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Miller's Theorem

Welcome to the sixth article in the "Circuit Intuitions" column series. As the title suggests, each article provides insights and intuitions into circuit design and analysis. These articles are aimed at undergraduate students but may serve the interests of other readers as well. I would appreciate your comments and feedback, as well as your requests and suggestions for future articles in this series. Please e-mail your comments to me at: ali@ece.utoronto.ca.

In the first article in this series, we said "looking into a node" we see Thevenin or Norton equivalent circuits of that node with respect to ground. In this article, we will introduce Miller's theorem to find the equivalent circuit of an impedance (Z) that is connected between the input and output nodes of an amplifier, as shown in Figure 1. Miller's theorem states that, as far as the input and the output nodes are concerned, the impedance can be broken into Z_1 (connected from the input node to ground) and Z_2 (connected from the output node to ground). A simple proof for this theorem can be found in many textbooks, such as [1] and [2]. Here, we provide a proof followed by some intuitions.

As far as the input node is concerned, the current drawn by Z is equal to

$$I_1 = \frac{V_1 - AV_1}{Z} = \frac{V_1}{Z/(1-A)}.$$

If we were to find an equivalent impedance from the input node to ground (Z_1) that draws the same current from V_1, we would have

$$I_1 = \frac{V_1}{Z_1}.$$

Therefore, comparing the two equations, we can write

$$Z_1 = \frac{Z}{1-A}. \quad (1)$$

Similarly, if we write an expression for the current drawn by Z from the output node and equate it to the current being drawn by Z_2, we can write

$$Z_2 = \frac{Z}{1-1/A}. \quad (2)$$

Note that Z is typically a resistor or a capacitor, although it can be any impedance in general. For simplicity, we assume Z to be a resistor ($Z = R$) in the remainder of this article.

If A is negative (such as in any inverting amplifier), then the above equations tell us that both R_1 and R_2 will be positive. To see this intuitively, imagine Nodes 1 and 2 are electrically connected via a resistive string whose total resistance is R, as shown in Figure 2(a). Since one side of this string (Node 1) is connected to a positive voltage and the other side (Node 2) is connected to a negative voltage, there must be a location along the string where the voltage is zero. If we actually ground this location, as in Figure 2(b), we will not disturb the circuit, as no current will be drawn from this location to ground. This is simply because the current that flows through the left branch (R_1) flows directly to the right branch (R_2). The ground location naturally

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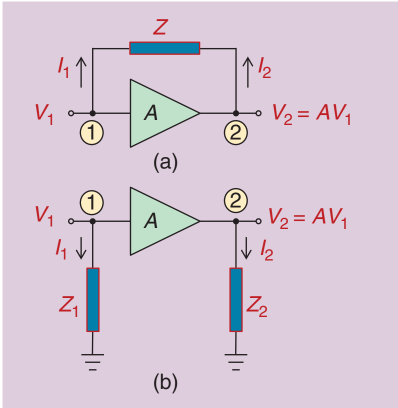

FIGURE 1: (a) Impedance Z is connected between the input and output nodes of a voltage amplifier with the gain of A and (b) Miller's equivalent circuit.

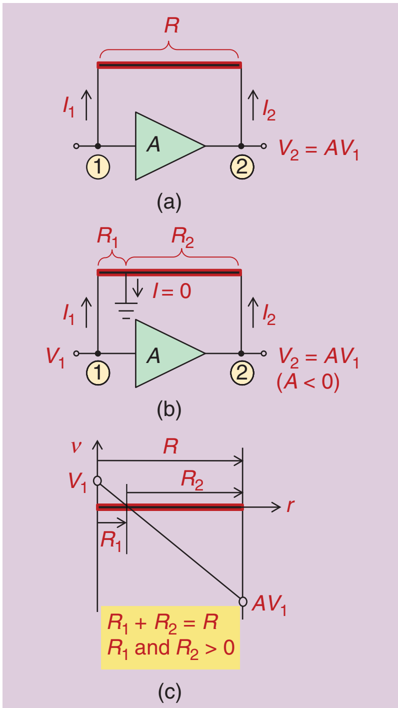

FIGURE 2: (a) R is displayed as a resistive string. (b) If A is negative, R can be broken into positive R_1 and R_2 such that the common node is at $0V$, and $R_1 + R_2 = R$, (c) voltage along the resistive string.

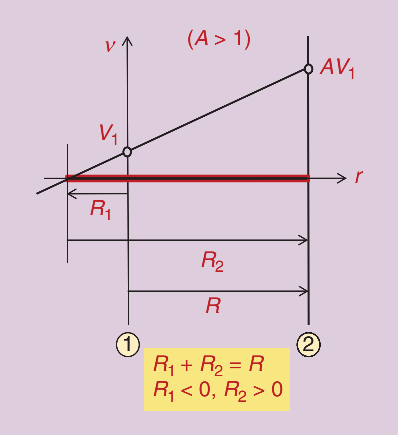

FIGURE 3: When $A > 1$, R can be split into one negative (R_1) and one positive resistance (R_2) such that $R = R_1 + R_2$.

splits the total resistance R into R_1 and R_2, which are the string resistances from Nodes 1 and 2 to the 0 V location, respectively. This is shown pictorially in Figure 2(c), where we have assumed V_1 to be a positive voltage and $V_2 = AV_1$ to be a negative voltage. The voltage along the resistive string decreases linearly from V_1 (at Node 1) to AV_1 (at Node 2). The intersection of this line with the resistance axis breaks the resistor into two pieces, R_1 and R_2. One can easily verify through the two similar triangles that the ratio of R_2 to R_1 is $|A|$. Given this and the fact that $R_1 + R_2 = R$, we will arrive at the same equations as in (1) and (2).

It would be interesting to use this graphical approach to gain intuition in the case when A is positive and greater than one. In this case, both ends of the resistive string have voltages of the same sign, and hence there exists no intermediate location with 0 V. This is shown in Figure 3 for the case where both nodes have positive voltages. If we connect V_1 and V_2 via a line, we will find that the location with 0 V lies outside the $[0\ R]$ region at a negative resistance (R_1) with respect to Node 1 and at a positive resistance (R_2) with respect to Node 2. This makes intuitive sense because if we apply a positive voltage to Node 1, given $V_2 > V_1$, there will be a current moving toward Node 1 (not leaving Node 1 as we expect with a positive resistance). For this reason, Node 1 experiences a negative resistance (R_1). From the perspective of Node 2, the current always leaves the node (when V_2 is positive) indicating a positive resistance. The resistance, however, between this node and ground (R_2) is now larger than R as indicated in Figure 3. Similar to the previous case, given the two similar triangles in this figure, we can verify that $|R_2/R_1| = A$ and $R_1 + R_2 = R$ to arrive at the same equations as (1) and (2). In other words, we can still split R into R_1 and R_2 but with R_1 being negative for the common node of the two resistors to be at 0 V.

In either case, the sum of the two resistances in series is equal to the original resistance.

To summarize, in applying Miller's theorem, we are essentially splitting the resistor between two nodes as two resistors in series such that their common node will have 0 V! If the two node voltages have opposite signs, we will end up with two positive resistors. If they have the same sign, we will end up with one negative and one positive resistor. In either case, the sum of the two resistances in series is equal to the original resistance.

References

[1] A. S. Sedra and K. C. Smith, *Microelectronic Circuits*, 7th ed. London, U.K.: Oxford Univ. Press, 2014.

[2] B. Razavi, *Fundamentals of Microelectronics*. Hoboken, NJ: Wiley, 2008.

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