Miller’s Approximation

Welcome to the seventh article in this column series. As the title suggests, each article provides insights and intuitions into circuit design and analysis. These articles are aimed at undergraduate students but may serve the interests of other readers as well. If you read this article, I would appreciate your comments and feedback, as well as your requests and suggestions for future articles in this series. Please e-mail your comments to me at: ali@ece.utoronto.ca.

In the previous article, we presented an intuitive view of Miller’s theorem, especially as it applies to resistors. In this article, we use Miller’s theorem to estimate the bandwidth of an amplifier with a capacitor between its input and output nodes. Figure 1 shows an ideal voltage amplifier (i.e., one with infinite input impedance and zero output impedance) with a constant voltage gain of \(-A_0\) and a capacitor \(C_{M} \) between its input and output nodes. Miller’s theorem says that we can replace \(C_{M} \) with two capacitors, \(C_{IM} \) and \(C_{OM} \), connected from the input and output node, respectively, to ground, where:

\[
C_{IM} = C_{M}(1 + A_0) \tag{1}
\]

\[
C_{OM} = C_{M}(1 + 1/A_0). \tag{2}
\]

If we assume \(A_0 \gg 1\), then Miller’s theorem tells us that the capacitor between the two nodes appears as much larger at the input node (by a factor of \(~A_0\) ) but as the same capacitor (by a factor of \(~1\) ) at the output node. This makes intuitive sense because a small voltage increment at the input results in a much larger decrement at the output, which in turn attracts a large amount of charge on the capacitor plates, as if the capacitor were much larger! From the perspective of the output node, however, a change in the output voltage corresponds to a much smaller change at the input. We can then simply assume the input node is grounded. This is equivalent to saying the capacitor seen from the output is the same as \(C_{M} \).

When the amplifier is ideal but its gain \((A)\) is frequency dependent (i.e., not constant) or when the amplifier is nonideal (e.g., has a finite output impedance), there may be confusion as how to apply Miller’s theorem or how useful it may be. We focus on this in the remainder of this article.

First, let us examine a slightly generalized case where the amplifier is ideal but it has a frequency-dependent gain \(A(f)\), instead of constant \(-A_0\), with a single pole frequency, \(f_p\). In other words, assume

\[
A(f) = \frac{-A_0}{1 + jf/f_p}. \tag{3}
\]

Note that since this is an ideal amplifier, adding \(C_{M} \) across it will not change the voltage transfer function. We can therefore apply Miller’s theorem (1), to arrive at the following equation:

\[
C_{IM} = C_{M}(1 + A_0) \frac{1 + jf}{f_p(1 + A_0)} \frac{1 + jf/f_p}{1 + jf/f_p}. \tag{4}
\]

This equation simply states that \(C_{IM} \) is now frequency dependent. At low frequencies (below \(f_p\), \(C_{M} \) is simply multiplied by the dc gain of the amplifier. At midfrequencies (between \(f_p\) and \(~A_0/f_p\)), when the voltage gain drops, \(C_{IM} \) drops also. Beyond the amplifier’s unity-gain frequency (i.e., \(~A_0/f_p\)), \(C_{IM} \) approaches \(C_{M} \). This makes intuitive sense because, at very high frequencies, the gain of the amplifier will approach zero and the output node becomes grounded, producing \(C_{M} \) at the input node.

We do not need to derive an equation for the output node capacitance \((C_{OM})\) in this case because \(C_{OM} \) will have no impact on \(V_2\), given that the amplifier is assumed to have zero output impedance.

**Figure 1:** Miller’s theorem: (a) capacitance \(C_{M} \) is connected between the input and output nodes of an ideal voltage amplifier with a constant gain of \(-A_0\), and (b) Miller’s equivalent circuit.
To summarize, a $C_{12}$ across a frequency-dependent ideal amplifier may only impact the capacitive load seen at the input node of the amplifier but will have no impact on the output node of the amplifier.

Now, let us consider a more general case where the amplifier is nonideal (i.e., it has a finite input impedance and a nonzero output impedance). We further assume this amplifier is driven by another nonideal amplifier as shown in Figure 2(a).

Let us denote by $A_{11}(j\omega)$ and $A_{22}(j\omega)$ the voltage transfer functions of the two stages without the presence of $C_{12}$. A linear model for this two-stage amplifier along with the added $C_{12}$ is shown in Figure 2(b). In this figure, $R_1$ and $C_1$ represent the total resistance and capacitance of node 1, respectively, and $g_1$ represents the transconductance of the first stage. $R_2$, $C_2$, and $g_2$ are the corresponding parameters of the second stage.

An important observation in this case is that the addition of $C_{12}$ will change $A_{22}(j\omega)$ to $A_{12}(j\omega)$, which is unknown and needs to be determined. However, to replace $C_{12}$ with its Miller’s equivalents, as shown in Figure 2(c), we need $A_{12}(j\omega)$ in the first place. What can we do?

In Miller’s approximation, we simply use the dc gain of the second stage ($A_{22}$) in (1) and (2) to find an estimate of $C_{1M}$ and $C_{2M}$. We then proceed to write an expression for $A_{11}(j\omega)$ and $A_{12}(j\omega)$.

How valid is this approximation? $A_{12}(j\omega) = A_{11}(j\omega)A_{22}(j\omega)$ has two poles corresponding to the two capacitive nodes in the circuit and a zero due to $C_{12}$. We do not know the exact locations of the poles, but we do know that in the vicinity of the first pole (i.e., the dominant pole) $A_{12}(j\omega)$ can be approximated by the dc gain of $A_{12}(j\omega)$. $A_{22}$. The actual gain at the dominant pole frequency is 3 dB lower, but we accept this approximation. Therefore, in calculating the location of the first pole, Miller’s approximation does provide a reasonable estimate, as follows:

$$f_{p1} = \frac{1}{2\pi R_1(C_1 + C_{12}(1 + A_{22}))}.$$

At the frequency of the second pole, however, the gain has already dropped substantially from the dc gain, and hence Miller’s approximation does not yield a good estimate for $C_{2M}$, and consequently for $f_{p2}$. In fact, if $f_{p2} \gg f_{p1}$, our estimate of $f_{p2}$ may be off by an order of magnitude.

Miller’s approximation also misses the zero that exists in the circuit of Figure 2(b). A quick inspection of this circuit reveals that a zero lies at a frequency where the current through $C_{12}$ becomes equal to $g_2V_1$. When this occurs, the current through the parallel combination of $C_2$ and $R_2$ becomes zero, creating a zero in the transfer function. In other words, we can write

$$f_z = \frac{g_2}{2\pi C_{12}}.$$

The existence of the zero makes intuitive sense because, as we increase the frequency, there will be a point where the impedances of $C_{12}$ and $C_2$ will be much smaller than $R_2$. At these frequencies, the transfer function of the second stage becomes that of a capacitive divider, which has a constant gain and does not drop with increasing frequency, a clear sign of the existence of a zero.

But none of these points should imply that Miller’s theorem is inaccurate; they only tell us the limitation of Miller’s approximation. In fact, one can show that if we use the proper frequency-dependent gain in calculating $C_{1M}$ and $C_{2M}$ (as we did in the case of an ideal amplifier), we can accurately find the original poles. But this will defeat the purpose of using Miller’s theorem for a simple, quick estimation of the circuit bandwidth.

If Miller’s approximation cannot be used to derive an expression for the second pole of the circuit in Figure 2(b), what can we do instead? The answer will actually depend on the relative ratios of the capacitors. Here, we provide a quick and intuitive method for the case when $C_{12} \gg C_1$ and $C_2$. In this case, $C_{12}$, which is referred to as the Miller compensation capacitor, improves the circuit stability when placed in a feedback loop [1], [2].

Given $C_{12} \gg C_1$ and $C_2$, as we increase frequency, $C_{12}$ will short the two nodes together, creating a total capacitance of $C_1 + C_2$ in parallel with a total resistance $1/g_2$, where we have assumed $1/g_2$ is much smaller than $R_1$ and $R_2$. Therefore, we can write an expression for the second pole as follows:

$$f_{p2} = \frac{g_2}{2\pi(C_1 + C_2)}.$$

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small-signal bandwidth must be far greater than the input frequency.

2) To which node(s) should the $n$-wells of $M_3$ and $M_8$ in Figure 10 be connected?

They should be connected to node $P$ to ensure the source and drain junctions of these transistors are not forward biased.

3) How high can $V_X$ in Figure 10 go to avoid stressing $M_{14}$?

When $M_{14}$ is off, its source voltage reaches approximately $V_{DD} - V_{TH}$. For the source–drain potential difference to remain less than $V_{DD}$, $V_X$ must not exceed $2V_{DD} - V_{TH}$.

References

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This equation, along with equations for $f_{p1}$ and $f_s$, can now be used to form the equation for the overall voltage transfer function of the two-stage amplifier.

It is worth noting that as we increase $C_{12}$, $f_{p1}$ and $f_s$ (as found by their respective equations) will move farther apart, a phenomenon referred to as pole splitting [1], [2].

In summary, Miller’s approximation uses the dc gain of the amplifier to provide a relatively accurate estimation of its dominant pole (i.e., the circuit bandwidth). This approximation, however, becomes inaccurate when determining the second pole of the amplifier; other intuitive methods exist for this purpose.

For further discussions and intuition into Miller’s theorem, we refer the readers to [3].

References