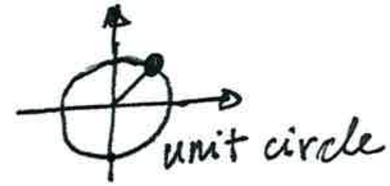
Sinusoidal Signals



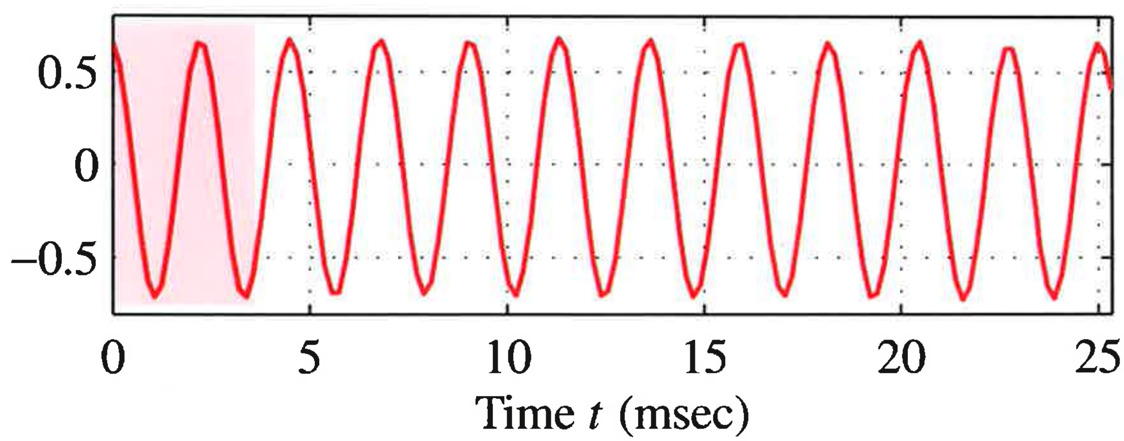
sin t

sin x

Objectives

- Sinusoid Formula/Sinusoid plot
- Relationship between time-shift and phase-shift
- Why Sinusoids ?
  - Tuning Fork - "A" note is a 440 Hz sinusoid)

A-440 Tuning Fork Signal



McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

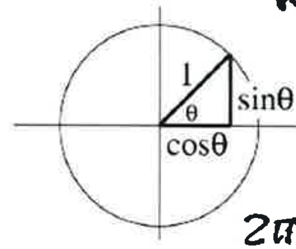
$T = [sec]$

$\omega = [rad/sec] = \frac{2\pi}{T}$

$\sin\left(\frac{2\pi}{T} t\right)$

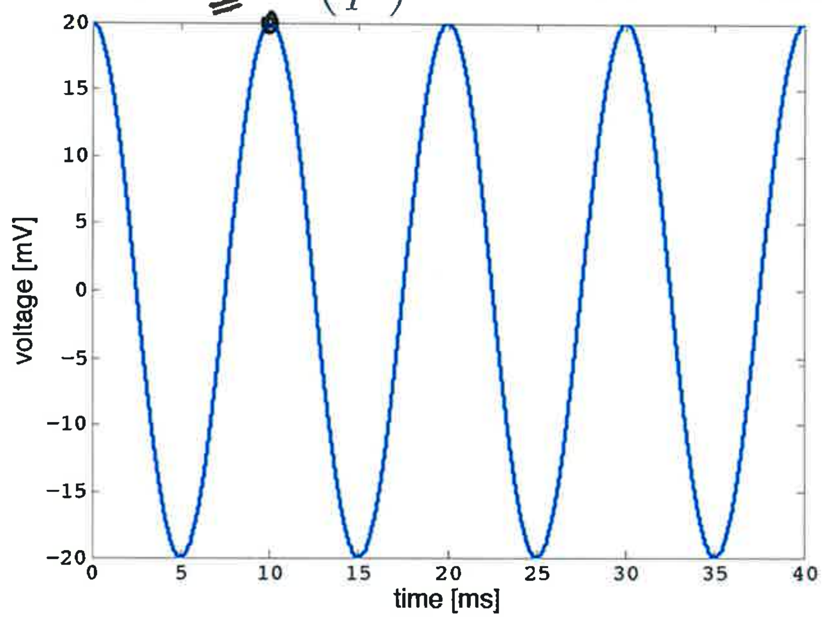
$\cos\left(\frac{2\pi}{T} \cdot t\right) = \cos \omega t$

Sinusoidal Signal

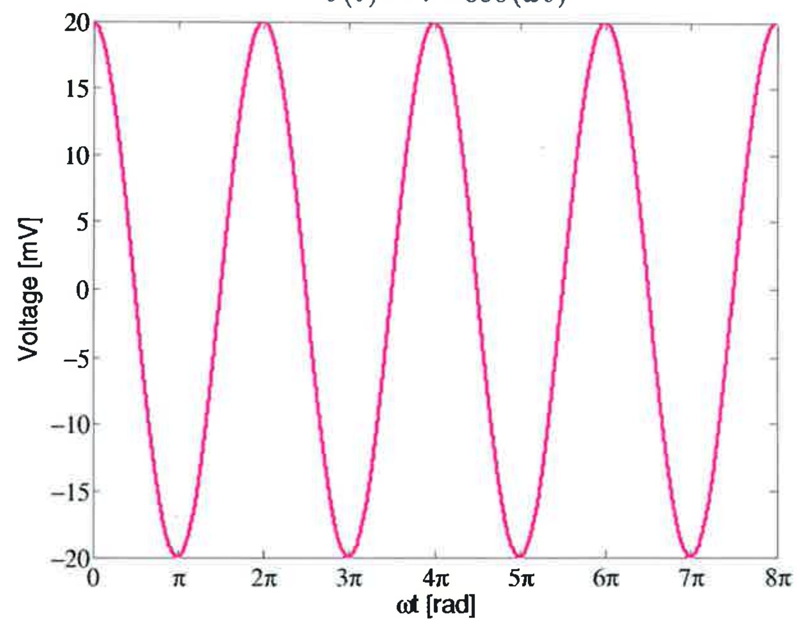


$2\pi f = \omega$

$$v(t) = \hat{V} \cdot cos\left(\frac{2\pi}{T} t\right) = \hat{V} \cdot cos(2\pi f t) = \hat{V} \cdot cos(\omega t)$$



$$v(t) = \hat{V} \cdot cos(\omega t)$$



ωt ~ phase

T = period [sec]

f = 1/T [cycles/sec = Hz]

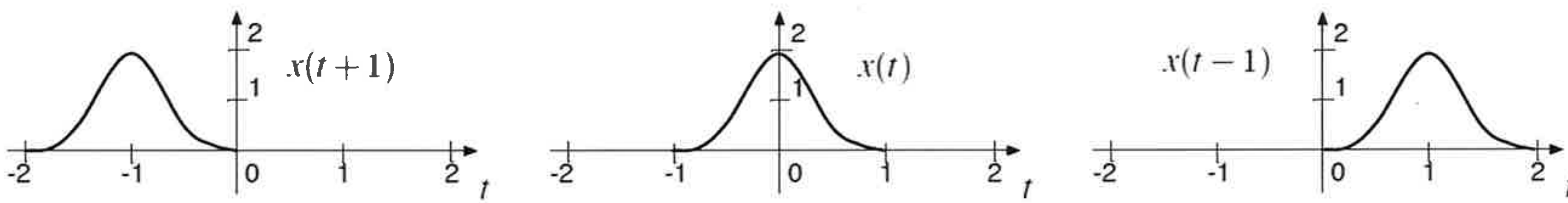
ω = 2π/T = 2πf = [rad/sec]

Example:
T = 10 ms
f = 100 Hz
ω = 628 rad/sec
$\hat{V} = 20\, mV$

Time Shift

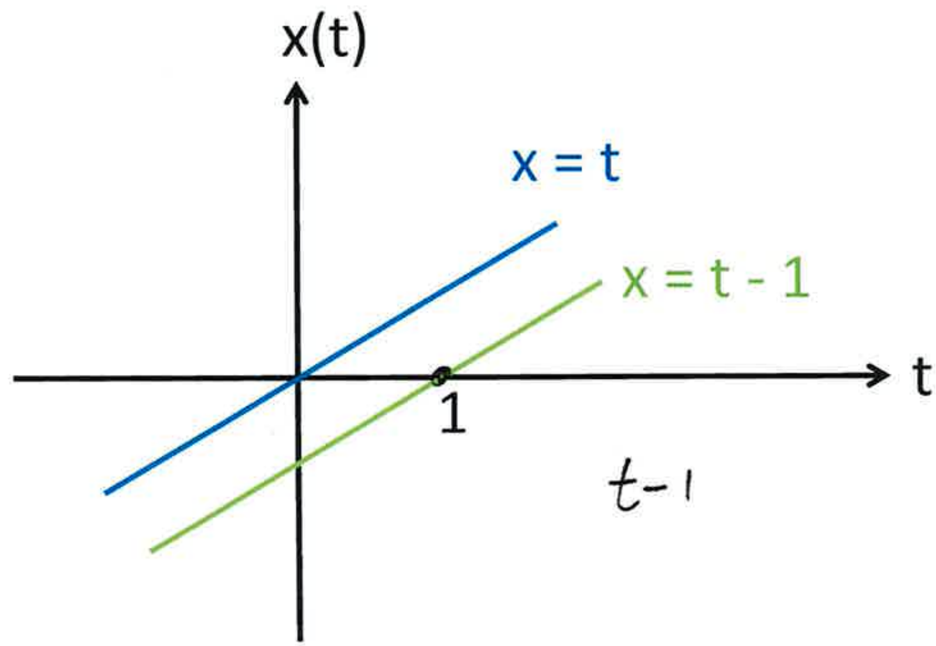
For a continuous-time signal $x(t)$, and a time $t_1 > 0$,

- Replacing t with $t - t_1$ gives a *delayed* signal $x(t - t_1)$
- Replacing t with $t + t_1$ gives an *advanced* signal $x(t + t_1)$



- May seem counterintuitive. Think about where $t - t_1$ is zero.

Time Shift

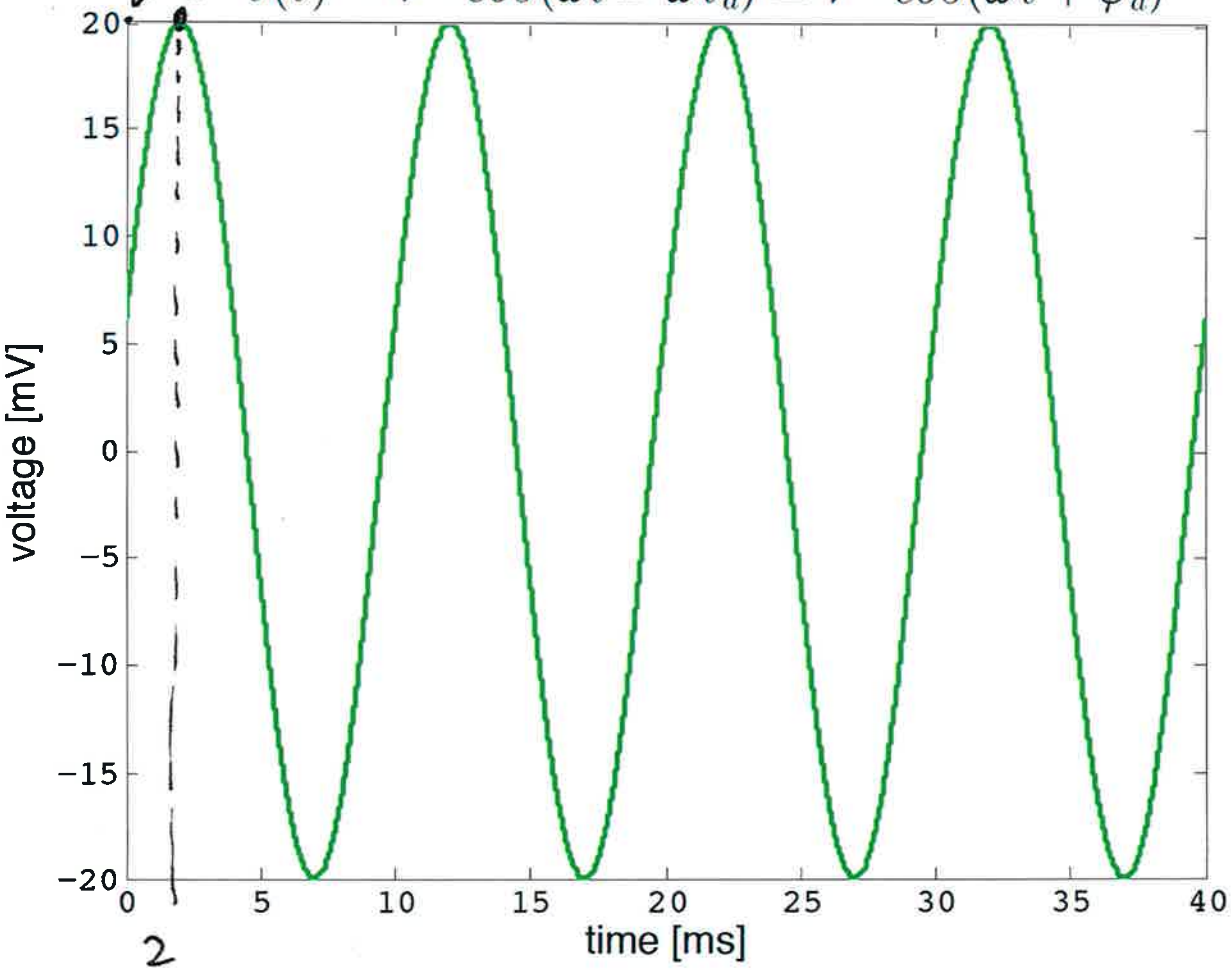


Time Shift

$t_d = 2ms$

Time shifted sinusoid: $t \rightarrow t - t_d$

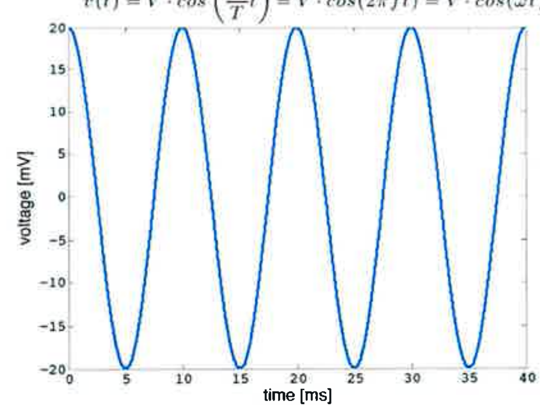
$$v(t) = \hat{V} \cdot cos(\omega t - \omega t_d) = \hat{V} \cdot cos(\omega t + \varphi_d)$$



t_d = time delay

φ_d = phase shift

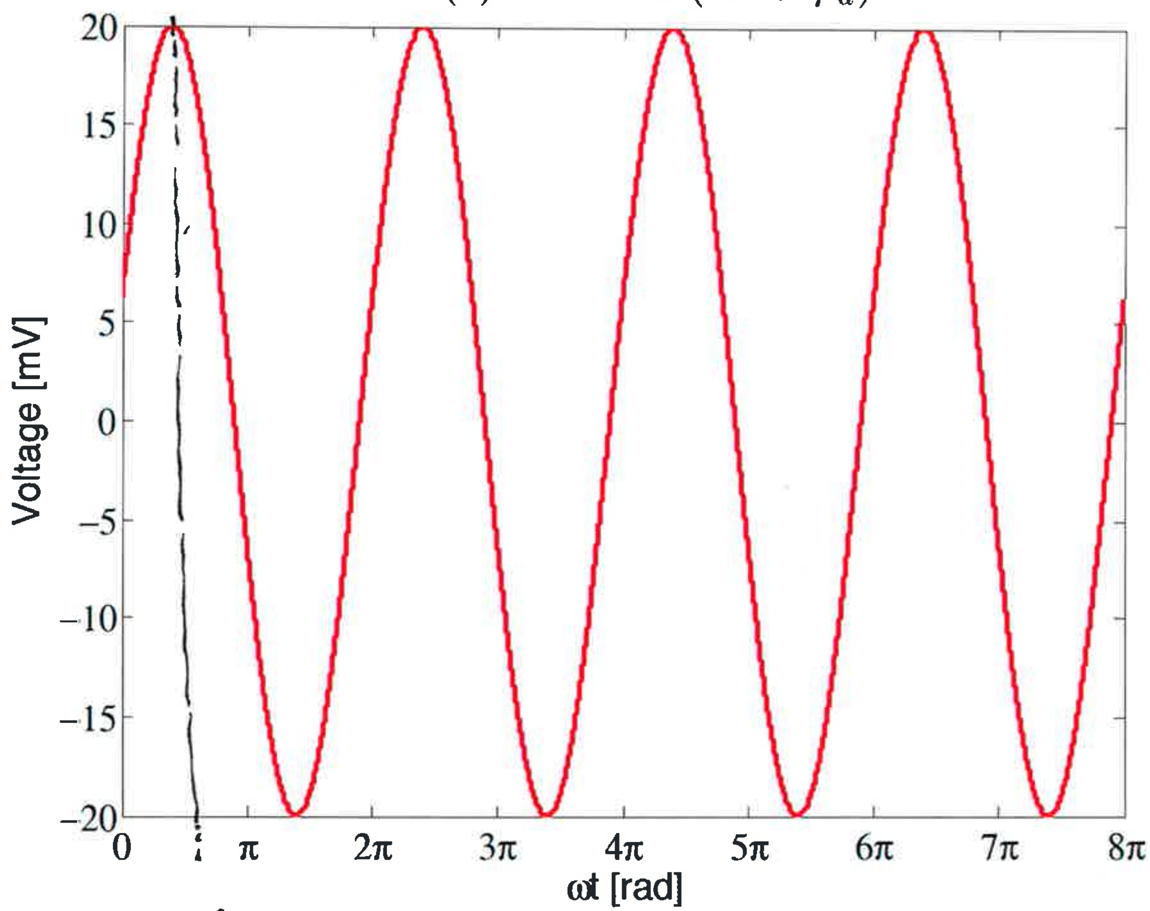
$$\boxed{\varphi_d = -2\pi \frac{t_d}{T} \; [rad]}$$



original sinusoid

Time Shift ⟷ Phase Shift

$$v(t) = \widehat{V} \cdot cos(\omega t + \varphi_d)$$



$72^0 \approx 0.4\pi$

Example:

T = 10 ms

t_d = 2 ms

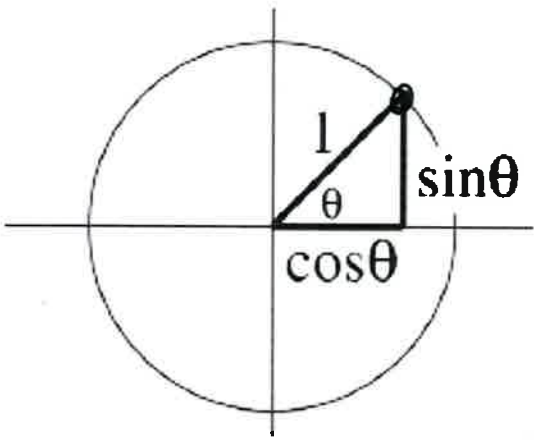
φ_d = −0.4π rad ≈ −72°

time delay

↕

negative phase shift

Euler's Formula



$$1e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

angular velocity = angular freq.

$\underline{\theta = \omega t}$ ⟵ Angle changes with time

Interpret $\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$ as a Rotating Vector

More general case (rotating along non-unity circle):
$R\, e^{j\omega t} = R\cos(\omega t) + j\, R\sin(\omega t)$

$\cos \omega t$ = REAL PART

$V_p \cdot e^{j(\omega t+\varphi)} =$

$= V_p \cdot e^{j\omega t} \cdot e^{j\varphi}$

$$\cos(\omega t) = \Re\mathrm{EAL}\{e^{j\omega t}\}$$

$V_p = V_M = \hat{V}$

General sinusoid: $v(t) = V_p \cos(\omega t + \varphi)$

$$V_p \cos(\omega t + \varphi) =$$

$$= \Re\mathrm{EAL}\{V_p e^{j(\omega t+\varphi)}\} = \Re\mathrm{EAL}\{\underbrace{V_p e^{j\varphi}}_{=\tilde{U}} e^{j\omega t}\} =$$

$$= \Re\mathrm{EAL}\{\tilde{U} e^{j\omega t}\}$$

$\dot{U}$ $\bar{U}$

(Complex Amplitude = Phasor)

$V_p,\ \varphi$

Complex Amplitude = Phasor

- If we know the speed at which we rotate (angular speed = angular frequency [rad/s])
- All we need to represent a sinusoid are two parameters: $\mathbf{V_p}$ and $\boldsymbol{\varphi}$
- To get back to the "verbose" representation:

$$\Re\text{EAL}\ \{\underbrace{V_p e^{j\varphi}}\ \underbrace{e^{j\omega t}}\} = \underline{V_p \cos(\omega t + \varphi)}$$

Why complex numbers ?

- They are way simpler to deal with than trigonometric expressions !!
- Do you remember what is $\cos(\alpha_1 + \alpha_2)$?

$RE\{e^{j(\alpha 1 + \alpha 2)}\} = RE\{e^{j\alpha 1}\, e^{j\alpha 2)}\}$

$$e^{j\alpha 1} \cdot e^{j\alpha 2} = (\cos\alpha_1 + j\sin\alpha_1)(\cos\alpha_2 + j\sin\alpha_2) =$$

$$= \cos\alpha 1 \overset{\cos\alpha 2}{\cdot} + j\cos\alpha 2 \cdot \sin\alpha_1 + j\sin\alpha 2 \cdot \cos\alpha 1 - \sin\alpha 1 \sin\alpha 2$$

$$= \cos\alpha 1 \cdot \cos\alpha 2 - \sin\alpha_1 \cdot \sin\alpha_2 + j(\cdots)$$

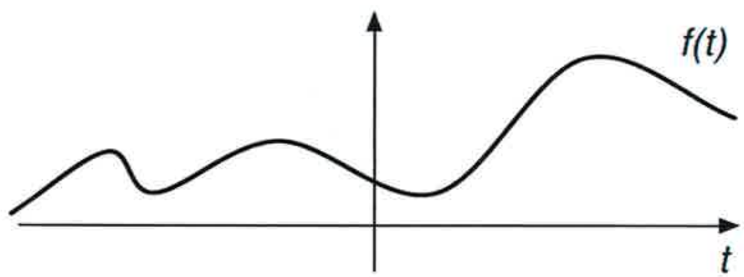
$$\underline{\cos(\alpha 1 + \alpha 2) = \cos\alpha 1 \cos\alpha 2 - \sin\alpha 1 \cdot \sin\alpha 2}$$

Why are we obsessed about sinusoids ?

- Given a sinusoid at the input, the output of an **LTI system** will be a sinusoid with same frequency but possibly different phase and amplitude
- **LTI systems** behavior can be fully characterized in fairly simple terms by applying sinusoids at their input

Signals

- Think of a signal as a function of time, t
  - e.g. the output voltage of some circuit
- Domain of a signal: all the t's for which is defined



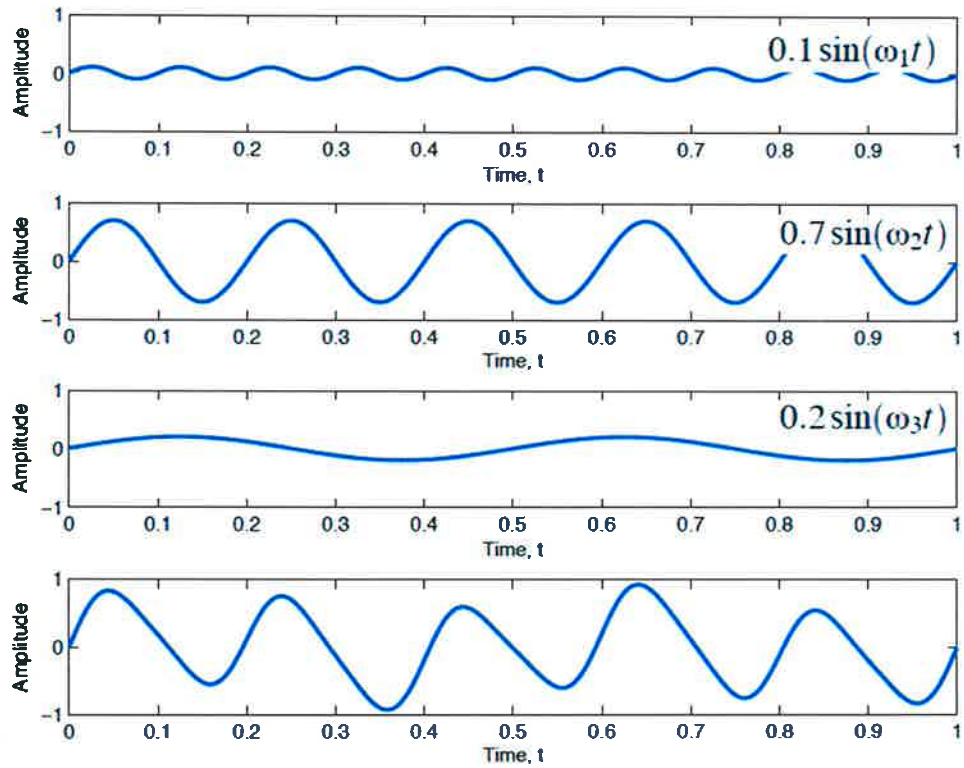
Useful for saying what is happening at a particular time

Not so useful for capturing the overall characteristics of the signal.

Idea : Frequency Domain Representation of Signals

Nice idea, but ... can we really do it ?

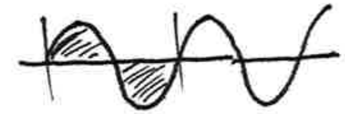
- Represent signal as a combination of sinusoids



$$f(t) = 0.1\sin(\omega_1 t) + 0.7\sin(\omega_2 t) + 0.2\sin(\omega_3 t)$$

source: S. Boyd

T



Measuring the "size" of a signal

size of a signal u is measured in many ways

• mean value: $\lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)\,dt$

for example, if $u(t)$ is defined for $t \geq 0$:

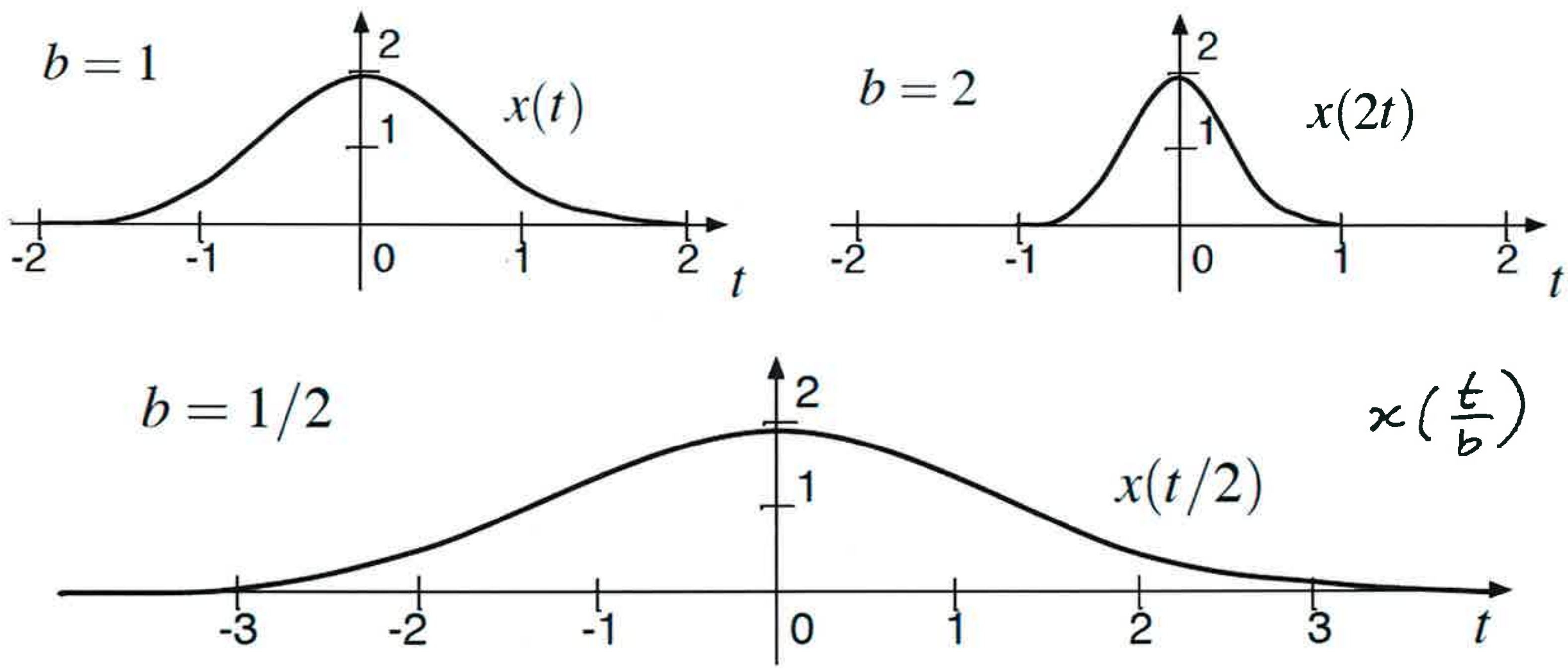
- *integral square* (or *total energy*): $\int_0^\infty u(t)^2\,dt$
- squareroot of total energy $\sqrt{\int_0^\infty u(t)^2\,dt}$
- *integral-absolute value*: $\int_0^\infty |u(t)|\,dt$
- *peak* or *maximum absolute value* of a signal: $\max_{t\geq 0} |u(t)|$
- *root-mean-square* (RMS) value: $\left(\lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)^2\,dt\right)^{1/2}$
- *average-absolute* (AA) value: $\lim_{T\to\infty} \frac{1}{T}\int_0^T |u(t)|\,dt$

for some signals these measures can be infinite, or undefined

Operations on Signals: Time Scaling, Continuous Time

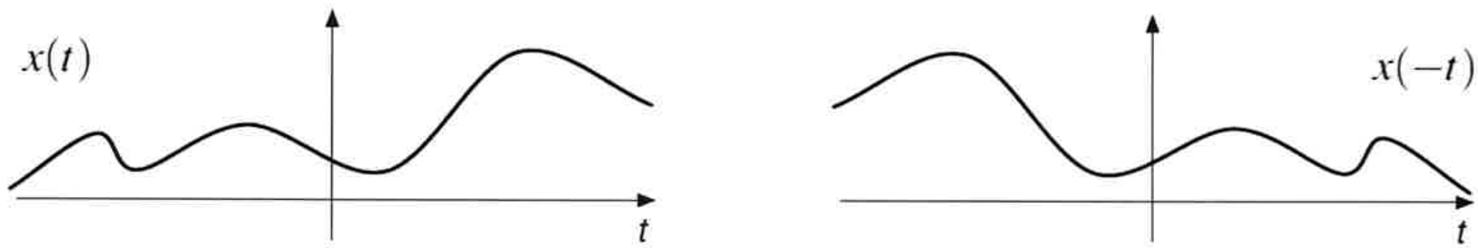
$x(t)$ $\quad$ $x(bt)$

A signal $x(t)$ is scaled in time by multiplying the time variable by a positive constant b, to produce $x(bt)$. A positive factor of b either expands $(0 < b < 1)$ or compresses $(b > 1)$ the signal in time.



Time Reversal

- Continuous time: replace t with $-t$, time reversed signal is $x(-t)$



- Special case of time scaling with $b = -1$.

(eventually LTI systems)

Systems

$x \to \boxed{S} \to y$

- A system transforms *input signals* into *output signals*.

- A system is a *function* mapping input signals into output signals.

$x \to \boxed{F} \to y$

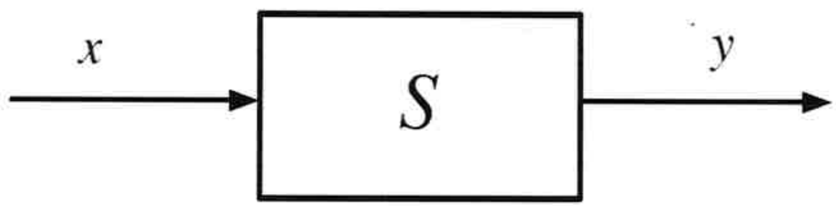
$y = S(x)$ $\quad$ $y = F(x)$

$y = Sx$ $\quad$ $y = Fx$

- Notation:
  - $y = Sx$ or $y = S(x)$, meaning the system S acts on an input signal x to produce output signal y.
  - $y = Sx$ does not (in general) mean multiplication!

Block Diagrams

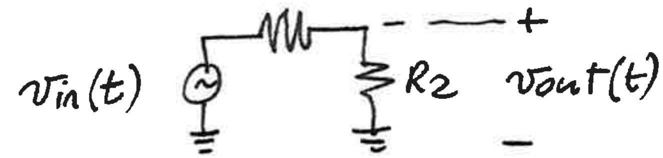
We will often describe systems by drawing pictures. These are *block diagrams*. A block diagram for a simple system is



- Lines with arrows denote signals (*not* wires).
- Boxes denote systems; arrows show inputs & outputs.
- Special symbols for some systems.

Examples

attenuator



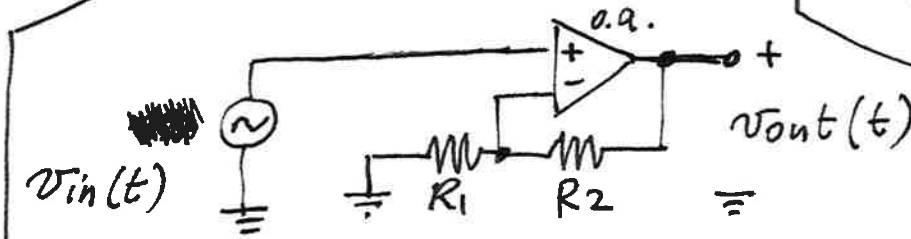
$$v_{out}(t) = \frac{R_2}{R_1+R_2} \cdot v_{in}(t)$$

gain

Scaling system: $y(t) = ax(t)$

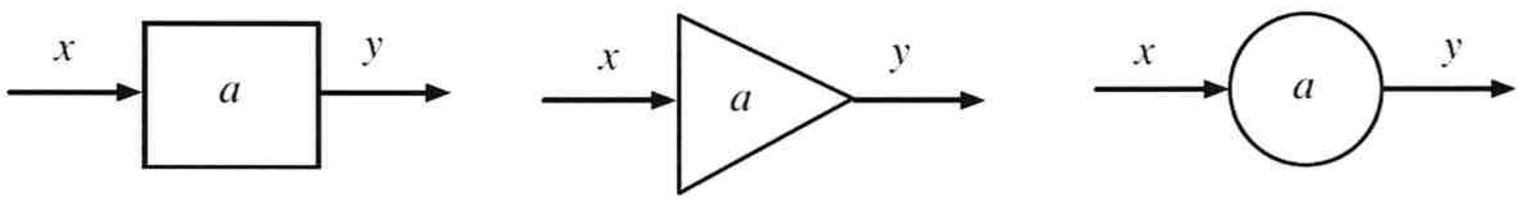
- *amplifier* if $|a| > 1$.
- *attenuator* if $|a| < 1$.
- *inverting* if $a < 0$.
- a is called the *gain* or *scale factor*.
- Common scaling system symbols:

amplifier

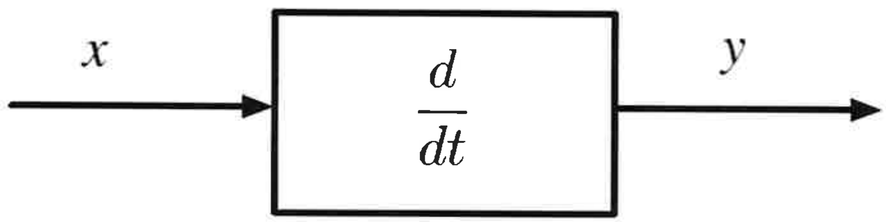


$$v_{out}(t) = \left(1 + \frac{R_2}{R_1}\right) \cdot v_{in}(t)$$

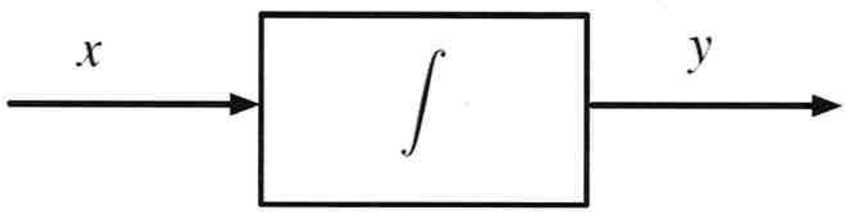
a large $\rightarrow R_2/R_1$ large



Differentiator: $y(t) = x'(t)$



Integrator: $y(t) = \int_a^t x(\tau)\, d\tau$ (a is often 0 or $-\infty$)



time shift system: $y(t) = x(t - T)$

- called a *delay system* if $T > 0$
- called a *predictor system* if $T < 0$

Linearity

A system F is **linear** if the following two properties hold:

1. **homogeneity:** if x is any signal and a is any scalar,

$$F(ax) = aF(x)$$

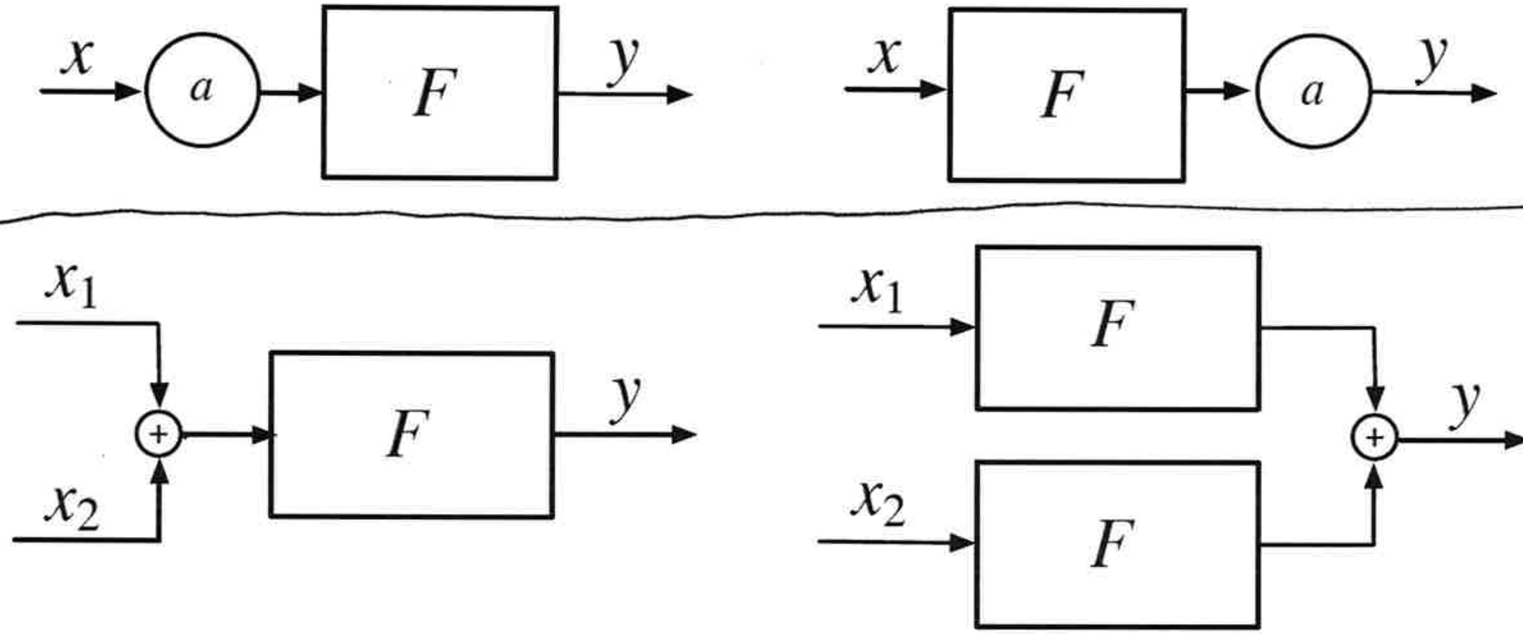
2. **superposition:** if x and $\tilde{x}$ are any two signals,

$$F(x + \tilde{x}) = F(x) + F(\tilde{x})$$

In words, linearity means:

- Scaling before or after the system is the same.
- Summing before or after the system is the same.

Linearity means the following pairs of block diagrams are equivalent, *i.e.*, have the same output for any input(s)



Equivalent Definition of Linearity: Superposition and homogeneity can be combined. If x and $\tilde{x}$ are any two signals, and a and b are constants, a system is linear if

$$F(ax + b\tilde{x}) = aF(x) + bF(\tilde{x})$$

Time-Invariance

- A system is time-invariant if a time shift in the input only produces the same time shift in the output.

- For a system F,

$$y(t) = Fx(t)$$

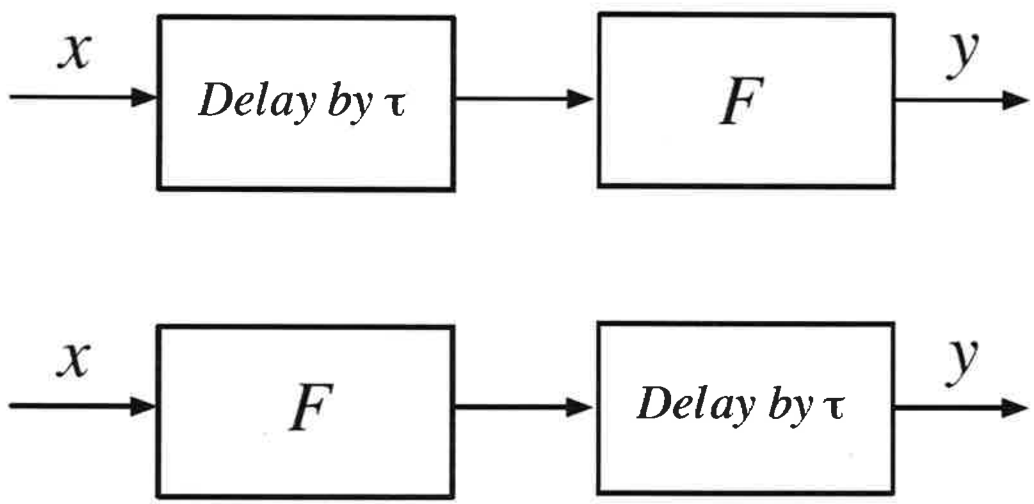
  implies that

$$y(t - \tau) = Fx(t - \tau)$$

  for any time shift τ.

<u>LTI</u>

- Implies that delay and the system F commute. These block diagrams are equivalent:



- Time invariance is an important system property, it greatly simplifies the analysis

Fourier Series (Exponential Representation)

If $f(t)$ is a well-behaved signal which is either (1) periodic with period T_0, or (2) defined only on an interval $\{\tau \leq t < \tau + T_0\}$ of length T_0 (finite-duration), then $f(t)$ can be written as a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/T_0$, and

$$D_n = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) e^{-jn\omega_0 t}\, dt$$

for all integer n. The sequence $\{D_n\}$ constitutes the *Fourier coefficients* of $f(t)$.

Source: Ulaby and Maharbitz

T_0

DC value = const. value

Fourier Series (Cosine/Sine Representation)

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t),$$

(cosine/sine representation),

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t)\, dt,$$

← mean value $f(t)$

$$f(t) = \sum_{n=0}^{\infty} (a_n \cdot \cos n\omega_0 t + b_n \cdot \sin n\omega_0 t)$$

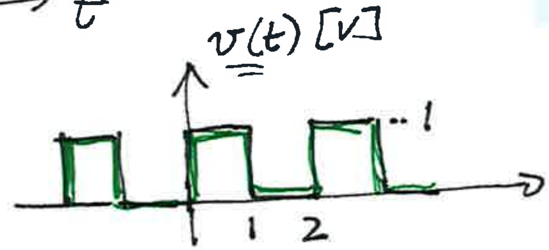
$$a_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos n\omega_0 t\, dt,$$

$$\omega_0 = \frac{2\pi}{T_0}$$

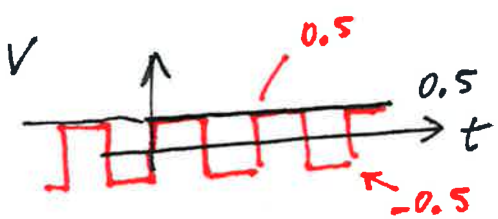
$$A \cos \omega_0 t \longrightarrow \omega_0 = 0 \longrightarrow A$$

$$b_n = \frac{2}{T_0} \int_0^{T_0} f(t) \sin n\omega_0 t\, dt.$$

$v_0 = 0$



$v_0 = 0.5\,V$



For the f(t) selected all a_i for $i=1,2,\cdots,\infty$ happens to be 0

Example (1)

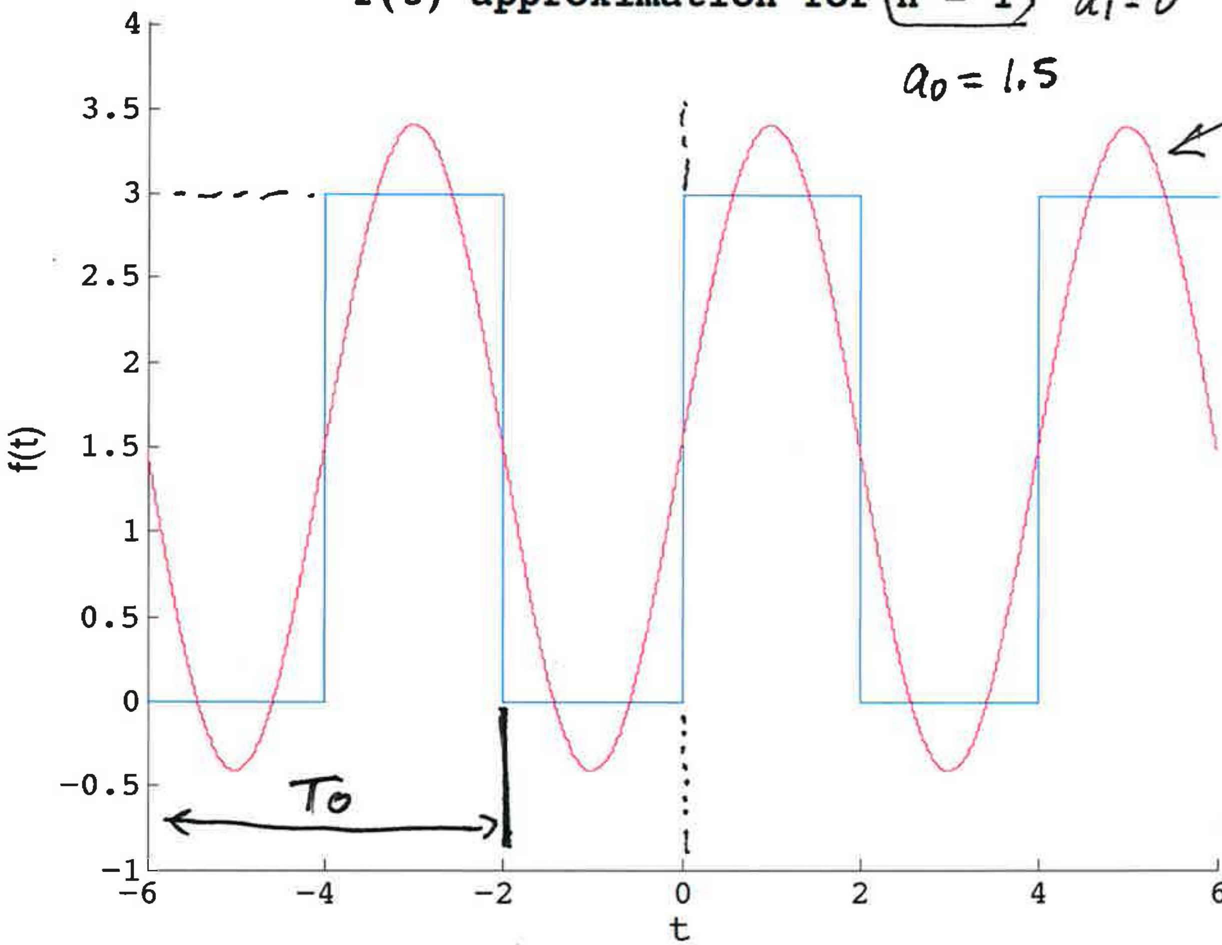
<u>square wave with T = 4 s</u>

$1.5 + b_1 \cdot \sin\frac{2\pi}{T_0}t \;\Big|\; \sum_{n=1}^{\infty}$

a_0

f(t) approximation for n = 1 $a_1 = 0$

$a_0 = 1.5$



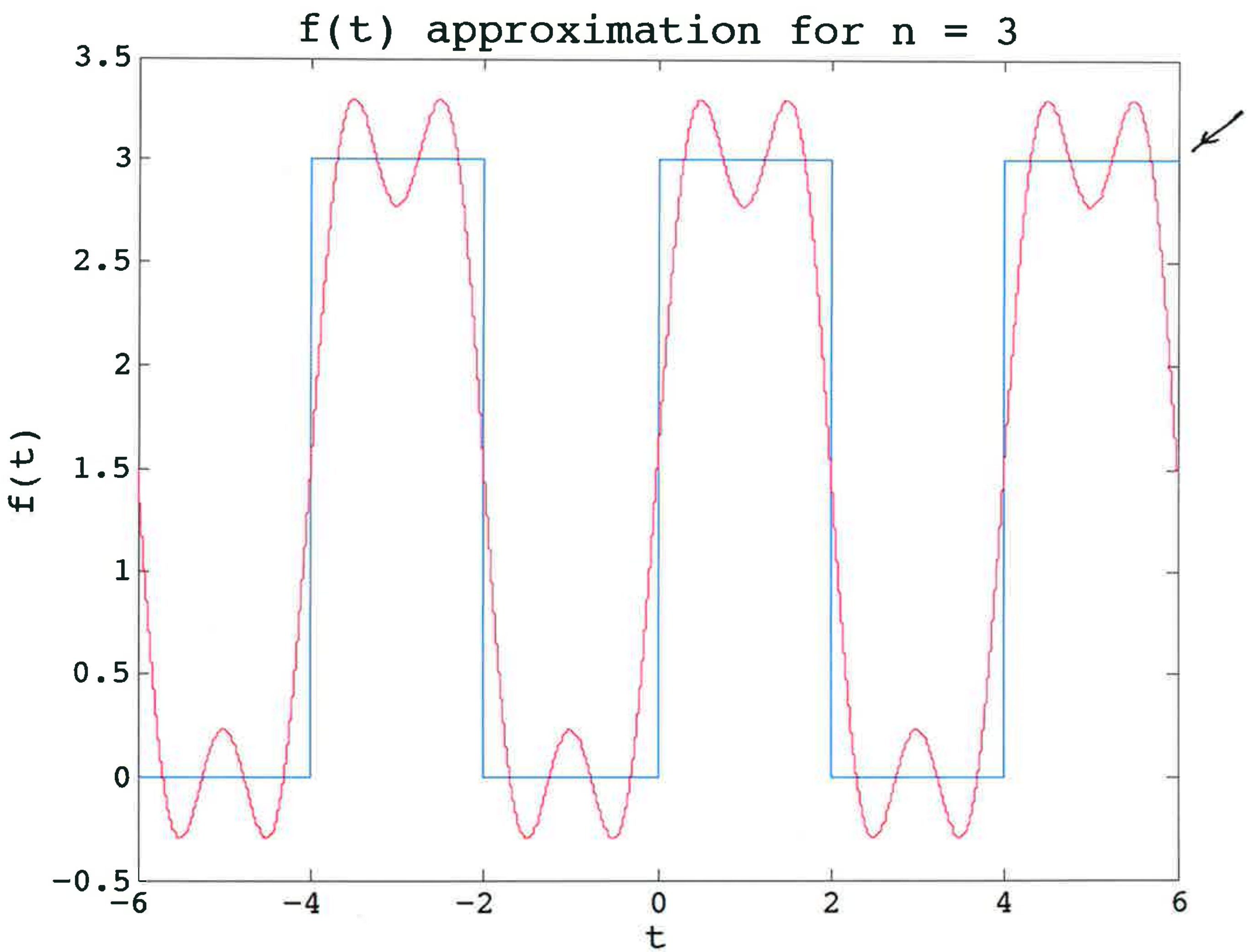
$T_0 = 4s$

$f_0 = \frac{1}{T_0}$

Example (2)

square wave with T = 4 s

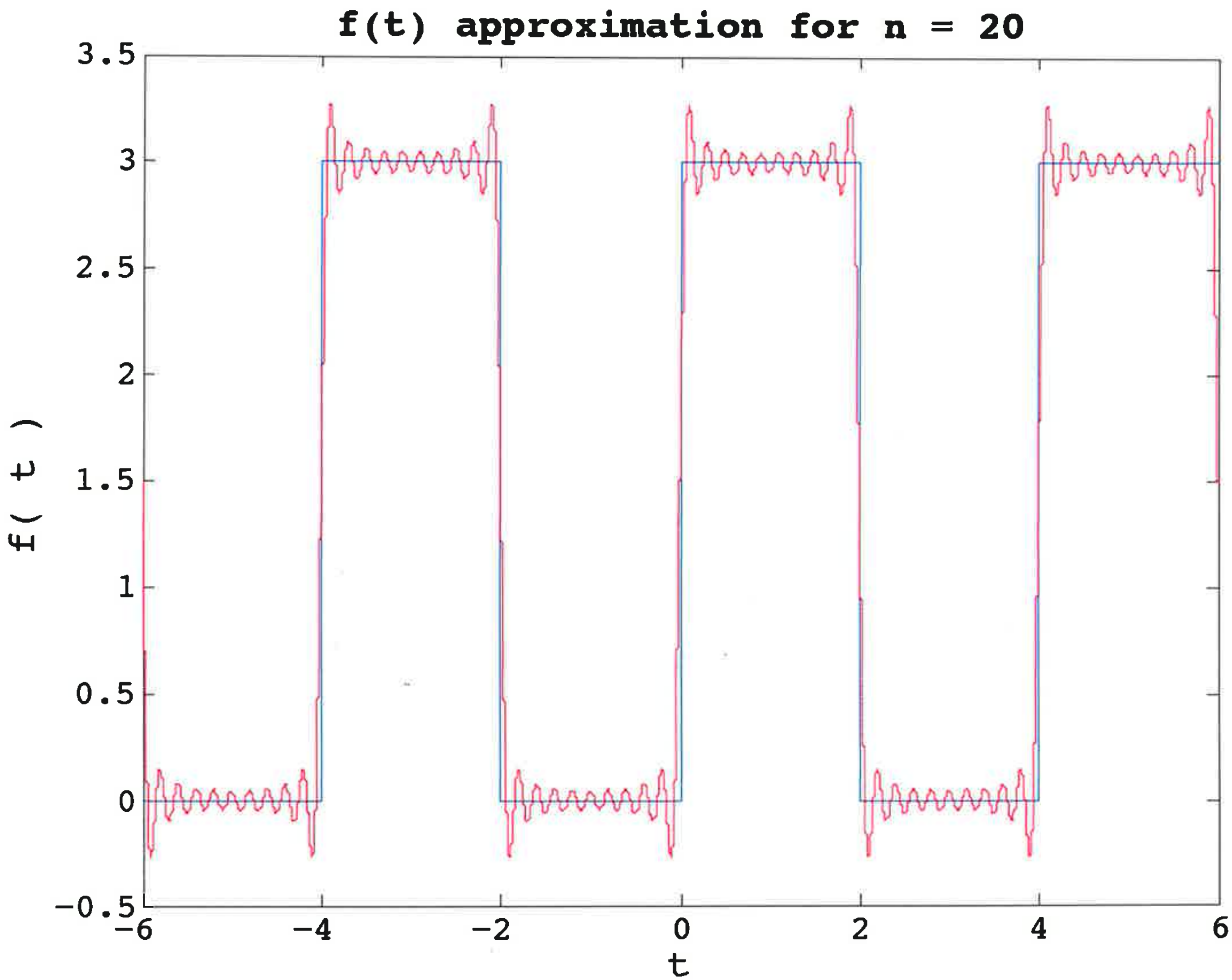
$a_0, b_1, b_2, b_3,$



Example (3)

square wave with T = 4 s

$n \to \infty$



Fourier Transform

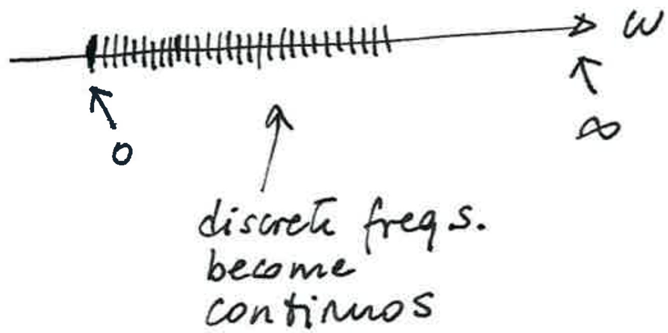
If $f(t)$ is a well-behaved aperiodic signal, then $f(t)$ can be written as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega, \qquad \Sigma \to \int$$

where

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

The function $F(j\omega)$ is called the *Fourier transform* of $f(t)$.



Frequency Domain Analysis

Freqency domain representation of continuous time signals in general means a Fourier series or Fourier transform.