IMPROVED FORMULAS FOR QUADRATIC ROOTS

$$as^2 + bs + c = 0$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 \gg 4ac$ the $\sqrt{\ }$ goes away

$$\boxed{T(s) = \frac{1}{as^2 + bs + c}}$$

1. WE DON'T LIKE A SOLUTION WITH $\sqrt{\ }$
2. [redacted] $b^2 \gg 4ac$ LHP and (real)

$$s_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \longrightarrow$$

"sloppy approx"

$s_1 = 0$

$$s_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \longrightarrow s_2 \approx -\frac{2b}{2a} = -\frac{b}{a}$$

OK PRECISION

$$0 = as^2 + bs + c = a\left(s^2 + \frac{b}{a}s + \frac{c}{a}\right) = a\,(s-s1)(s-s2)$$

$$0 = as^2 + bs + c = c\left(1 + s\frac{b}{c} + s^2\frac{a}{c}\right) \rightarrow s_1 \approx -\frac{c}{b}$$

Example

$a=1$, $b=45000$, $c=1$ $\rightarrow$ $s^2 + bs + 1 = 0$

$$\frac{-45000 \pm \sqrt{45000^2 - 4}}{2} = s_{1,2}$$

$b^2 \gg 4ac$

$45000^2 \gg 4$

"sloppy approx"

$s_1 = 0$, -2.000×10^{-5}, MATLAB -2.22×10^{-5}

HP 11C

$s2 =$ "sloppy" -45000, HP 11C -45000, MATLAB -45000 $\approx -\frac{b}{a}$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} =$$

$$= -\frac{b}{2a}\left(1 \pm \frac{\cancel{2a}}{b}\cdot\frac{1}{\cancel{2a}}\sqrt{b^2-4ac}\right) =$$

$$= -\frac{b}{2a}\left(1 \pm \sqrt{1-4\frac{ac}{b^2}}\right) = -\frac{b}{2a}\left(1 \pm \sqrt{1-4Q^2}\right)$$

$$Q^2 = \frac{ac}{b^2}$$

$$b^2 \gg 4ac \longleftrightarrow \frac{b^2}{4ac} \gg 1 \longleftrightarrow Q^2 \ll 0.25 \longleftrightarrow \boxed{Q \ll 0.5}$$

<u>back to the example</u>

$$s_{1,2} = -\frac{45000}{2}\left(1 \pm \sqrt{1-\frac{4}{45000^2}}\right)$$

	"sloppy approx"	HPC 11	MATLAB
$s_1 =$	0	-2.25×10^{-5}	-2.22×10^{-5}
$s_2 =$	-45000	-45000	-45000

Approximated formulas for quadratic roots

$$s_{1,2} = -\frac{b}{2a}\left[1 \pm \sqrt{1-4Q^2}\right]$$

$a=1,\ b=45000,\ c=1$

$$Q^2 = \frac{ac}{b^2}$$

$$s_{1,2} = -\frac{b}{2}\left[1 \pm \sqrt{1-\frac{4}{b^2}}\right]$$

$$s_2 \cong -\frac{b}{a} = -45000$$

$$\cancel{a}(s-s_1)(s-s_2) = \cancel{a}\left(s^2 + s\frac{b}{a} + \frac{c}{a}\right)$$

$$s^2 - s\cdot s_1 - s\cdot s_2 + \underbrace{s_1 s_2} = s^2 + s\frac{b}{a} + \underbrace{\frac{c}{a}}$$

$$s_1 s_2 = \frac{c}{a}$$

$$s_1 = \frac{c}{a}\cdot\frac{1}{s_2} \approx -\frac{c}{\cancel{a}}\cdot\frac{\cancel{a}}{b} \approx -\frac{c}{b} = -\frac{1}{45000} =$$

$$= -2.22\times 10^{-5}$$

BETTER FORMAT

$$s_2 = -\frac{b}{2a}\underbrace{\left(1+\sqrt{1-4Q^2}\right)}_{=F} = -\frac{b}{2a}\cdot F$$

s2 is acceptable for all values of a, b, c

$$s_1 = \frac{+c}{a}\cdot\frac{1}{s2} = -\frac{c}{a}\cdot\frac{2a}{b}\cdot\frac{1}{F} = -\frac{2c}{b}\cdot\frac{1}{F}$$

s_1 is not acceptable for $Q \ll 0.5$ ($b^2 \gg 4ac$)

$$Q^2 = \frac{ac}{b^2}$$

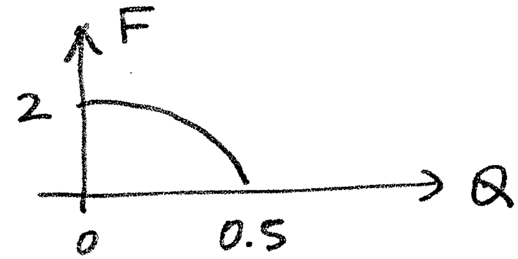
→ if $Q \leq 0.5 \longrightarrow$ F is real (LHP real roots)

- $Q \ll 0.5 \longrightarrow F \approx 2 \longrightarrow s_2 \cong -\frac{b}{a}$, $s_1 \approx -\frac{c}{b}$

→ if $Q > 0.5 \longrightarrow$ F is complex (LHP roots complex)

$Q \leq 0.3 \quad F \approx 2$ within about 10% error

$s_2 \cong -\frac{b}{a} \qquad s_1 \simeq -\frac{c}{b}$



$$T(s) = \frac{1}{1 + b_1 \cdot s + b_2 \cdot s^2}$$

$$b_1 = b/c$$
$$b_2 = a/c$$

$$as^2 + bs + c = 0 = c\left(1 + \frac{b}{c}s + \frac{a}{c} \cdot s^2\right)$$

$$s_1 \cdot s_2 \cdot a \,\frac{(s - s_1)}{s_1}\,\frac{(s - s_2)}{s_2} = 0$$

$$a \cdot s_1 \cdot s_2 \cdot \left(\frac{s}{s_1} - 1\right)\left(\frac{s}{s_2} - 1\right) = 0$$

$$\underbrace{a \cdot s_1 \cdot s_2}\, \cdot \left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right) = 0 = \underbrace{c}\left(1 + \frac{b}{c}s + \frac{a}{c} \cdot s^2\right)$$

$$\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right) = \left(1 + \underbrace{\frac{b}{c}}_{\equiv b_1} s + \underbrace{\frac{a}{c}}_{\equiv b_2} s^2\right)$$

$$T(s) = \frac{1}{1 + b_1 s + b_2 s^2} = \frac{1}{\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right)}$$

$$T(s) = \frac{1}{1 - \frac{s}{s_1} - \frac{s}{s_2} + \frac{s^2}{s_1 \cdot s_2}} = \frac{1}{1 + b_1 \cdot s + b_2 \cdot s^2}$$

s_1 and s_2 are widely spaced

$$T(s) \approx \frac{1}{1 - \frac{s}{s_1} + \frac{s^2}{s_1 s_2}} = \frac{1}{1 + b_1 s + b_2 s^2}$$

$$s_1 \approx -\frac{1}{b_1}$$

$$s_1 s_2 = \frac{1}{b_2} \longrightarrow s_2 = \frac{1}{b_2 \cdot s_1} \approx -\frac{b_1}{b_2}$$

$s_1 \approx -\dfrac{1}{b1}$ $\quad$ $s2 \approx -\dfrac{b1}{b2}$ $\quad$ $b1 = b/c$, $\; b2 = a/c$

$$\left.\begin{aligned} s1 &\approx -\frac{c}{b} \\ s2 &\approx -\frac{b}{a} \end{aligned}\right) \rightarrow \quad Q \ll 0.5 \longleftrightarrow |s1| \ll |s2|$$

In summary:

$$1 + b_1 s + b_2 s^2 = 0$$

$$\left.\begin{aligned} s1 &= -\frac{1}{b1} \cdot \frac{2}{F} \\ s_2 &= -\frac{b1}{b2} \cdot \frac{F}{2} \end{aligned}\right) \rightarrow \quad \begin{aligned} &Q \ll 0.5 \;(F \approx 2) \longleftrightarrow |s_1| \ll |s_2| \\ &s1 \approx -\frac{1}{b1} \\ &s2 \approx -\frac{b1}{b2} \end{aligned}$$

ADVANTAGES OF THE "IMPROVED" FORMULAS

1. BOTH ROOTS CAN BE COMPUTED WITH EQUAL ACCURACY
2. FOR REAL ROOTS ($\leftrightarrow Q \leq 0.5$) IT IS POSSIBLE TO GET A VERY GOOD APPROX (WITHIN ABOUT 10% OR LESS AS LONG AS $Q \leq 0.3$) WHERE THERE IS NO $\sqrt{\quad}$ ANYMORE IN THE RESULT AND EACH ROOT IS A SIMPLE RATIO OF COEFFICIENTS OF THE ORIGINAL QUADRATIC EQN.

$$s_1 = -\frac{1}{b_1}\cdot\frac{2}{F} \qquad \xrightarrow[(|s_1| \ll |s_2|)]{Q \ll 0.5\,(F \approx 2)} \qquad s_1 \approx -\frac{1}{b_1}$$

$$s_2 = -\frac{b_1}{b_2}\cdot\frac{F}{2} \qquad\qquad\qquad\qquad\qquad s_2 \approx -\frac{b_1}{b_2}$$

$$s_1 = -\frac{c}{b}\cdot\frac{2}{F} \qquad \xrightarrow[(|s_1| \ll |s_2|)]{Q \ll 0.5\,(F \approx 2)} \qquad s_1 \approx -\frac{c}{b}$$

$$s_2 = -\frac{b}{a}\cdot\frac{F}{2} \qquad\qquad\qquad\qquad\qquad s_2 \approx -\frac{b}{a}$$

using parameters that have "physical meaning"

$$Q = \frac{ac}{b^2}$$

- $Q \le 0.5$

$$s_{1,2} = -\frac{b}{2a}\left(1 \pm \sqrt{1 - 4Q^2}\right)$$

- $Q \le 0.3$ $\quad (F \simeq 2) \longleftrightarrow |s_1| << |s_2|$

$$s_1 \cong -\frac{1}{b_1} = -c/b$$

$$s_2 \approx -\frac{b_1}{b_2} = -b/a$$

- $Q > 0.5$

$$s_{1,2} = -\frac{b}{2a}\left(1 \pm j\sqrt{4Q^2 - 1}\right)$$

$$\frac{b}{2a} = ?$$

$$Q^2 = \frac{ac}{b^2} \longrightarrow 2a = \frac{2Q^2 b^2}{c}$$

$$\frac{b}{2a} = \frac{\cancel{b} \cdot c}{2Q^2 b^{\cancel{2}}} = \frac{1}{2Q^2} \cdot \frac{c}{b} = \frac{1}{2Q^2} \cdot \frac{1}{b_1}$$

$$b_1 = \frac{b}{c} \qquad b_2 = \frac{a}{c}$$

$$1 + b_1 \cdot s + b_2 \cdot s^2 \leftarrow$$

$$\frac{b_2}{b_1^2} = Q^2 = \frac{a/c}{b^2/c^2} = \frac{ac}{b^2} \longrightarrow b_2 = Q^2 \cdot b_1^2 \triangleq \frac{1}{\omega_0^2}$$

$$b_1^2 = \frac{1}{Q^2 \omega_0^2} \longrightarrow b_1 = \frac{1}{Q\omega_0}$$

$$b_2 = \frac{1}{\omega_0^2} \qquad 1 + b_1 s + b_2 s^2 = 1 + \frac{s}{Q\omega_0} + \frac{s^2}{\omega_0^2} = 0$$

$$\frac{b}{2a} = \frac{1}{2Q^2} \cdot \frac{1}{b_1} = \frac{1}{2Q^2} \cdot Q \cdot \omega_0 = \frac{\omega_0}{2Q}$$

REMEMBER

- For $Q > 0.5$

$$s_{1,2} = -\frac{\omega_0}{2Q}\left(1 \pm j\sqrt{4Q^2 - 1}\right)$$

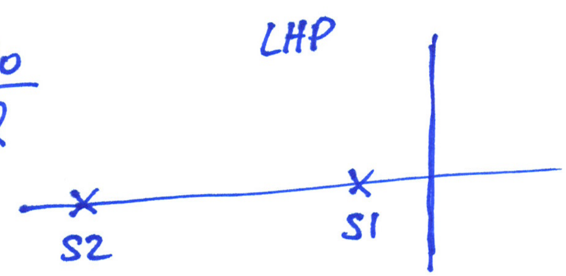
- For $Q \leq 0.5$

$$s_{1,2} = -\frac{\omega_0}{2Q}\left(1 \pm \sqrt{1 - 4Q^2}\right)$$

$\rightarrow$ $Q \ll 0.5$ (for all practical purposes $Q \leq 0.3$)

$$s_1 \approx -\frac{1}{b_1} = -\omega_0 Q$$

$$s_2 \approx -\frac{b_1}{b_2} = -\frac{\omega_0}{Q}$$



$$T(s) = \frac{1}{1 + b_1 s + b_2 s^2} \approx \frac{1}{1 + b_1 s} \longrightarrow \begin{array}{l} 1 + b_1 \cdot s = 0 \\ s_1 = -\frac{1}{b_1} = -\omega_0 Q \end{array}$$

for small s