First and Second Order Circuits

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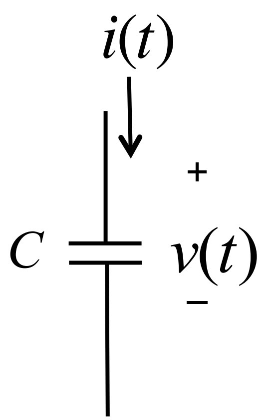
Capacitors and Inductors

- intuition: bucket of charge

$$q = Cv \rightarrow i = C\frac{dv}{dt}$$

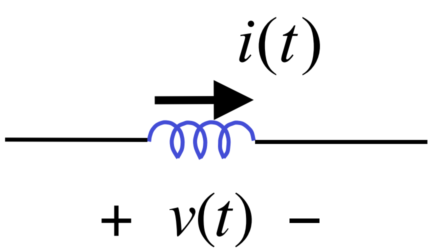
- Resist change of voltage
- DC open circuit
- Store voltage (charge)
- Energy stored = 0.5 C v(t)^2



- intuition: water hose

$$\lambda = Li \rightarrow v = L\frac{di}{dt}$$

- Resist change of current
- DC short circuit
- Store current (magnetic flux)
- Energy stored = 0.5 L i(t)^2

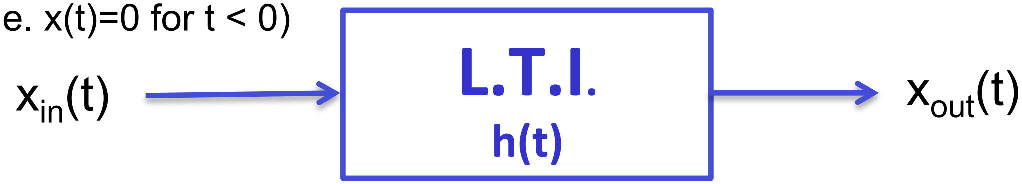


Characterization of an LTI system's behavior

- Techniques commonly used to characterize an LTI system:

  1. Observe the response of the system when excited by a step input (time domain response)

Assumption:
$x_{in}(t)$ is causal (i.e. x(t)=0 for t < 0)



$$x_{out} = \int_{0-}^{t} x_{in}(\tau)h(t-\tau)d\tau \equiv x_{in}(t) * h(t)$$

  2. Observe the response of the system when excited by sinusoidal inputs (frequency response)

$$X_{out}(s) = X_{in}(s) \cdot H(s)$$

Frequency Response

- The merit of frequency-domain analysis is that it is easier than time domain analysis:

$$L[x(t)] = \int_{0-}^{\infty} e^{-st} x(t) dt = X(s)$$

← One sided Laplace Transform (assumption: x(t) is causal or is made causal by multiplying it by u(t))

- The transfer function of any of the LTI circuits we consider
  - Are rational with m ≤ n
  - Are real valued coefficients a_j and b_i
  - Have poles and zeros that are either real or complex conjugated
  - Furthermore, if the system is stable
    - All denominator coefficients are positive
    - The real part of all poles are negative

$$H(s) = \frac{a_0 + a_1 s + \ldots + a_m s^m}{1 + b_1 s + \ldots + b_n s^n} = K \frac{(s+\omega_{z1})\ldots(s+\omega_{zm})}{(s+\omega_{p1})\ldots(s+\omega_{pn})} =$$

$$= K \frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)} \quad with\, K \equiv \frac{a_m}{b_n}$$

← root form "mathematicians" style

Frequency Response

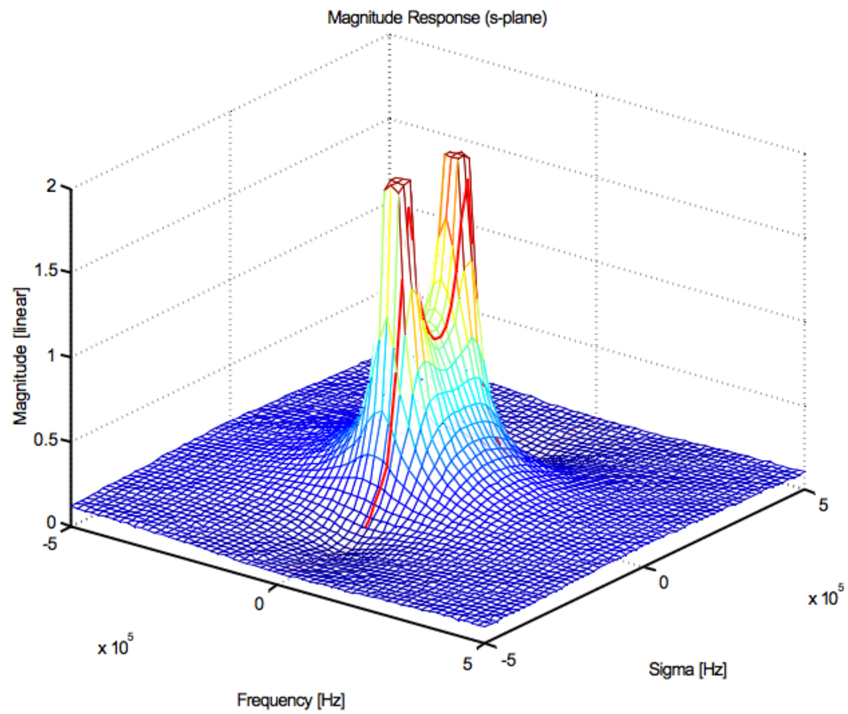
$$H(s) = a_0 \frac{\left(1+\frac{s}{\omega_{z1}}\right)\ldots\left(1+\frac{s}{\omega_{zm}}\right)}{\left(1+\frac{s}{\omega_{p1}}\right)\ldots\left(1+\frac{s}{\omega_{pn}}\right)} = a_0 \frac{\left(1-\frac{s}{z_1}\right)\ldots\left(1-\frac{s}{z_m}\right)}{\left(1-\frac{s}{p_1}\right)\ldots\left(1-\frac{s}{p_n}\right)}$$

← "EE" style

$NOTE:$

$p_i = -\omega_{pi} \quad (poles)$

$z_i = -\omega_{zi} \quad (zeros)$



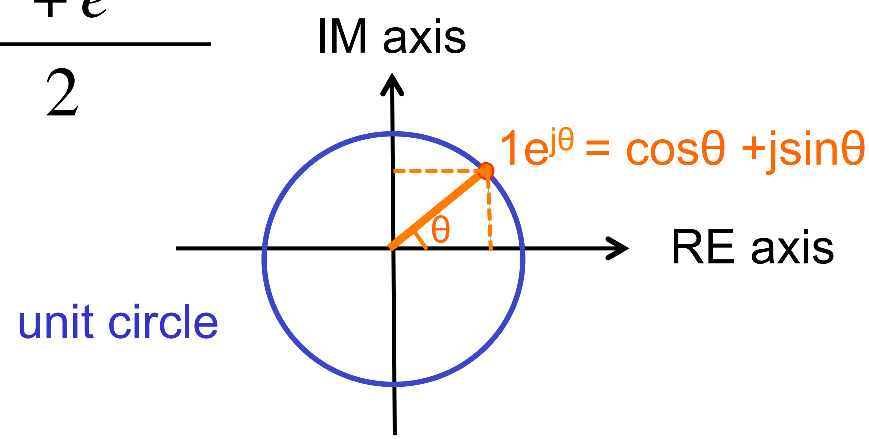
Magnitude and Phase (1)

- When an LTI system is exited with a sinusoid the output is a sinusoid of the same frequency. The magnitude of the output is equal to the input magnitude multiplied by the magnitude response ($|H(j\omega_{in})|$). The phase difference between the output and input sinusoid is equal to the phase response (ϕ=phase[$H(j\omega_{in})$])

$$x_{in}(t) = A_{in}\cos(\omega t) = A_{in}\frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\updownarrow \mathcal{F}$$

$$X_{in}(j\omega) = F[x_{in}(t)]$$



$$H(j\omega) = |H(j\omega)|e^{j\omega t_0}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Euler's formula

$$X_{out}(j\omega) = X_{in}(j\omega)\cdot H(j\omega)$$

Magnitude and Phase (2)

$$X_{out}(j\omega) = X_{in}(j\omega) \cdot H(j\omega) = X_{in}(j\omega) \cdot |H(j\omega)| \cdot e^{j\omega t_0}$$

$\mathcal{F}^{-1}$ ↕

Time Shift Property:

$\mathcal{F}[x(t-t_0)] = X(f)\ e^{-j2\pi ft0}$

$$x_{out}(t) = \left|H(j\omega_{in})\right| x_{in}(t + t_0) =$$

$$= A_{in}\left|H(j\omega_{in})\right|\cos[\omega(t + t_0)] =$$

$$= A_{in}\left|H(j\omega_{in})\right|\cos(\omega t + \omega t_0)$$

First order circuits

- A first order transfer function has a first order denominator

$$H(s) = \frac{A_0}{1 + \frac{s}{\omega_p}}$$

First order low pass transfer function. This is the most commonly encountered transfer function in electronic circuits

$$H(s) = A_0 \frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_p}}$$

General first order transfer function.

Step Response of first order circuits (1)

- *Case 1*: First order low pass transfer function $$H(s) = \frac{A_0}{1 + \frac{s}{\omega_p}}$$

$$x_{in}(t) = A_{in} \cdot u(t) \quad \leftrightarrow \quad X_{in}(s) = \frac{A_{in}}{s}$$

$$X_{out}(s) = \frac{A_{in}}{s} \frac{A_0}{1 + \frac{s}{\omega_p}} = A_{in} A_0 \left[\frac{1}{s} - \frac{1}{s + \omega_p} \right]$$

$$\updownarrow$$

$$x_{out}(t) = A_{in} A_0 u(t) \left[1 - e^{-t/\tau} \right] \quad with\ \tau = 1/\omega_p$$

Step Response of first order circuits (2)

- *Case 2*: General first order transfer function

$$H(s) = A_0 \frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}}$$

$$x_{in}(t) = A_{in} \cdot u(t) \qquad \leftrightarrow \qquad X_{in}(s) = \frac{A_{in}}{s}$$

$$X_{out}(s) = \frac{A_{in}A_0}{s} \frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}}$$

$$\updownarrow$$

$$x_{out}(t) = A_{in}A_0 u(t)\left[1-\left(1-\frac{\omega_p}{\omega_z}\right)e^{-t/\tau}\right] \quad where\ \tau = 1/\omega_p$$

Step Response of first order circuits (3)

- Notice $x_{out}(t)$ "short term" and "long term" behavior

$$x_{out}(0+) = A_{in}A_0 \frac{\omega_p}{\omega_z}$$

$$x_{out}(\infty) = A_{in}A_0$$

- The short term and long term behavior can also be verified using the Laplace transform

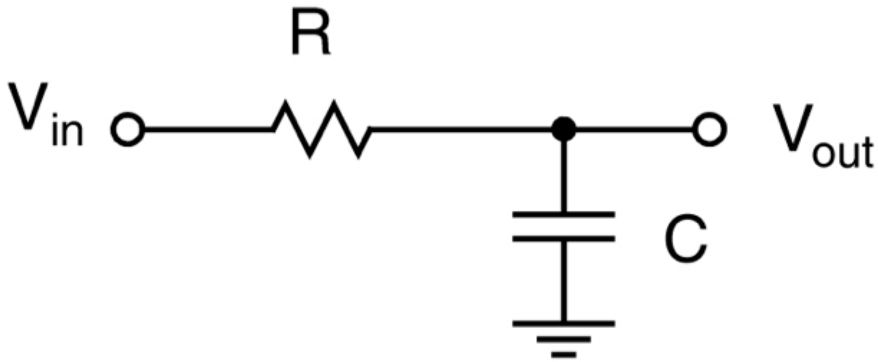
$$x_{out}(0+) = \lim_{s\to\infty} s \cdot X_{out}(s) = \lim_{s\to\infty} s \frac{A_{in}}{s} \cdot H(s) = A_{in}A_0 \frac{\omega_p}{\omega_z}$$

$$x_{out}(\infty) = \lim_{s\to 0} s \cdot X_{out}(s) = \lim_{s\to 0} s \frac{A_{in}}{s} \cdot H(s) = A_{in}A_0$$

Equation for step response to any first order circuit

$$x_{out}(t) = \underbrace{x_{out}(\infty)}_{\text{Steady response}} - \underbrace{[x_{out}(\infty) - x_{out}(0+)] \cdot e^{-t/\tau}}_{\text{Transitory response}} \quad where\ \tau = 1/\omega_p$$

Example #1



- R = 1KΩ, C= 1μF.
- Input is a 0.5V step at time 0

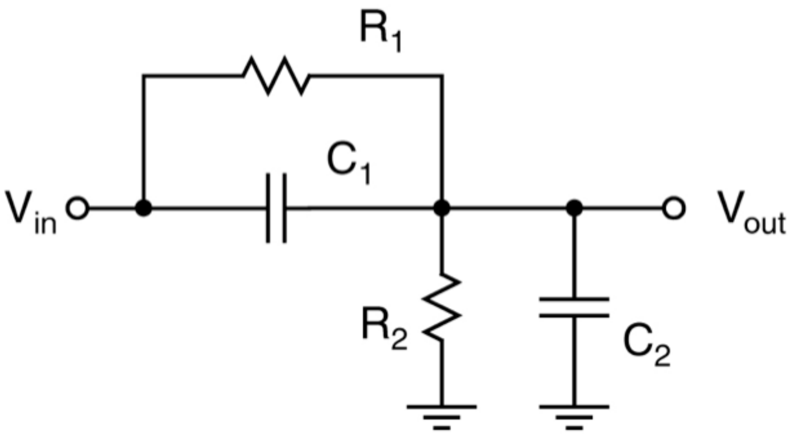
Source: Carusone, Johns and Martin

$$H(s) = \frac{1}{1 + sRC}$$

$$\tau = RC = 1\mu s \Leftrightarrow \omega_p = \frac{1}{\tau} = 1Mrad/s \Leftrightarrow f_{-3dB} = \frac{\omega_p}{2\pi} \cong 159KHz$$

$$V_{out}(t) = 0.5 \cdot \left(1 - e^{-t/\tau}\right) u(t)$$

Example #2 (1)



- R1 = 2KΩ, R2=10KΩ
- C1= 5pF. C2=10pF
- Input is a 2V step at time 0

Source: Carusone, Johns and Martin

$$H(s \to 0) = \frac{R_2}{R_1 + R_2} \equiv A_0; \qquad H(s \to \infty) = \frac{C_1}{C_1 + C_2} \equiv A_\infty$$

$$\tau_p = (R_1 \parallel R_2) \cdot (C_1 \parallel C_2) = \frac{R_1 R_2}{R_1 + R_2}(C_1 + C_2)$$

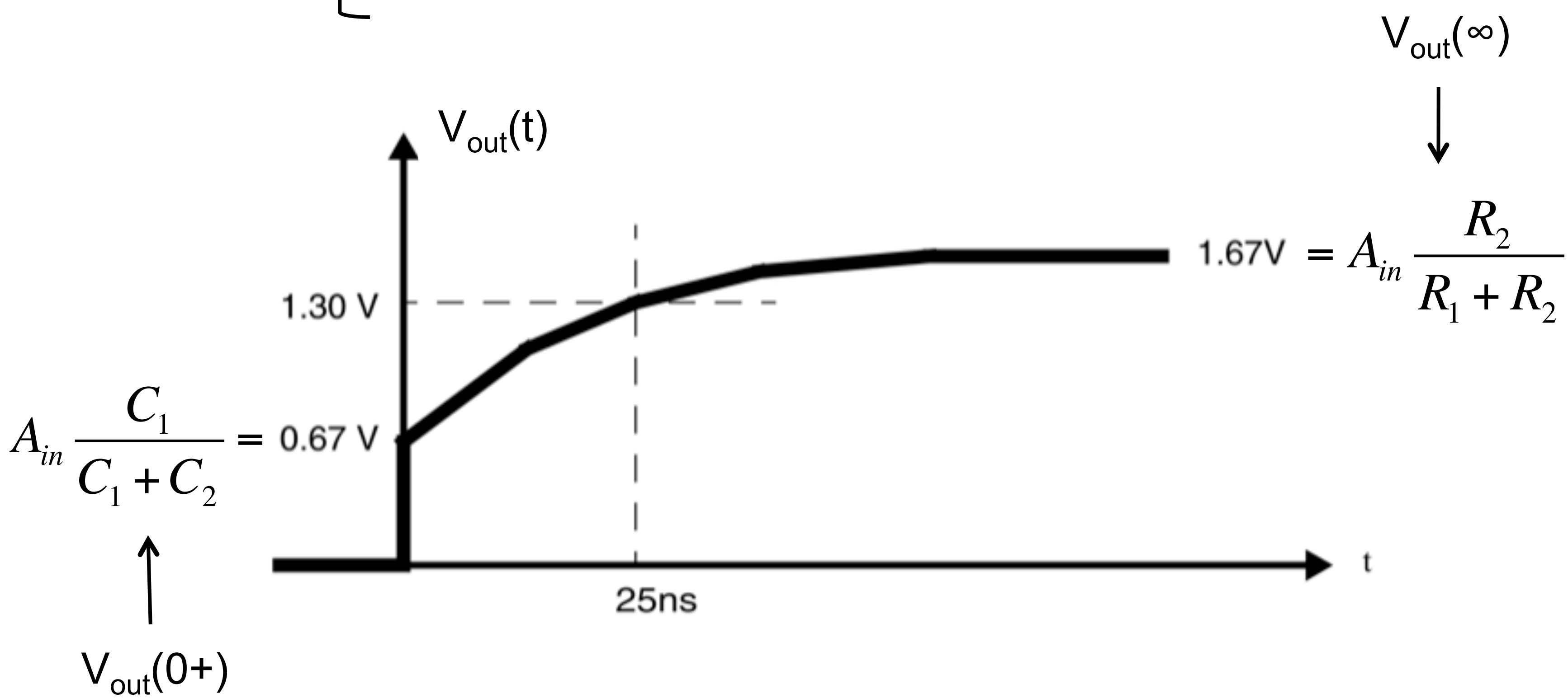
$$H(s) = \frac{R_2}{R_1 + R_2} \cdot \left[\frac{1 + sR_1C_1}{1 + s\frac{R_1R_2}{R_1 + R_2}(C_1 + C_2)} \right]$$

← By inspection

$$\tau_z = R_1 C_1$$

Example #2 (2)

$$V_{out}(t) = \begin{cases} V_{out}(0+) + \left[V_{out}(\infty) - V_{out}(0+) \cdot \left(1 - e^{-t/\tau p}\right)\right] & for\, t > 0 \\ 0 & for\, t \le 0 \end{cases}$$



Example #3

- Consider an amplifier having a small signal transfer function approximately given by

$$A(s) = \frac{A_0}{1 + \frac{s}{\omega_p}}$$

- $A_0 = 1 \times 10^5$
- $\omega_p = 1 \times 10^3$ rad/s

- Find approx. unity gain BW and phase shift at the unity gain frequency

since $A_0 >> 1$:

$$A(s) \approx \frac{A_0}{\frac{s}{\omega_p}} = \frac{A_0 \omega_p}{s} \quad \Longleftrightarrow \quad A(j\omega) \approx \frac{A_0 \omega_p}{j\omega}$$

$$\left| \frac{A_0 \omega_p}{j\omega_u} \right| = 1 \Rightarrow \omega_u \cong A_0 \omega_p \qquad Phase[A(j\omega_u)] \approx Phase\left[\frac{A_0 \omega_p}{j\omega_u} \right] = -90^\circ$$

Second-order low pass Transfer Function

$$H(s) = \frac{a_0}{1 + b_1 s + b_2 s^2} = \frac{a_0}{1 + \frac{s}{\omega_0 Q} + \frac{s^2}{\omega_0^2}}$$

$$b_1 \equiv \frac{1}{\omega_0 Q}; \quad b_2 \equiv \frac{1}{\omega_0^2}$$

- Interesting cases:
  - Poles are real
    - one of the poles is dominant $\longrightarrow$ $\omega_{3dB} \cong \frac{1}{b_1} \quad \left(b_1 = \sum \tau_j\right)$
  - Poles are complex

Poles Location

- Roots of the denominator of the transfer function: $1+\frac{s}{\omega_0 Q}+\frac{s^2}{\omega_0^2}=0$

- Complex Conjugate poles (overshooting in step response)

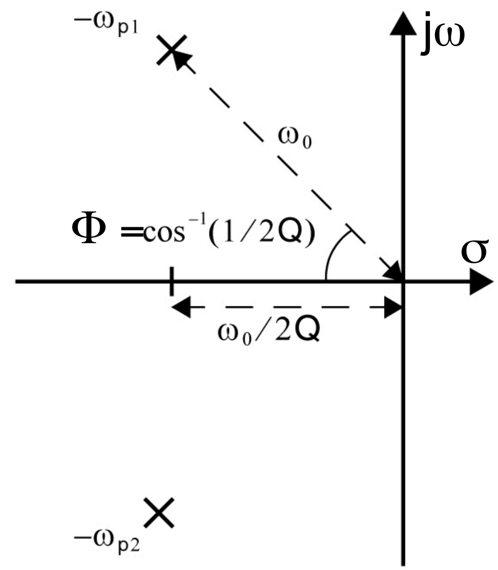
$$for\quad Q>0.5 \Rightarrow p_{1,2}=-\frac{\omega_0}{2Q}\left(1\mp j\sqrt{4Q^2-1}\right)=-\omega_R \mp j\omega_I$$

  - For Q = 0.707 (Φ=45°), the −3dB frequency is ω_0 (Maximally Flat Magnitude or Butterworth Response)
  - For Q > 0.707 the frequency response has peaking

- Real poles (no overshoot in the step response)

$$for\quad Q\le 0.5 \Rightarrow p_{1,2}=-\frac{\omega_0}{2Q}\left(1\mp \sqrt{1-4Q^2}\right)$$

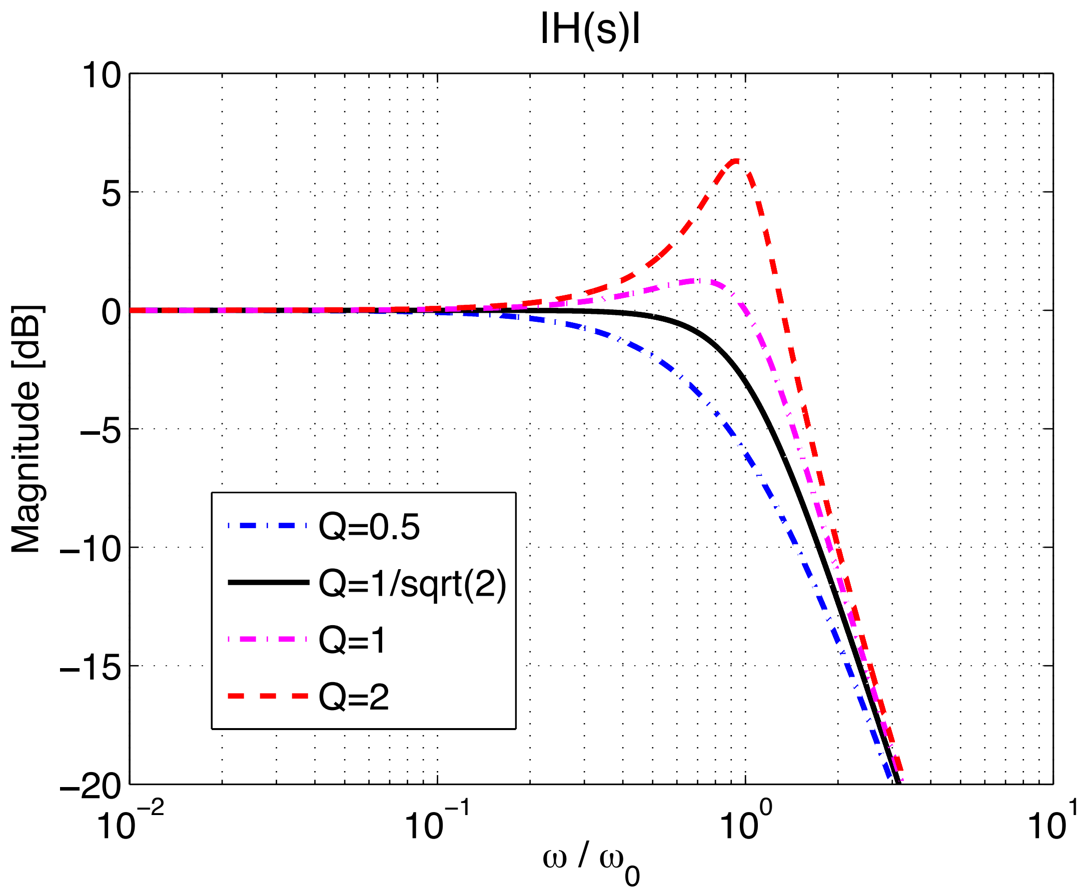
Poles location with Q> 0.5



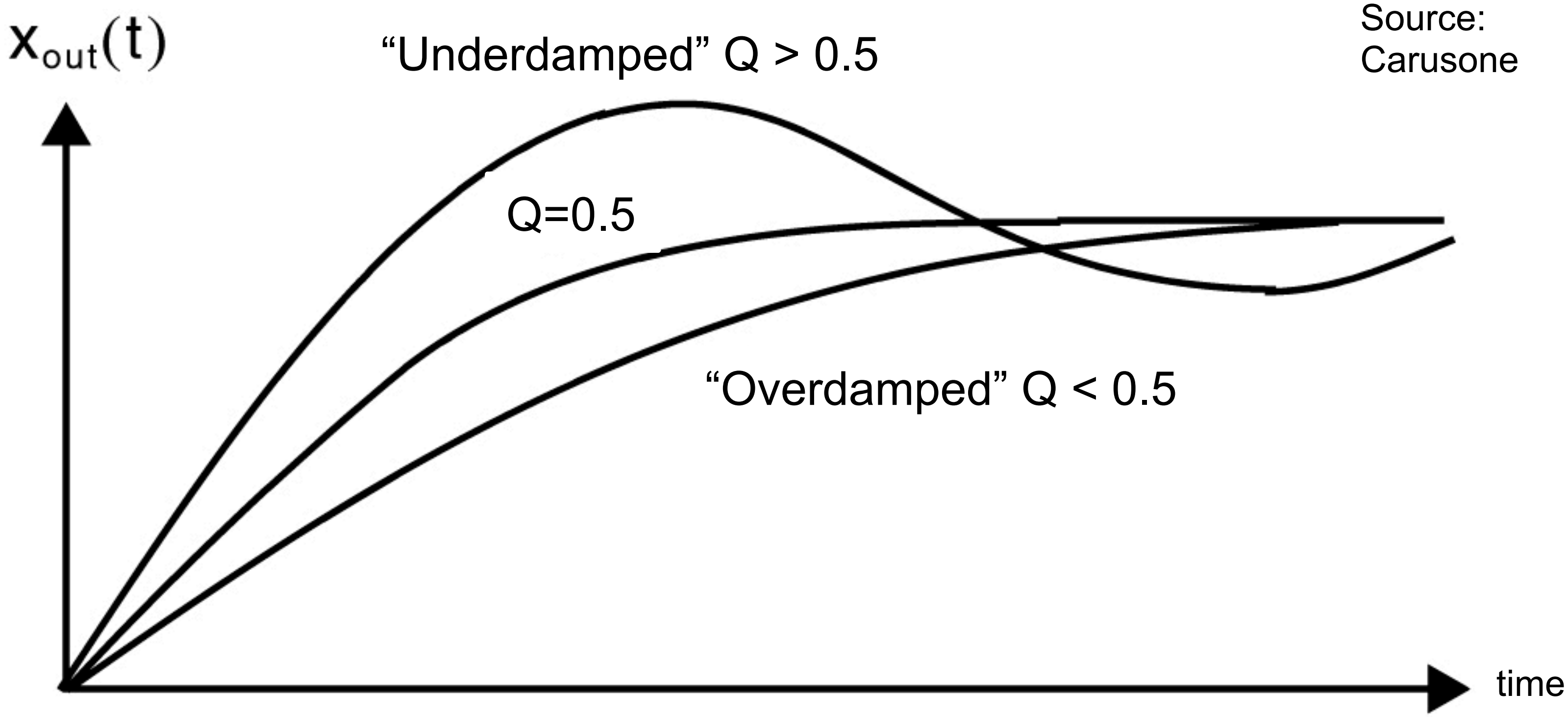
Source: Carusone

Frequency Response

$$H(s) = \frac{1}{1 + \frac{s}{\omega_0 Q} + \frac{s^2}{\omega_0^2}}$$



Step Response



- Ringing for Q > 0.5
- The case Q=0.5 is called maximally damped response (fastest settling without any overshoot)

Widely- Spaced Real Poles

$$H(s) = \frac{a_0}{1 + b_1 s + b_2 s^2} = \frac{a_0}{1 + \frac{s}{\omega_0 Q} + \frac{s^2}{\omega_0^2}} \qquad b_1 \equiv \frac{1}{\omega_0 Q}; \quad b_2 \equiv \frac{1}{\omega_0^2}$$

Real poles occurs when Q ≤ 0.5:

$$for \quad Q \le 0.5 \Rightarrow p_{1,2} = -\frac{\omega_0}{2Q}\left(1 \mp \sqrt{1 - 4Q^2}\right)$$

Real poles widely-spaced (that is one of the poles is dominant) implies:

$$p_1 \equiv -\frac{\omega_0}{2Q} - \frac{\omega_0}{2Q}\sqrt{1 - 4Q^2} \ll p_2 \equiv -\frac{\omega_0}{2Q} + \frac{\omega_0}{2Q}\sqrt{1 - 4Q^2}$$

$$\Updownarrow$$

$$0 \ll 2\sqrt{1 - 4Q^2} \Leftrightarrow 0 \ll \sqrt{1 - 4Q^2} \Leftrightarrow 0 \ll 1 - 4Q^2 \Leftrightarrow Q^2 \ll \frac{1}{4} \Leftrightarrow \frac{b_2}{b_1^2} \ll \frac{1}{4}$$

Widely-Spaced Real Poles

$$H(s) = \frac{a_0}{1 + b_1 s + b_2 s^2} = \frac{a_0}{\left(1 - \frac{s}{p_1}\right) \cdot \left(1 - \frac{s}{p_2}\right)} = \frac{a_0}{1 - \frac{s}{p_1} - \frac{s}{p_2} + \frac{s^2}{p_1 p_2}} \cong \frac{a_0}{1 - \frac{s}{p_1} + \frac{s^2}{p_1 p_2}}$$

$$\Rightarrow \quad \mathrm{p}_1 \cong -\frac{1}{b_1} \qquad \mathrm{p}_2 \cong \frac{1}{p_1 b_2} = -\frac{b_1}{b_2}$$

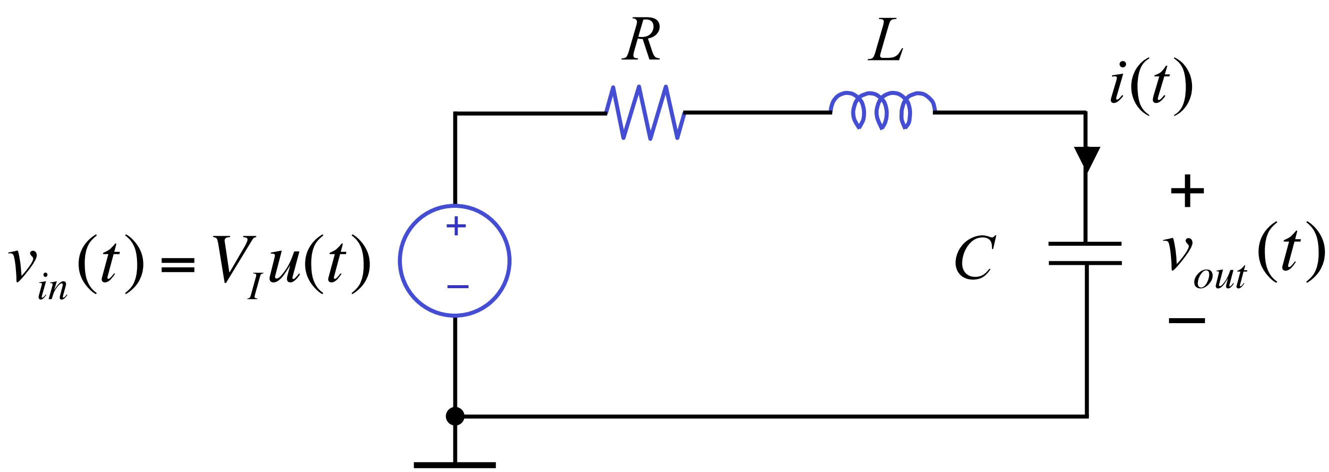
This means that in order to estimate the -3dB bandwidth of the circuit, all we need to know is b_1 !

$$H(s) \cong \frac{a_0}{1 - \frac{s}{p_1}} \qquad \Rightarrow \omega_{\text{-}3dB} \cong |p_1| \cong \frac{1}{b_1}$$

ZVTC method: $b_1 = \sum \tau_j$

$$\Rightarrow \omega_{-3dB} \cong \frac{1}{b_1} = \frac{1}{\sum \tau_j}$$

Example: Series RLC circuit (1)



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{1 + sRC + s^2 LC} \equiv \frac{1}{1 + \frac{s}{\omega_0 Q} + \frac{s^2}{\omega_0^2}}$$

$$\omega_0^2 = \frac{1}{LC}; \quad Q = \frac{1}{\omega_0 RC} = \frac{\sqrt{L/C}}{R} \equiv \frac{Z_0}{R}$$

Example - Series RLC: Poles location (2)

$$1+\frac{s}{\omega_0 Q}+\frac{s^2}{\omega_0^2}=0 \Leftrightarrow s^2+s\cdot\frac{\omega_0}{Q}+\omega_0^2=0 \Leftrightarrow s_{1,2}=\frac{-\frac{\omega_0}{Q}\mp\sqrt{\left(\frac{\omega_0}{Q}\right)^2-4\omega_0^2}}{2} \Leftrightarrow$$

$$\Leftrightarrow s_{1,2}=-\underbrace{\frac{\omega_0}{2Q}}\left(1\mp\sqrt{1-4Q^2}\right) \qquad \frac{\omega_0}{2Q}\equiv\alpha$$

- Two possible cases:
  - For Q ≤ ½ real poles:

$$s_{1,2}=-\frac{\omega_0}{2Q}\left(1\mp\sqrt{1-4Q^2}\right)\equiv-\alpha\left(1\mp\sqrt{1-4Q^2}\right)$$

  - For Q > ½ complex poles

$$s_{1,2}=-\frac{\omega_0}{2Q}\left(1\mp j\sqrt{4Q^2-1}\right)\equiv-\alpha\left(1\mp j\sqrt{4Q^2-1}\right)$$

Example – series RLC

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{1 + sRC + s^2LC} \equiv \frac{1}{1 + s\frac{Q}{\omega_0} + \frac{s^2}{\omega_0^2}}$$

$$\omega_0^2 = \frac{1}{LC}; \quad Q = \frac{1}{\omega_0 RC} = \frac{\sqrt{L/C}}{R} \equiv \frac{Z_0}{R}; \quad \frac{\omega_0}{2Q} \equiv \alpha$$

$$\Downarrow$$

$$C = \frac{1}{L \cdot \omega_0^2}; \quad L = \frac{1}{C \cdot \omega_0^2}$$

$$Q = \frac{1}{RC\omega_0} = \frac{L}{R} \cdot \frac{1}{\omega_0}$$

$$\alpha = \frac{\omega_0}{2Q} = \frac{R}{2L}$$

Example – series RLC

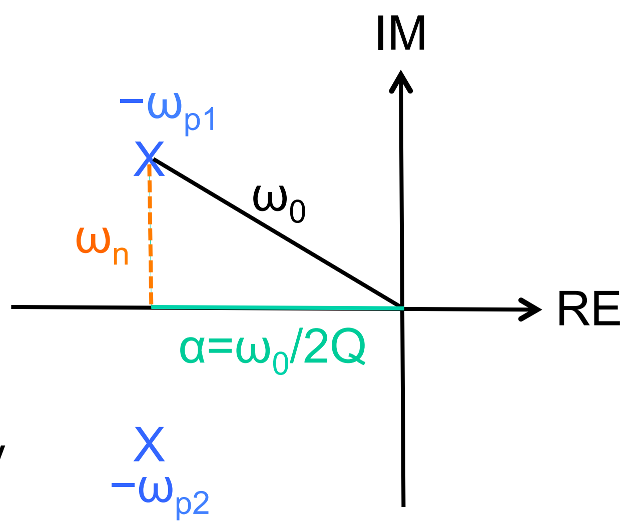
$$s^2 + s\cdot\frac{\omega_0}{Q} + \omega_0^2 = 0 \Leftrightarrow s^2 + s\cdot 2\alpha + \omega_0^2 = 0 \Leftrightarrow s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

– For Q ≤ ½ real poles:

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} \qquad \left(for\ \frac{\omega_0}{\alpha} << 1\ \ the\ poles\ are\ widely\ spaced \right)$$

– For Q > ½ complex poles

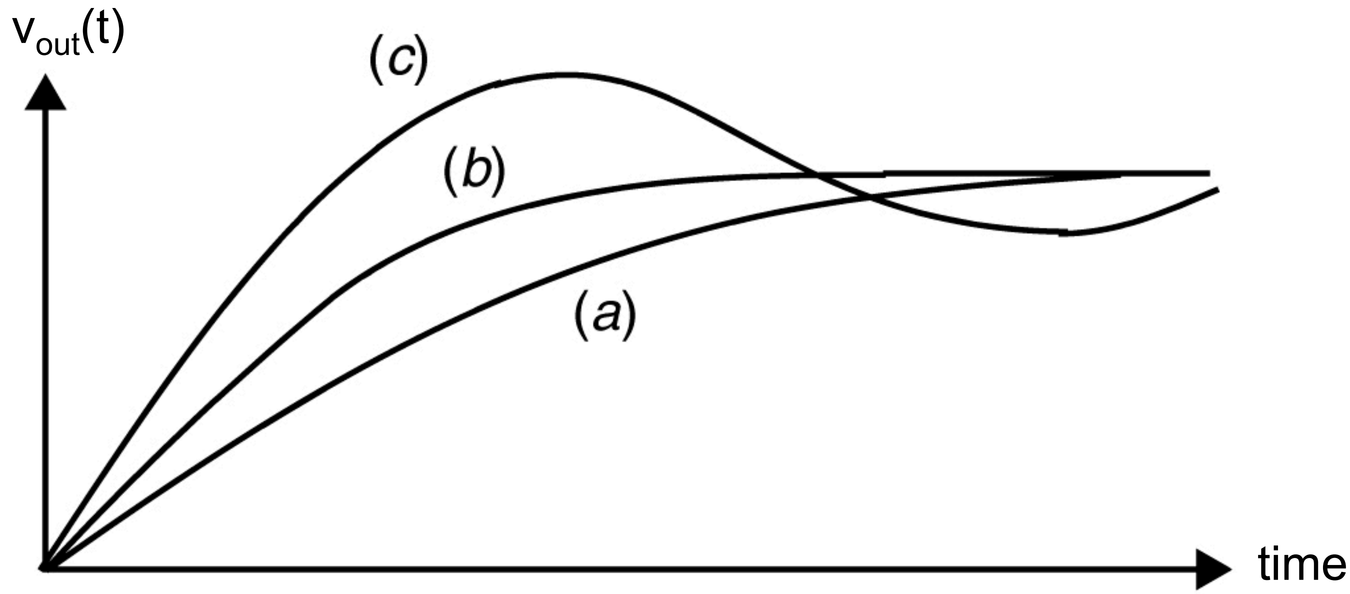
$$s_{1,2} = -\alpha \mp j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \mp j\omega_n$$



ω_n = natural (damped) frequency

ω_0 = resonant frequency

Example – series RLC: Step response



(a) Overdamped Q < 0.5 $(\alpha > \omega_0 \Leftrightarrow \zeta \equiv \alpha / \omega_0 > 1)$

(b) Critically damped Q=0.5 $(\alpha = \omega_0 \Leftrightarrow \zeta \equiv \alpha / \omega_0 = 1)$

(c) Underdamped Q > 0.5 $(\alpha < \omega_0 \Leftrightarrow \zeta \equiv \alpha / \omega_0 < 1)$ ω_n = ringing frequency

α = damping factor (rate of decay)
ω_0 = resonance frequency
ζ = Damping ratio

Example – RLC series: Quality Factor Q

For a system under sinusoidal excitation at a frequency ω, the most fundamental definition for Q is:

$$Q = \omega \frac{energy\, stored}{average\, power\, dissipated}$$ ⟵ dimensionless

At the resonant frequency ω_0, the current through the network is simply V_{in}/R. Energy in such a network sloshes back and forth between the inductance and the inductor, with a constant sum. The peak inductor current at resonance is $I_{pk}=V_{pk}/R$, so the energy stored by the network can be computed as:

$$E_{stored} = \frac{1}{2} L \cdot I_{pk}^2$$

The average power dissipated in the resistor at resonance is: $P_{avg} = \frac{1}{2} R \cdot I_{pk}^2$

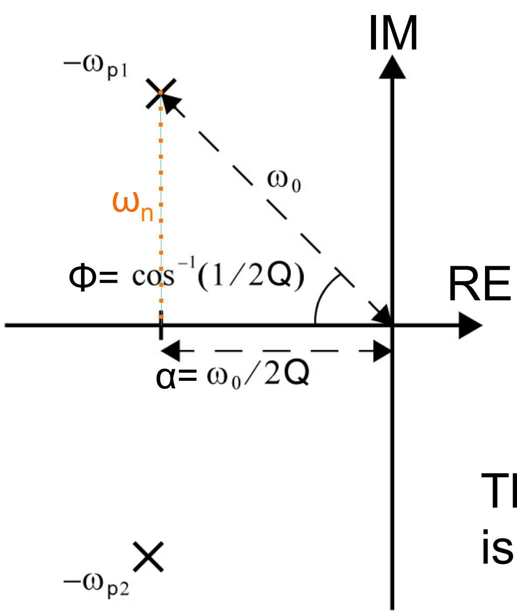
$$Q = \omega_0 \frac{E_{stored}}{P_{avg}} = \frac{1}{\sqrt{LC}} \cdot \frac{L}{R} = \frac{\sqrt{L/C}}{R} = \frac{Z_0}{R}$$

Z_0 = Characteristic impedance of the network

At resonance $|Z_C| = |Z_L| = \omega_0 L = (\omega_0 C)^{-1} = \sqrt{L/C} \equiv Z_0$

Example – series RLC: Ringing and Q

Since Q is a measure of the rate of energy loss, one expect a higher Q to be associated with more persistent ringing than a lower Q.



R large ⟷ α large ⟷ Q small

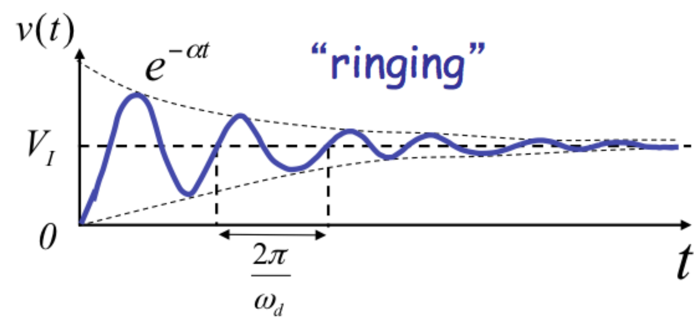
↕

Ringing doesn't last long and its excursion is small

Q=0.5 → Φ = 0°
Q=0.707 → Φ = 45°
Q=1 → Φ = 60°
Q=10 → Φ ≈ 87°

The decaying envelope is proportional to:

$$e^{-\alpha t} = e^{-\frac{\omega_0 t}{2Q}}$$

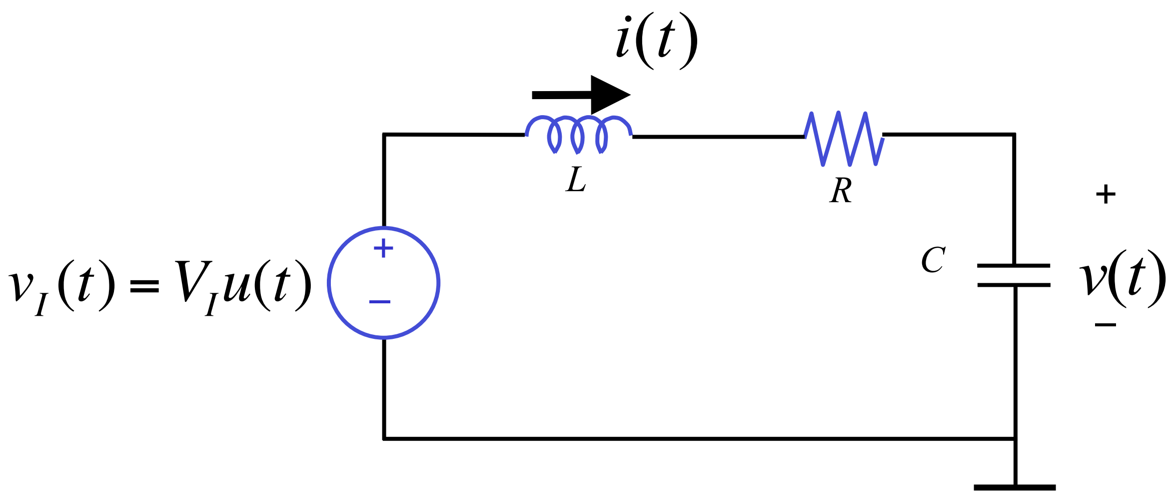


Rule of thumb: Q is roughly equal to the number of cycles of ringing.

Frequency of the ringing oscillations: $1/f_n = T_n = 2\pi/\omega_n$

Example – RLC series by intuition (1)

- We can predict the behavior of the circuit without solving pages of differential equations or Laplace transforms. All we need to know is the characteristics equation (denominator of H(s)) and the initial conditions



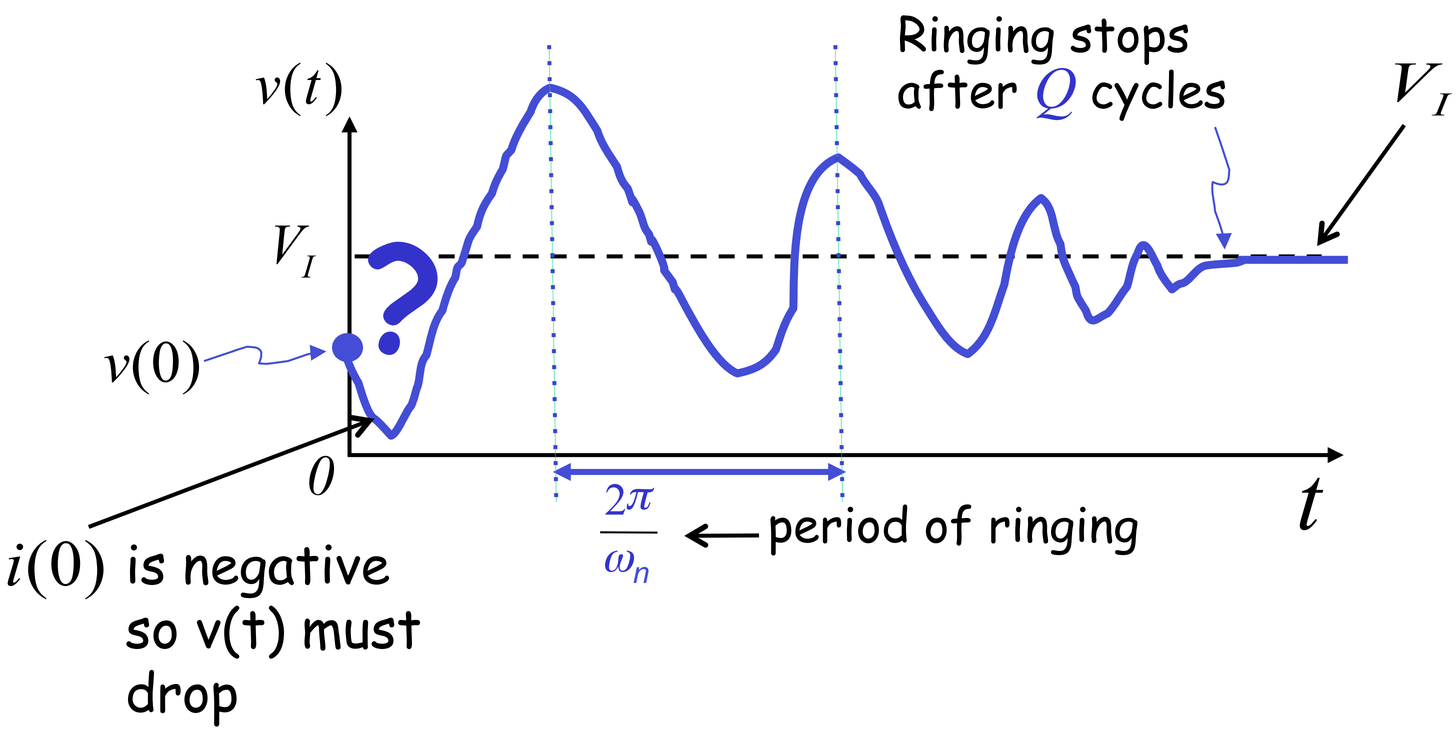
Assume the values of R,L,C are such that $\omega_0 > \alpha$ (underdamping).

Initial conditions:
$i(0) = -I_0$
$v(0) = +V_0$

- The voltage across the capacitor cannot jump: $v(0+)=v(0)=+V_0$
- The current through the inductor cannot jump: $i(0+)=i(0)=-I_0$
- The output voltage starts at V_0, it ends at $v(\infty)=V_I$ and it rings about Q times before settling at V_I

Example – RLC series using intuition (2)

- The only question left is to decide if v(t) will start off shooting down or up ?
- But, … we know that i(0+) is negative → this means the current flows from the capacitor toward the inductor → which means the capacitor must be discharging → the voltage across the capacitor must be dropping



Example – RLC series using intuition (3)

- In practice for the under damped case it useful to compute two parameters
  - Overshoot
  - Settling time

$$OS = \exp\left(\frac{-\pi}{\omega_n}\alpha\right)$$

← Normalized overshoot = % overshoot w.r.t. final value

$$t_s \cong -\frac{1}{\alpha}\ln\left(\varepsilon\frac{\omega_n}{\omega_0}\right)$$

ε is the % error that we are willing to tolerate w.r.t. the ideal final value

First order vs. Second order circuits Behavior

- First order circuits introduce exponential behavior
- Second order circuits introduce sinusoidal and exponential behavior combined
- Fortunately we will not need to go on analyzing 3^{rd}, 4^{th}, 5^{th} and so on circuits because they are not going to introduce fundamentally new behavior