left(q-\frac{1}{q}\right) x_{k,s}x_{i,s}x_{l,r}x_{j,r}$$

$$= q\left(q-\frac{1}{q}\right)\left(\frac{1}{q}-q\right) x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$
$$+ \frac{1}{q}\left(q-\frac{1}{q}\right) x_{i,r}x_{k,r}x_{j,s}x_{l,s}$$
$$- q\left(q-\frac{1}{q}\right) x_{i,s}x_{k,r}\left[\left(\frac{1}{q}-q\right) x_{j,s}x_{l,r} + x_{j,r}x_{l,s}\right]$$
$$- q\left(q-\frac{1}{q}\right)\left[\left(\frac{1}{q}-q\right) x_{i,s}x_{k,r} + x_{i,r}x_{k,s}\right] x_{j,s}x_{l,r}$$
$$+ q\left(q-\frac{1}{q}\right) x_{i,s}x_{k,s}x_{j,r}x_{l,r}$$

$$= \frac{1}{q}\left(q-\frac{1}{q}\right) x_{i,r}x_{k,r}x_{j,s}x_{l,s}$$
$$- q\left(q-\frac{1}{q}\right) x_{i,s}x_{k,r}x_{j,r}x_{l,s}$$

$$- q\left(q-\frac{1}{q}\right)\left(\frac{1}{q}-q\right)x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$
$$- q\left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}x_{j,s}x_{l,r}$$
$$+ q\left(q-\frac{1}{q}\right)x_{i,s}x_{k,s}x_{j,r}x_{l,r}$$

$$= \left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}x_{j,r}x_{l,s}$$
$$- q\left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}x_{j,s}x_{l,r}$$
$$- q\left(q-\frac{1}{q}\right)x_{i,s}x_{k,r}x_{j,r}x_{l,s}$$
$$+ q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$

$$- \left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}x_{j,r}x_{l,s}$$
$$- q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$

$$+ \frac{1}{q}\left(q-\frac{1}{q}\right)x_{i,r}x_{k,r}x_{j,s}x_{l,s}$$
$$- q\left(q-\frac{1}{q}\right)\left(\frac{1}{q}-q\right)x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$
$$+ q\left(q-\frac{1}{q}\right)x_{i,s}x_{k,s}x_{j,r}x_{l,r}$$

$$= \left(q-\frac{1}{q}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)\left(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r}\right)$$

$$+\frac{1}{q}\left(q-\frac{1}{q}\right)x_{i,r}x_{k,r}x_{j,s}x_{l,s}$$
$$-\left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}x_{j,r}x_{l,s}$$
$$-\left(q-\frac{1}{q}\right)x_{i,s}x_{k,r}x_{j,s}x_{l,r}$$
$$+q\left(q-\frac{1}{q}\right)x_{i,s}x_{k,s}x_{j,r}x_{l,r}$$

$$=\left(q-\frac{1}{q}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)\left(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r}\right)$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{k,r}x_{l,s}x_{j,s}$$
$$-\left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}\left[\left(q-\frac{1}{q}\right)x_{l,r}x_{j,s}+x_{l,s}x_{j,r}\right]$$
$$-\left(q-\frac{1}{q}\right)x_{i,s}x_{k,r}x_{l,r}x_{j,s}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{i,s}x_{k,s}x_{l,r}x_{j,r}$$

$$=\left(q-\frac{1}{q}\right)\left(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r}\right)\left(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r}\right)$$

$$+\left(q-\frac{1}{q}\right)x_{i,r}x_{k,r}x_{l,s}x_{j,s}$$
$$-\left(q-\frac{1}{q}\right)\left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}x_{l,r}x_{j,s}$$
$$-\left(q-\frac{1}{q}\right)x_{i,r}x_{k,s}x_{l,s}x_{j,r}$$

$$- \left(q - \frac{1}{q}\right) x_{i,s}x_{k,r}x_{l,r}x_{j,s}$$

$$+ q^2 \left(q - \frac{1}{q}\right) x_{i,s}x_{k,s}x_{l,r}x_{j,r}$$

$$= \left(q - \frac{1}{q}\right) (x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}) (x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r} \left[\left(q - \frac{1}{q}\right) x_{l,r}x_{k,s} + x_{l,s}x_{k,r}\right] x_{j,s}$$

$$- \left(q - \frac{1}{q}\right) \left(q - \frac{1}{q}\right) x_{i,r}x_{l,r}x_{k,s}x_{j,s}$$

$$- q \left(q - \frac{1}{q}\right) x_{i,r}x_{l,s}x_{k,s}x_{j,r}$$

$$- q \left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,r}x_{j,s}$$

$$+ q^2 \left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,s}x_{j,r}$$

$$= \left(q - \frac{1}{q}\right) (x_{i,r}x_{k,s} - qx_{i,s}x_{k,r}) (x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r}x_{l,s}x_{k,r}x_{j,s}$$

$$- q \left(q - \frac{1}{q}\right) x_{i,r}x_{l,s}x_{k,s}x_{j,r}$$

$$- q \left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,r}x_{j,s}$$

$$+ q^2 \left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,s}x_{j,r}$$

$$= \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r}x_{l,s}x_{k,r}x_{j,s}$$
$$- q\left(q - \frac{1}{q}\right) x_{i,r}x_{l,s}x_{k,s}x_{j,r}$$
$$- q\left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,r}x_{j,s}$$
$$+ q^2\left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,s}x_{j,r}$$

$$= \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})$$
$$+ \left(q - \frac{1}{q}\right)(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})(x_{k,r}x_{j,s} - qx_{k,s}x_{j,r})$$

$$= \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,r}x_{l,s} - qx_{j,s}x_{l,r})$$
$$- q\left(q - \frac{1}{q}\right)(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})(x_{j,r}x_{k,s} - qx_{j,s}x_{k,r})$$

$$= \left(q - \frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{j,l}_{r,s} - q\left(q - \frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{r,s}$$

□

Proposition C.18. *Given the quantum minor determinants* $\xi^{i,j}_{r,s}$ *and* $\xi^{k,l}_{t,u}$ *where* $i < j < k < l$ *and* $r < s < t < u$

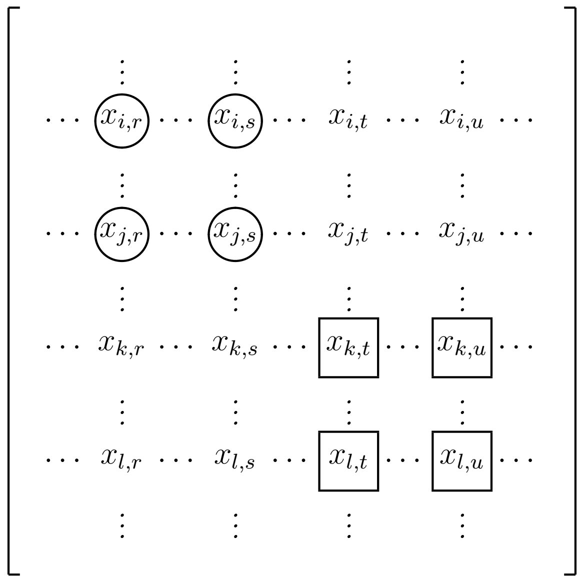

Figure C.17: Relative positions of $\xi^{i,j}_{r,s}$ *and* $\xi^{k,l}_{t,u}$

then

$$\begin{aligned}\xi^{i,j}_{r,s}\xi^{k,l}_{t,u} = \xi^{k,l}_{t,u}\xi^{i,j}_{r,s} + \left(q - \frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{j,l}_{t,u} + \left(q - \frac{1}{q}\right)\xi^{i,k}_{t,u}\xi^{j,l}_{r,s} \\ - q\left(q - \frac{1}{q}\right)\xi^{i,l}_{t,u}\xi^{j,k}_{r,s} - q\left(q - \frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{t,u} \qquad \text{(C.19)}\end{aligned}$$

Proof.

$$\begin{aligned}\xi^{i,j}_{r,s}\xi^{k,l}_{t,u} &= \left(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}\right)\left(x_{k,t}x_{l,u} - qx_{k,u}x_{l,t}\right) \\ &= x_{i,r}x_{j,s}x_{k,t}x_{l,u}\end{aligned}$$

$$- q x_{i,r} x_{j,s} x_{k,u} x_{l,t}$$

$$- q x_{i,s} x_{j,r} x_{k,t} x_{l,u}$$

$$+ q^2 x_{i,s} x_{j,r} x_{k,u} x_{l,t}$$

$$= x_{i,r} \left[\left(q - \frac{1}{q} \right) x_{k,s} x_{j,t} + x_{k,t} x_{j,s} \right] x_{l,u}$$
$$- q x_{i,r} \left[\left(q - \frac{1}{q} \right) x_{k,s} x_{j,u} + x_{k,u} x_{j,s} \right] x_{l,t}$$
$$- q x_{i,s} \left[\left(q - \frac{1}{q} \right) x_{k,r} x_{j,t} + x_{k,t} x_{j,r} \right] x_{l,u}$$
$$+ q^2 x_{i,s} \left[\left(q - \frac{1}{q} \right) x_{k,r} x_{j,u} + x_{k,u} x_{j,r} \right] x_{l,t}$$

$$= \left(q - \frac{1}{q} \right) x_{i,r} x_{k,s} x_{j,t} x_{l,u}$$
$$- q \left(q - \frac{1}{q} \right) x_{i,r} x_{k,s} x_{j,u} x_{l,t}$$
$$- q \left(q - \frac{1}{q} \right) x_{i,s} x_{k,r} x_{j,t} x_{l,u}$$
$$+ q^2 \left(q - \frac{1}{q} \right) x_{i,s} x_{k,r} x_{j,u} x_{l,t}$$

$$+ x_{i,r} x_{k,t} x_{j,s} x_{l,u}$$

$$- q x_{i,r} x_{k,u} x_{j,s} x_{l,t}$$

$$- q x_{i,s} x_{k,t} x_{j,r} x_{l,u}$$

$$+ q^2 x_{i,s} x_{k,u} x_{j,r} x_{l,t}$$

$$= \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})$$

$$+ x_{i,r}x_{k,t}x_{j,s}x_{l,u}$$

$$- qx_{i,r}x_{k,u}x_{j,s}x_{l,t}$$

$$- qx_{i,s}x_{k,t}x_{j,r}x_{l,u}$$

$$+ q^2x_{i,s}x_{k,u}x_{j,r}x_{l,t}$$

$$= \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})$$

$$\begin{aligned}
&+ \left[\left(q - \frac{1}{q}\right)x_{i,t}x_{k,r} + x_{k,t}x_{i,r}\right]\left[\left(q - \frac{1}{q}\right)x_{j,u}x_{l,s} + x_{l,u}x_{j,s}\right] \\
&- q\left[\left(q - \frac{1}{q}\right)x_{i,u}x_{k,r} + x_{k,u}x_{i,r}\right]\left[\left(q - \frac{1}{q}\right)x_{j,t}x_{l,s} + x_{l,t}x_{j,s}\right] \\
&- q\left[\left(q - \frac{1}{q}\right)x_{i,t}x_{k,s} + x_{k,t}x_{i,s}\right]\left[\left(q - \frac{1}{q}\right)x_{j,u}x_{l,r} + x_{l,u}x_{j,r}\right] \\
&+ q^2\left[\left(q - \frac{1}{q}\right)x_{i,u}x_{k,s} + x_{k,u}x_{i,s}\right]\left[\left(q - \frac{1}{q}\right)x_{j,t}x_{l,r} + x_{l,t}x_{j,r}\right]
\end{aligned}$$

$$= \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})$$

$$+ x_{k,t}x_{i,r}x_{l,u}x_{j,s}$$

$$- qx_{k,u}x_{i,r}x_{l,t}x_{j,s}$$

$$- qx_{k,t}x_{i,s}x_{l,u}x_{j,r}$$

$$+ q^2x_{k,u}x_{i,s}x_{l,t}x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{k,t}x_{i,r}x_{j,u}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{k,u}x_{i,r}x_{j,t}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{k,t}x_{i,s}x_{j,u}x_{l,r}$$

$$+ q^2\left(q - \frac{1}{q}\right) x_{k,u}x_{i,s}x_{j,t}x_{l,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,t}x_{k,r}x_{l,u}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,u}x_{k,r}x_{l,t}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,t}x_{k,s}x_{l,u}x_{j,r}$$

$$+ q^2\left(q - \frac{1}{q}\right) x_{i,u}x_{k,s}x_{l,t}x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right)^2 x_{i,t}x_{k,r}x_{j,u}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right)^2 x_{i,u}x_{k,r}x_{j,t}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right)^2 x_{i,t}x_{k,s}x_{j,u}x_{l,r}$$

$$+ q^2\left(q - \frac{1}{q}\right)^2 x_{i,u}x_{k,s}x_{j,t}x_{l,r}$$

$$= \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})$$

$$+ x_{k,t}\left[\left(q - \frac{1}{q}\right)x_{i,u}x_{l,r} + x_{l,u}x_{i,r}\right]x_{j,s}$$

$$- qx_{k,u}\left[\left(q - \frac{1}{q}\right)x_{i,t}x_{l,r} + x_{l,t}x_{i,r}\right]x_{j,s}$$

$$- qx_{k,t}\left[\left(q - \frac{1}{q}\right)x_{i,u}x_{l,s} + x_{l,u}x_{i,s}\right]x_{j,r}$$

$$+ q^2x_{k,u}\left[\left(q - \frac{1}{q}\right)x_{i,t}x_{l,s} + x_{l,t}x_{i,s}\right]x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right)x_{k,t}x_{i,r}x_{j,u}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right)x_{k,u}x_{i,r}x_{j,t}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right)x_{k,t}x_{i,s}x_{j,u}x_{l,r}$$

$$+ q^2\left(q - \frac{1}{q}\right)x_{k,u}x_{i,s}x_{j,t}x_{l,r}$$

$$+ \left(q - \frac{1}{q}\right)x_{i,t}x_{k,r}x_{l,u}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right)x_{i,u}x_{k,r}x_{l,t}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right)x_{i,t}x_{k,s}x_{l,u}x_{j,r}$$

$$+ q^2\left(q - \frac{1}{q}\right)x_{i,u}x_{k,s}x_{l,t}x_{j,r}$$

$$+\left(q-\frac{1}{q}\right)^2 x_{i,t}x_{k,r}x_{j,u}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)^2 x_{i,u}x_{k,r}x_{j,t}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)^2 x_{i,t}x_{k,s}x_{j,u}x_{l,r}$$
$$+q^2\left(q-\frac{1}{q}\right)^2 x_{i,u}x_{k,s}x_{j,t}x_{l,r}$$

$$=\left(q-\frac{1}{q}\right)(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r})(x_{j,t}x_{l,u}-qx_{j,u}x_{l,t})$$

$$+x_{k,t}x_{l,u}x_{i,r}x_{j,s}$$
$$-qx_{k,u}x_{l,t}x_{i,r}x_{j,s}$$
$$-qx_{k,t}x_{l,u}x_{i,s}x_{j,r}$$
$$+q^2x_{k,u}x_{l,t}x_{i,s}x_{j,r}$$

$$+\left(q-\frac{1}{q}\right)x_{k,t}x_{i,u}x_{l,r}x_{j,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{k,u}x_{i,t}x_{l,r}x_{j,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{k,t}x_{i,u}x_{l,s}x_{j,r}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{k,u}x_{i,t}x_{l,s}x_{j,r}$$

$$
\begin{aligned}
&+\left(q-\frac{1}{q}\right) x_{k,t}x_{i,r}x_{j,u}x_{l,s}\\
&-q\left(q-\frac{1}{q}\right) x_{k,u}x_{i,r}x_{j,t}x_{l,s}\\
&-q\left(q-\frac{1}{q}\right) x_{k,t}x_{i,s}x_{j,u}x_{l,r}\\
&+q^2\left(q-\frac{1}{q}\right) x_{k,u}x_{i,s}x_{j,t}x_{l,r}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(q-\frac{1}{q}\right) x_{i,t}x_{k,r}x_{l,u}x_{j,s}\\
&-q\left(q-\frac{1}{q}\right) x_{i,u}x_{k,r}x_{l,t}x_{j,s}\\
&-q\left(q-\frac{1}{q}\right) x_{i,t}x_{k,s}x_{l,u}x_{j,r}\\
&+q^2\left(q-\frac{1}{q}\right) x_{i,u}x_{k,s}x_{l,t}x_{j,r}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(q-\frac{1}{q}\right)^2 x_{i,t}x_{k,r}x_{j,u}x_{l,s}\\
&-q\left(q-\frac{1}{q}\right)^2 x_{i,u}x_{k,r}x_{j,t}x_{l,s}\\
&-q\left(q-\frac{1}{q}\right)^2 x_{i,t}x_{k,s}x_{j,u}x_{l,r}\\
&+q^2\left(q-\frac{1}{q}\right)^2 x_{i,u}x_{k,s}x_{j,t}x_{l,r}
\end{aligned}
$$

$$
\begin{aligned}
=\ &(x_{k,t}x_{l,u}-qx_{k,u}x_{l,t})\,(x_{i,r}x_{j,s}-qx_{i,s}x_{j,r})\\
&+\left(q-\frac{1}{q}\right)(x_{i,r}x_{k,s}-qx_{i,s}x_{k,r})(x_{j,t}x_{l,u}-qx_{j,u}x_{l,t})
\end{aligned}
$$

$$+\left(q-\frac{1}{q}\right)\left(x_{k,t}x_{i,u}-qx_{k,u}x_{i,t}\right)\left(x_{l,r}x_{j,s}-qx_{l,s}x_{j,r}\right)$$

$$+\left(q-\frac{1}{q}\right)x_{k,t}x_{i,r}x_{j,u}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{k,u}x_{i,r}x_{j,t}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{k,t}x_{i,s}x_{j,u}x_{l,r}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{k,u}x_{i,s}x_{j,t}x_{l,r}$$

$$+\left(q-\frac{1}{q}\right)x_{i,t}x_{k,r}x_{l,u}x_{j,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{i,u}x_{k,r}x_{l,t}x_{j,s}$$
$$-q\left(q-\frac{1}{q}\right)x_{i,t}x_{k,s}x_{l,u}x_{j,r}$$
$$+q^2\left(q-\frac{1}{q}\right)x_{i,u}x_{k,s}x_{l,t}x_{j,r}$$

$$+\left(q-\frac{1}{q}\right)^2x_{i,t}x_{k,r}x_{j,u}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)^2x_{i,u}x_{k,r}x_{j,t}x_{l,s}$$
$$-q\left(q-\frac{1}{q}\right)^2x_{i,t}x_{k,s}x_{j,u}x_{l,r}$$
$$+q^2\left(q-\frac{1}{q}\right)^2x_{i,u}x_{k,s}x_{j,t}x_{l,r}$$

$$= (x_{k,t}x_{l,u} - qx_{k,u}x_{l,t})(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})$$
$$+ \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})$$
$$- q\left(q - \frac{1}{q}\right)(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})(x_{l,r}x_{j,s} - qx_{l,s}x_{j,r})$$

$$+ \left(q - \frac{1}{q}\right)\left[\left(\frac{1}{q} - q\right)x_{i,t}x_{k,r} + x_{i,r}x_{k,t}\right]x_{j,u}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right)\left[\left(\frac{1}{q} - q\right)x_{i,u}x_{k,r} + x_{i,r}x_{k,u}\right]x_{j,t}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right)\left[\left(\frac{1}{q} - q\right)x_{i,t}x_{k,s} + x_{i,s}x_{k,t}\right]x_{j,u}x_{l,r}$$
$$+ q^2\left(q - \frac{1}{q}\right)\left[\left(\frac{1}{q} - q\right)x_{i,u}x_{k,s} + x_{i,s}x_{k,u}\right]x_{j,t}x_{l,r}$$

$$+ \left(q - \frac{1}{q}\right)x_{i,t}\left[\left(q - \frac{1}{q}\right)x_{l,r}x_{k,u} + x_{l,u}x_{k,r}\right]x_{j,s}$$
$$- q\left(q - \frac{1}{q}\right)x_{i,u}\left[\left(q - \frac{1}{q}\right)x_{l,r}x_{k,t} + x_{l,t}x_{k,r}\right]x_{j,s}$$
$$- q\left(q - \frac{1}{q}\right)x_{i,t}\left[\left(q - \frac{1}{q}\right)x_{l,s}x_{k,u} + x_{l,u}x_{k,s}\right]x_{j,r}$$
$$+ q^2\left(q - \frac{1}{q}\right)x_{i,u}\left[\left(q - \frac{1}{q}\right)x_{l,s}x_{k,t} + x_{l,t}x_{k,s}\right]x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right)^2 x_{i,t}x_{k,r}x_{j,u}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right)^2 x_{i,u}x_{k,r}x_{j,t}x_{l,s}$$
$$- q\left(q - \frac{1}{q}\right)^2 x_{i,t}x_{k,s}x_{j,u}x_{l,r}$$

$$+ q^2 \left(q - \frac{1}{q}\right)^2 x_{i,u}x_{k,s}x_{j,t}x_{l,r}$$

$$= (x_{k,t}x_{l,u} - qx_{k,u}x_{l,t})(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})$$

$$+ \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})$$

$$- q\left(q - \frac{1}{q}\right)(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})(x_{l,r}x_{j,s} - qx_{l,s}x_{j,r})$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r}x_{k,t}x_{j,u}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,r}x_{k,u}x_{j,t}x_{l,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,s}x_{k,t}x_{j,u}x_{l,r}$$

$$+ q^2\left(q - \frac{1}{q}\right) x_{i,s}x_{k,u}x_{j,t}x_{l,r}$$

$$+ \left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,t}x_{l,r}x_{k,u}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,u}x_{l,r}x_{k,t}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,t}x_{l,s}x_{k,u}x_{j,r}$$

$$+ q^2\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,u}x_{l,s}x_{k,t}x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,t}x_{l,u}x_{k,r}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,u}x_{l,t}x_{k,r}x_{j,s}$$

$$- q\left(q - \frac{1}{q}\right) x_{i,t}x_{l,u}x_{k,s}x_{j,r}$$
$$+ q^2\left(q - \frac{1}{q}\right) x_{i,u}x_{l,t}x_{k,s}x_{j,r}$$

$$= (x_{k,t}x_{l,u} - qx_{k,u}x_{l,t})(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r})$$
$$+ \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})$$
$$- q\left(q - \frac{1}{q}\right)(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})(x_{l,r}x_{j,s} - qx_{l,s}x_{j,r})$$

$$+ \left(q - \frac{1}{q}\right) x_{i,r}x_{l,s}x_{k,t}x_{j,u}$$
$$- q\left(q - \frac{1}{q}\right) x_{i,r}x_{l,s}x_{k,u}x_{j,t}$$
$$- q\left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,t}x_{j,u}$$
$$+ q^2\left(q - \frac{1}{q}\right) x_{i,s}x_{l,r}x_{k,u}x_{j,t}$$

$$+ \left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,t}x_{k,u}x_{l,r}x_{j,s}$$
$$- q\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,u}x_{k,t}x_{l,r}x_{j,s}$$
$$- q\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,t}x_{k,u}x_{l,s}x_{j,r}$$
$$+ q^2\left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right) x_{i,u}x_{k,t}x_{l,s}x_{j,r}$$

$$+ \left(q - \frac{1}{q}\right) x_{i,t}x_{l,u}x_{k,r}x_{j,s}$$

$$
\begin{aligned}
& - q\left(q - \frac{1}{q}\right) x_{i,u}x_{l,t}x_{k,r}x_{j,s} \\
& - q\left(q - \frac{1}{q}\right) x_{i,t}x_{l,u}x_{k,s}x_{j,r} \\
& + q^2\left(q - \frac{1}{q}\right) x_{i,u}x_{l,t}x_{k,s}x_{j,r}
\end{aligned}
$$

$$
\begin{aligned}
= \ & (x_{k,t}x_{l,u} - qx_{k,u}x_{l,t})(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}) \\
& + \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t}) \\
& - q\left(q - \frac{1}{q}\right)(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})(x_{l,r}x_{j,s} - qx_{l,s}x_{j,r}) \\
& + \left(q - \frac{1}{q}\right)(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})(x_{k,t}x_{j,u} - qx_{k,u}x_{j,t}) \\
& + \left(q - \frac{1}{q}\right)\left(q - \frac{1}{q}\right)(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})(x_{l,r}x_{j,s} - qx_{l,s}x_{j,r}) \\
& + \left(q - \frac{1}{q}\right)(x_{i,t}x_{l,u} - qx_{i,u}x_{l,t})(x_{k,r}x_{j,s} - qx_{k,s}x_{j,r}) \\
= \ & (x_{k,t}x_{l,u} - qx_{k,u}x_{l,t})(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}) \\
& + \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t}) \\
& - \frac{1}{q}\left(q - \frac{1}{q}\right)(x_{i,t}x_{k,u} - qx_{i,u}x_{k,t})(x_{l,r}x_{j,s} - qx_{l,s}x_{j,r}) \\
& + \left(q - \frac{1}{q}\right)(x_{i,r}x_{l,s} - qx_{i,s}x_{l,r})(x_{k,t}x_{j,u} - qx_{k,u}x_{j,t}) \\
& + \left(q - \frac{1}{q}\right)(x_{i,t}x_{l,u} - qx_{i,u}x_{l,t})(x_{k,r}x_{j,s} - qx_{k,s}x_{j,r}) \\
= \ & (x_{k,t}x_{l,u} - qx_{k,u}x_{l,t})(x_{i,r}x_{j,s} - qx_{i,s}x_{j,r}) \\
& + \left(q - \frac{1}{q}\right)(x_{i,r}x_{k,s} - qx_{i,s}x_{k,r})(x_{j,t}x_{l,u} - qx_{j,u}x_{l,t})
\end{aligned}
$$

$$+\left(q-\frac{1}{q}\right)(x_{i,t}x_{k,u}-qx_{i,u}x_{k,t})(x_{j,r}x_{l,s}-qx_{j,s}x_{l,r})$$
$$-q\left(q-\frac{1}{q}\right)(x_{i,t}x_{l,u}-qx_{i,u}x_{l,t})(x_{j,r}x_{k,s}-qx_{j,s}x_{k,r})$$
$$-q\left(q-\frac{1}{q}\right)(x_{i,r}x_{l,s}-qx_{i,s}x_{l,r})(x_{j,t}x_{k,u}-qx_{j,u}x_{k,t})$$

$$=\xi^{k,l}_{t,u}\xi^{i,j}_{r,s}+\left(q-\frac{1}{q}\right)\xi^{i,k}_{r,s}\xi^{j,l}_{t,u}+\left(q-\frac{1}{q}\right)\xi^{i,k}_{t,u}\xi^{j,l}_{r,s}$$
$$-q\left(q-\frac{1}{q}\right)\xi^{i,l}_{t,u}\xi^{j,k}_{r,s}-q\left(q-\frac{1}{q}\right)\xi^{i,l}_{r,s}\xi^{j,k}_{t,u}$$

□

Proposition C.19. *Given the quantum minor determinants $\xi^{i,j}_{t,u}$ and $\xi^{k,l}_{r,s}$ where $i < j < k < l$ and $r < s < t < u$*

$$\begin{bmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & x_{i,r} & \cdots & x_{i,s} & \cdots & \enclose{circle}{x_{i,t}} & \cdots & \enclose{circle}{x_{i,u}} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & x_{j,r} & \cdots & x_{j,s} & \cdots & \enclose{circle}{x_{j,t}} & \cdots & \enclose{circle}{x_{j,u}} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \boxed{x_{k,r}} & \cdots & \boxed{x_{k,s}} & \cdots & x_{k,t} & \cdots & x_{k,u} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \boxed{x_{l,r}} & \cdots & \boxed{x_{l,s}} & \cdots & x_{l,t} & \cdots & x_{l,u} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{bmatrix}$$

Figure C.18: Relative positions of $\xi^{i,j}_{t,u}$ and $\xi^{k,l}_{r,s}$

then

$$\xi^{i,j}_{t,u}\xi^{k,l}_{r,s} = \xi^{k,l}_{r,s}\xi^{i,j}_{t,u} \tag{C.20}$$

Proof.

$$\begin{aligned} \xi^{i,j}_{t,u}\xi^{k,l}_{r,s} = &\ (x_{i,t}x_{j,u} - qx_{i,u}x_{j,t})(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r}) \\ = &\ x_{i,t}x_{j,u}x_{k,r}x_{l,s} \\ &- qx_{i,t}x_{j,u}x_{k,s}x_{l,r} \\ &- qx_{i,u}x_{j,t}x_{k,r}x_{l,s} \\ &+ q^2x_{i,u}x_{j,t}x_{k,s}x_{l,r} \end{aligned}$$

$$= x_{i,t}x_{k,r}x_{j,u}x_{l,s}$$

$$- qx_{i,t}x_{k,s}x_{j,u}x_{l,r}$$

$$- qx_{i,u}x_{k,r}x_{j,t}x_{l,s}$$

$$+ q^2x_{i,u}x_{k,s}x_{j,t}x_{l,r}$$

$$=x_{k,r}x_{i,t}x_{l,s}x_{j,u}$$

$$- qx_{k,s}x_{i,t}x_{l,r}x_{j,u}$$

$$- qx_{k,r}x_{i,u}x_{l,s}x_{j,t}$$

$$+ q^2x_{k,s}x_{i,u}x_{l,r}x_{j,t}$$

$$= x_{k,r}x_{l,s}x_{i,t}x_{j,u}$$

$$- qx_{k,s}x_{l,r}x_{i,t}x_{j,u}$$

$$- qx_{k,r}x_{l,s}x_{i,u}x_{j,t}$$

$$+ q^2x_{k,s}x_{l,r}x_{i,u}x_{j,t}$$

$$= \left(x_{k,r}x_{l,s} - qx_{k,s}x_{l,r}\right)\left(x_{i,t}x_{j,u} - qx_{i,u}x_{j,t}\right)$$

$$= \xi^{k,l}_{r,s}\xi^{i,j}_{t,u}$$

□

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