MA421 Proofs

In the following proofs I have attempted to keep the same arguments that you had presented in class.

1. Using binomial coefficients show that the number of subsets of a finite set A is 2^n where n is the number of elements in A. (I.e. if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$).

Proof: (by Kelsey Bashaw)

If $|A| = n$

$\Rightarrow$ For a given set A there are $\binom{n}{k}$ subsets of A of size k where $0 \leq k \leq n$.

$$\Rightarrow \quad |\mathcal{P}(A)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$$

$|\mathcal{P}(A)|$ is the power set of A

From Theorem 1.9 we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

Now if we let $x = 1$ and $y = 1$ we have

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$$

$$\Rightarrow \quad 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-kk} = \sum_{k=0}^{n} \binom{n}{k}$$

$$\therefore \quad |\mathcal{P}(A)| = 2^n$$

2. For any positive integer n and $r = 1, 2, ..., n-1$, $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

Proof: (by Daniel Burke)

Write both sides in factorial form:

$$\frac{n!}{(n-r)!r!} = \frac{(n-1)!}{((n-1)-r)!r!} + \frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!}$$

$$\Leftrightarrow \quad \frac{n!}{(n-r)!r!} = \frac{(n-1)!}{((n-1)-r)!r!} + \frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!}$$

After cleaning up parenthesis on RHS

$$\Leftrightarrow \quad \frac{n!}{(n-r)!r!} = \frac{(n-1)!}{((n-1)-r)!r!} \cdot \frac{n-r}{n-r} + \frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!} \cdot \frac{r}{r}$$

Find common denominator

$$\Leftrightarrow \quad \frac{n!}{(n-r)!r!} = \frac{(n-1)!}{((n-1)-r)!r!} \cdot \frac{n-r}{n-r} + \frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!} \cdot \frac{r}{r}$$

Simplify using

$(r-1)!r = r!$ and

$(n-r-1)!(n-r) = (n-r)!$

$$\Leftrightarrow \quad \frac{n!}{(n-r)!r!} = \frac{(n-1)!(n-r)}{(n-r)!r!} + \frac{(n-1)!r}{(n-r)!r!}$$

$$\Leftrightarrow \quad \frac{n!}{(n-r)!r!} = \frac{(n-1)!(n-r+r)}{(n-r)!r!}$$

$$\Leftrightarrow \quad \frac{n!}{(n-r)!r!} = \frac{(n-1)!n}{(n-r)!r!}$$

More simplification

$$\Leftrightarrow \quad \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!r!}$$

$$\therefore \quad \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

3. Prove $\displaystyle\sum_{r=0}^{n}\binom{n}{r}^2 = \binom{2n}{n}$. Where $n \geq 0$.

Proof: (by Mat Fazzari)

$$\sum_{r=0}^{n}\binom{n}{r}^2$$
$$= \sum_{r=0}^{n}\binom{n}{r}\binom{n}{r}$$
$$= \sum_{r=0}^{n}\binom{n}{r}\binom{n}{n-r}$$

From Theorem 1.12 we have $\displaystyle\sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}$

so if we let $m = n = k$ we get

$$\sum_{r=0}^{n}\binom{n}{r}\binom{n}{n-r} = \binom{n+n}{n} = \binom{2n}{n}$$
$$\therefore \sum_{r=0}^{n}\binom{n}{r}^2 = \binom{2n}{n}$$

4. Let X and Y be continuous random variables with uniform density over $[0,1]$. i.e.

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases} \quad and \quad g(y) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

If X and $Y$are independent, find $P\left(\sqrt{(X-Y)^2} \leq \frac{1}{n}\right)$ where n is a positive integer.

5. If $P(A|B) < P(A)$ then $P(B|A) < P(B)$.

6. Show that $P(A \cup B) \geq 1 - P(A') - P(B')$ for any two events A and B in the sample space S.

7. If X is a random variable with distribution f and distribution function F, show that $F(-\infty) = 0$ where we define $F(-\infty) = \lim_{x \to -\infty} \sum_{t \leq x} f(t)$.

8. Show that if X and Y are continuous random variables with a joint density $f(x, y)$ then the conditional density of X given that $Y = a$ is indeed a density. (i.e. show that $u(x|Y = a)$ in definition 3.13 is a density.)

9. Let X and Y be continuous random variables with a joint density $f(x, y)$. Show that if X and Y are independent, then the condition density of X given $Y = y$ is simply the marginal density of X. (I.e. $u(x|Y = x) = g(x)$)

10. Let X be a discrete random variable with expected value $E[X] = \mu$. Show that $var(X) = E\left[X^2\right] - E[X]^2$.

11. If the random variable X has the variance σ^2, then $var\,(aX + b) = a^2\sigma^2$.

12. Prove $\dfrac{d^r M_X(t)}{dt^r} = \mu_r^{'}$ where $M_X(t)$ is the moment generating function of X.

13. If a and b are constants and $M_X(t)$ is the moment generating function for a random variable X then show (with referring to parts 1 and 2 of Theorem 4.10) that $M_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}} M_X\left(\dfrac{t}{a}\right)$.

14. We define the **characteristic function** of a distribution or density by $\varphi_X(t) = E\left[e^{itX}\right]$, where $i = \sqrt{-1}$. Show that $\varphi_X(t)$ exists for every density.

15. If X has characteristic function $\varphi_X(t)$ and both $E[X]$ and $E\left[X^2\right]$ exist then find expressions for $E[X]$ and $E\left[X^2\right]$ in terms of $\varphi_X(t)$.

16. Show for the continuous case that if X and Y are independent variables with joint density $f(x,y)$ then $cov\,[X,Y] = 0$.

17. Prove that the variance of the binomial distribution is $\sigma^2 = n\theta(1-\theta)$.

18. Prove that the mean of the negative binomial distribution is, $\dfrac{k}{\theta}$.

19. Prove that the moment generating function of the Poisson distribution is $M_X(t) = e^{\lambda\left(e^t - 1\right)}$ then show that the mean of the Poisson distribution is λ and the variance of the Poisson distribution is λ.