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A Capacitor Analogy, Part 3

Welcome to the 12th article in the "Circuit Intuitions" column series. As the title suggests, each article provides insights and intuitions into circuit design and analysis. These articles are aimed at undergraduate students but may serve the interests of other readers as well. If you read this article, I would appreciate your comments and feedback, as well as your requests and suggestions for future columns in this series. Please e-mail your comments to me at ali@ece.utoronto.ca

In the previous two articles in this series, "A Capacitor Analogy, Part 1" and "A Capacitor Analogy, Part 2," we described how a glass of water with cross-section area C and water height V is analogous to a capacitor with capacitance C and voltage V across the capacitor. At the end of Part 2, we asked readers to contemplate the process of charge sharing between two capacitors, as shown in Figure 1, when the switch is ideal (i.e., has zero resistance) to see if the energy wasted in this process is still half the initial stored energy. Also, we asked if the water sharing between two glasses follows the same process, i.e., has a similar solution. This article focuses on providing answers to these questions and exploring a similar problem in our glass of water analogy.

An ideal switch is characterized by the following equations:

$$\begin{cases} i = 0 & \text{when the switch is open} \\ v = 0 & \text{when the switch is closed} \end{cases}.$$

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Because the switch is either open or closed, it appears that either the current through the switch or the voltage across it is always zero. Consequently, it may be concluded that the power consumed by the switch is always zero!

This is, indeed, true for $t < 0$ (when the switch is open) and for $t > 0$ (when the switch is closed), but not for $t = 0$ (when we turn the switch on). To see this, let us begin by identifying an equation for the voltage across the switch as a function of time $v_s(t)$. For $t < 0$, we already know that $v_s(t) = v$. For $t > 0$, $v_s(t) = 0$ because the switch is closed. If we assume $v_s(t)$ is halfway between these voltages at $t = 0$, then we can write an equation for $v_s(t)$ as follows:

$$v_s(t) = \frac{V}{2}(1 - \operatorname{sgn}(t)),$$

where $\operatorname{sgn}(t)$ is defined as

$$\operatorname{sgn}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ +1 & t > 0 \end{cases}.$$

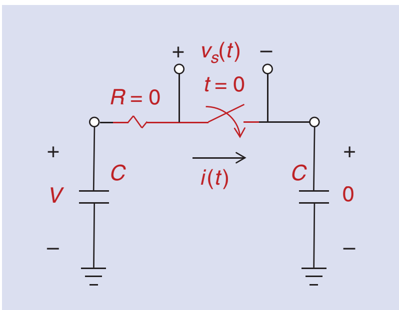

FIGURE 1: A capacitor charged to initial voltage V shares its charge with a discharged capacitor of equal capacitance via a switch with zero resistance.

Given $v_s(t)$, we can now write an equation for the current through the switch. Because the switch is placed in series with the two series capacitors (with an equivalent capacitance $C/2$), its current is the same as those of the two capacitors. Thus, we can write

$$i(t) = -\frac{C dv_s}{2dt} = \frac{C}{2} V\delta(t),$$

where $\delta(t)$ represents the Dirac delta function. We can now write an expression for the instantaneous power consumption of the switch $p(t), v_s(t)\, i(t)$, as follows:

$$p(t) = \frac{C}{4} V^2 (1 - \operatorname{sgn}(t))\delta(t).$$

This power is clearly zero for any time before and after zero, but it is infinity at $t = 0$. However, the integral of this power, which provides the energy consumption of the ideal switch in this circuit, is well defined. We can write

$$E_{\text{switch}} = \int_{-\infty}^{+\infty} p(t)\, dt = \frac{1}{4} CV^2.$$

This equation confirms that an ideal switch consumes half of the energy initially stored in the capacitor.

Let us now return to our analogy and see what happens if we use an ideal pipe and valve in water sharing between the two glasses. First, we need to imagine ways to reduce the friction, perhaps by using polished glasses with no surface roughness and by increasing the pipe diameter. One way to do this would be to replace the valve and pipe with a wall between two polished glasses and to assume that this wall can be removed instantly, corresponding to turning the valve on.

As shown in Figure 2, water is initially trapped on the left side of the wall. When we instantly remove the wall, the water rushes from left to right, gaining kinetic energy while losing some of its potential energy on the path, then losing the kinetic energy to the potential energy again and moving back to the left, then to right, and so on. In fact, if there is no resistance in the path of water motion that turns kinetic energy to heat, we would expect the energy to oscillate back and forth between potential energy and kinetic energy.

Readers can verify experimentally at home that this behavior is not limited to the ideal situation we have imagined. In fact, even with a simple U-tube and a valve, as shown in Figure 3, turning the valve on quickly will create oscillations in the water level, albeit damped because we could not remove friction completely.

Why is there such a difference in behavior? Why can we not observe a similar behavior in capacitors? Or can we? Readers are encouraged to ponder these questions before considering the answers that follow.

The energy stored in the two glasses can take the form of either potential energy (when the water is still) or kinetic energy (when the water is moving). The stored energy changes form as we open the valve and allow water to move (i.e., store energy in kinetic form). However, at any moment in time, part of the stored energy is in kinetic form, and part is in potential form. After the water settles, due to friction, only potential energy will be left in the two glasses.

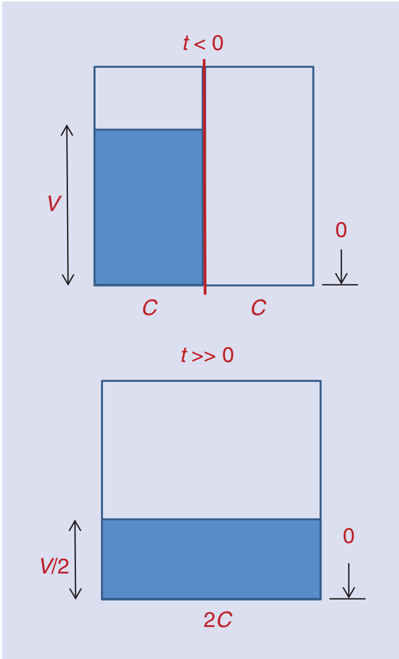

FIGURE 2: (a) An imaginary wall (in red) separates two glasses. At time zero, the wall disappears instantly, allowing the water to move to the right glass without much resistance. (b) After some time, the water level settles to *V*/2.

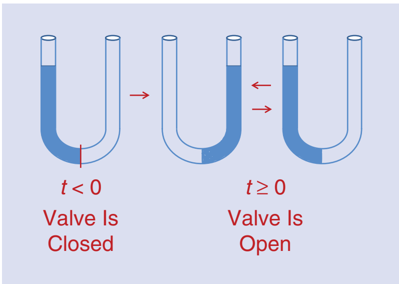

FIGURE 3: Upon opening the valve, water will oscillate between the two branches of a frictionless tube.

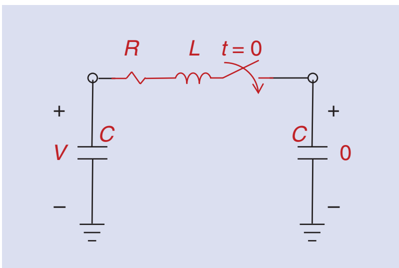

FIGURE 4: Inductance L is included as part of the charge-sharing circuit between two capacitors.

In our capacitor example, however, we have only considered potential energy, and that is the energy stored in the capacitor. How about the kinetic energy? What is the equivalent of kinetic energy in our capacitor example? The answer is the magnetic energy in an inductance that we have totally ignored so far in our capacitor circuit. A more accurate representation of our circuit must include an inductance L in series with the resistance R, as shown in Figure 4. Once we consider L, we can see that L stores the equivalent of kinetic energy (which is the energy stored in the magnetic field), and this gives the possibility of oscillation similar to the case of the two glasses. Let us now revisit the two-capacitor problem and see what happens when R approaches zero, while we assume a nonzero L in the circuit.

When the switch is open, all the energy is stored in the capacitor on the left. There is zero current in the circuit; therefore, there is zero energy stored in the inductor. As we turn on the switch, there will be a current in the circuit. Note that, initially, the inductor impedes the rise of the current and will force the current waveform to be continuous. However, as the current begins to increase, there will be more energy stored in the inductor, and some energy is wasted in the resistor. If the resistance is small enough, corresponding to an under-damped behavior ($R < 2\sqrt{2L/C}$), the stored energy will oscillate back and forth between the capacitor and the inductor until the current goes to zero, storing the remaining energy in the capacitor. Again, during this process, half the initial stored energy is lost to heat in the resistor, no matter how small the resistor is or how fast the oscillations settle. In case the resistor is large enough ($R \geq 2\sqrt{2L/C}$), there will be no oscillation, corresponding to either an over-damped or a critically damped behavior. In this case, a portion of the energy does move to the inductor, but it will be wasted in the process in the resistor, never to return to the capacitor. In this case, too, half the initial energy is wasted in the resistor

Finally, we reconsider the question of what happens when R is exactly zero, while we have a nonzero inductance in the circuit. In this case, the energy stored initially in the capacitor will swing back and forth between the capacitor and the inductor and, because it will have no place to be consumed, will result

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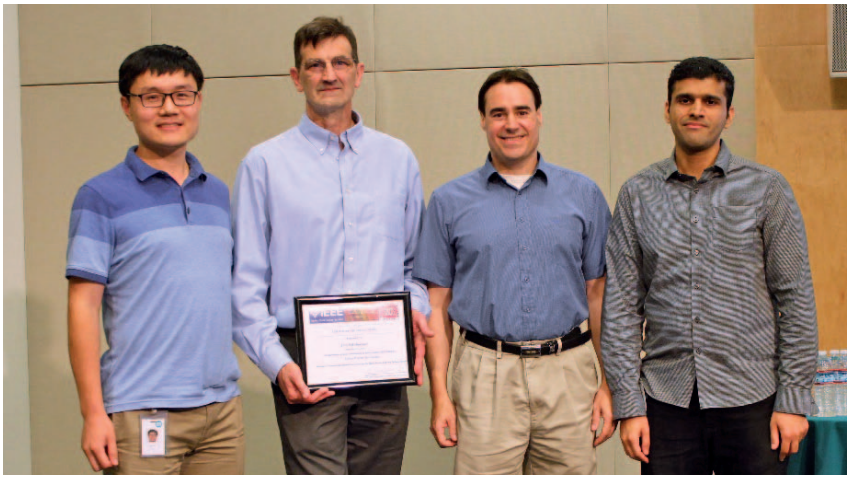

The speaker, David Robertson, receiving a certificate of appreciation from the Silicon Valley Chapter and the SSCS Society webinar program (from left): Haitao Li, David Robertson, Michael Perrott, and Mojtaba Sharifzadeh.

Prof. Shanthi Pavan from IIT-Madras and Silicon Valley Chapter officers and seminar attendees at Maxim Inc. in San Jose.

companies in Silicon Valley attended the talk, and some Maxim employees watched the live broadcast online. The speaker arranged the seminar as a comprehensive and interactive short course format, with the attendees engaged in learning. He covered many aspects of time-varying linear systems from top-level intuition to detailed mathematical equations as well as many interesting real-life design examples. This seminar will also be available as an SSCS webinar in the spring of 2017. Please feel free to attend the webinar and ask your long-time design questions live online from Prof. Pavan after the webinar in the dedicated Q&A session.

Abstract

An analog/mixed-signal designer encounters time-varying circuits everywhere—sample-and-holds, chopper stabilized amplifiers, mixers, switched-capacitor amplifiers and filters, discrete and continuous-time delta-sigma modulators, and N-path filters. The analysis of signals and noise in these circuits is often associated with messy mathematics and algebra.

This talk aims to demystify linear (periodically) time-varying circuits. Starting from first principles, intuition behind various aspects of time-varying circuits and systems will be given. This intuition is illustrated with case studies of practical circuits and systems, like chopper-stabilized amplifiers and continuous-time delta-sigma modulators.

—Mojtaba Sharifzadeh
Chapter Vice Chair
Silicon Valley SSCS Chapter

CIRCUIT INTUITIONS *(continued from p. 8)*

in a sustained oscillation. This is, indeed, the situation in an LC oscillator, where the inherent resistance in the inductance and wiring is compensated for by a negative resistance, so as to yield a zero effective resistance in the circuit.

In summary, we provided two answers for the question of charge sharing between two capacitors when separated by an ideal switch. If we assume $R = 0$ and $L = 0$, as shown in Figure 4, we expect the voltages across the capacitors to settle to their final values instantly. In this process, one half of the initial stored energy is consumed in the ideal switch, and the other half remains in the capacitors. If we assume $R = 0$ but $L \neq 0$, there will be no waste of energy in turning on the switch. The initial energy in the capacitor will remain indefinitely in the system, swinging back and forth between the capacitor and the inductor.