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A Capacitor Analogy, Part 2

Welcome to the 11th article in the "Circuit Intuitions" column series. As the title suggests, each article provides insights and intuitions into circuit design and analysis. These articles are aimed at undergraduate students but may serve the interests of other readers as well. If you read this article, I would appreciate your comments and feedback as well as your requests and suggestions for future articles in this series. Please e-mail your comments to me at ali@ece.utoronto.ca.

In the previous article in this series, "A Capacitor Analogy, Part 1," we described how a glass of water with cross-section area C and water height V is analogous to a capacitor with capacitance C and voltage V across the capacitor. In particular, we showed how the energy stored and wasted during the process of charging the capacitor are equal, respectively, to the potential energy stored and wasted during the process of filling the glass with water. In this column, we take this analogy one step further to see if it could solve a classic problem in charge sharing between two capacitors. We state the problem first, build and solve its analogous problem using a glass of water, and then return to the original problem.

Consider two identical capacitors, one charged to V and one charged to zero (fully discharged) with a switch between them as shown in Figure 1. The internal resistance of the switch and the wiring resistance are all lumped into a single resistor R. There are two questions (often asked in job interviews) that we would like to answer here: 1) what is the total energy stored in the capacitors before and after the switch is closed? and 2) where does the difference go?

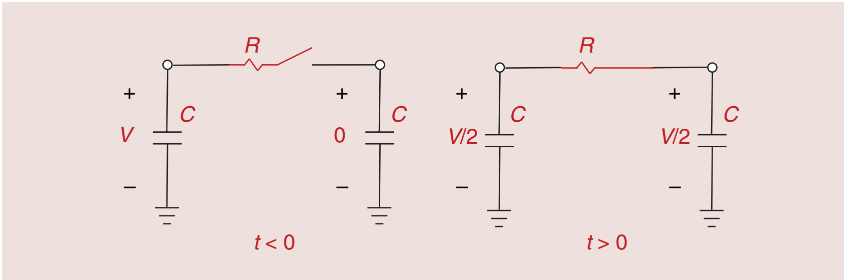

FIGURE 1: A capacitor charged to initial voltage V shares its charge with a discharged capacitor of equal capacitance via a switch.

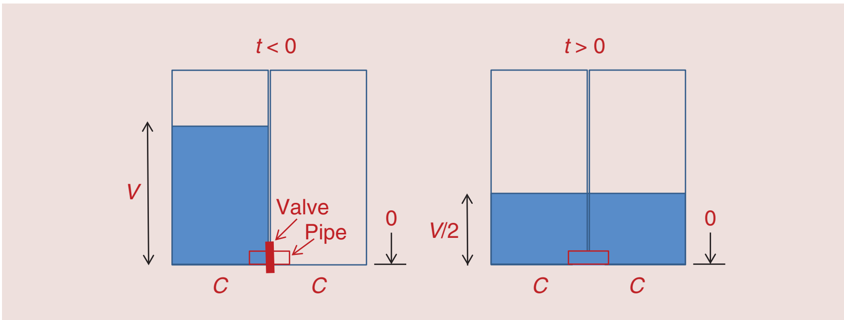

FIGURE 2: A glass with a cross-section area C and water height V shares its water with an empty glass of same size via a pipe and a valve.

To answer these questions, we first build an analogous problem using two glasses, a pipe, and a valve, as shown symbolically in Figure 2. Like a practical switch, the valve and the pipe represent an ideal switch and a series resistance, respectively. The valve and the pipe are assumed to hold no water, so as not to contribute to total capacitance, but allow the water to flow between the two glasses when the valve is open (similar to a switch being closed) and block the water flow when the valve is closed (similar to a switch being open). We will refer to open and closed valves as being ON and OFF, respectively, to be consistent with the terminology being used for an electrical switch. While the valve is OFF, we fill one glass with water to height V while keeping the other glass empty. If we now turn the valve ON, the water will flow from the full glass to the empty one until the water level is the same in both glasses. One can easily see that, in this process, the total amount of water is the same

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before and after we open the valve, but we have now created an effective glass with twice the cross-section area. This will result in a new water height that is half of the initial height $V/2$. This situation is identical to that of two capacitors. The total amount of charge (CV) will remain the same before and after we close the switch. However, the equivalent capacitance is $2C$ after we close the switch. Therefore, the voltage across the capacitors will settle to $V/2$.

What about the energy before and after the valve is turned ON? Before turning the valve ON, all the potential energy is stored in the left glass. We can write

$$E_{\text{before}} = C \times V \times V/2 + 0 = CV^2/2.$$

After turning the valve ON, we can write

$$E_{\text{after}} = 2C \times V/2 \times V/4 = CV^2/4,$$

where E_{before} and E_{after} represent the total potential energy stored in the system before and long after we turn on the valve. We have also assumed unity water density (ρ) and gravity (g). These same equations are also valid for the case of two capacitors. In both cases (see Figure 3), we lose half of the initial energy. The question is where the energy is lost. To answer this question, the reader is encouraged to think of an answer for the case of two glasses first and see if the answer can be extended to the case of two capacitors.

Both the glass and the pipe walls present friction to the flow of water from one glass to the other. When we first turn ON the valve, part of the potential energy is turned to kinetic energy, moving water from the left glass to the right one. The kinetic energy, however, is turned to heat as the water flow will have friction with the walls of the pipe and the glass. Once the water level is settled to its final value, all the kinetic energy is turned to heat, warming up the water and the glass accordingly. What is interesting in this case is that the total energy lost to heat is always half of the total initial energy, independent of the pipe diameter and the glass surface roughness (which contributes to its resistance to water movement). If the overall resistance to water movement is larger, it will take longer for the water to settle to its final height, but the energy will be lost at a lower rate. If the resistance is smaller, the water will move faster, but the rate of energy loss will be higher. Again, in all cases, exactly half of the initial energy is lost in the process of water sharing.

Two Capacitors (Charge Sharing)	Equation	Two Glasses (Water Sharing)
Total Charge at $t < 0$	$CV + 0$	Total Weight at $t < 0$
Total Charge at $t > 0$	$CV/2 + CV/2$	Total Weight at $t > 0$
Final Voltage at $t > 0$	$V/2$	Water Height at $t > 0$
Total Energy at $t < 0$	½ CV^2	Potential Energy at $t < 0$
Total Energy at $t > 0$	¼ CV^2	Potential Energy at $t > 0$

FIGURE 3: The equations governing the charge sharing between two capacitors are identical to those governing the water sharing between two glasses of water. We have assumed unity water density (ρ) and gravity (g) in writing these equations.

Let us now return to the case of two capacitors. Similar to the valve and the pipe, the switch and its associated resistance will impede the flow of charge from one capacitor to the other. The energy lost in this process is the energy dissipated as heat in the resistor. To see this quantitatively, let us write an equation for the current flowing from the left capacitor to the right. For $t > 0$, we have

$$i_R(t) = \frac{V}{R} e^{\frac{-2t}{RC}}.$$

And hence the power being dissipated in the resistor can be written as

$$p_R(t) = \frac{V^2}{R} e^{\frac{-4t}{RC}}.$$

These equations show that the lower the R, the larger the initial current and the initial power (proportional to $1/R$), but the smaller the time constant (proportional to R). Since the area under a decaying exponential curve is the product of its initial value and its time constant, the energy dissipated, which is the area under $p_R(t)$, will be independent of R:

$$E_R = \frac{V^2}{R} \times \frac{RC}{4} = \frac{1}{4} CV^2.$$

This equation confirms that the energy wasted in the process of charge sharing is independent of the switch resistance.

An interesting exercise to contemplate here is the case where R is exactly zero. Clearly, there will be no heat dissipation in this case as there is no resistance. What happens then? Will half of the initial stored energy be wasted again in the process of charge sharing? If yes, where is it wasted and in what form? The reader is encouraged to resort back to our analogy again and see if a similar case can be built in the analogous world and if an answer may emerge. In the interest of giving the readers a chance to explore these questions on their own, we will answer these questions in a future article.

In summary, the process of charge sharing of two capacitors using a switch is analogous to the process of water sharing among two adjacent glasses using a valve and a pipe. In both cases, where the two capacitors (glasses) have equal capacitances (cross-section areas), exactly half of the initial stored (potential) energy is lost in the process of charge (water) sharing. This loss is independent of the switch resistance (pipe diameter and friction).

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