# COUNTING AND PROBABILITY

**Axioms of Probability.** The assignment of probability to events in a sample space $S$ must obey the following rules:
1. $P(E) \geq 0$ for any event $E$
2. $P(S) = 1$
3. If $E_1, E_2, E_3, \ldots$ are disjoint events, then $P(E_1 \cup E_2 \cup E_3 \cup \ldots) = \sum_{i=1}^{\infty} P(E_i)$

**Theorem.** If $S$ is a sample space consisting of $N$ equally likely outcomes and $E$ is an event consisting of $N(E)$ outcomes, then $P(E) = \dfrac{N(E)}{N}$.

**1.** The experiment of flipping a coin 3 times has a sample space consisting of 8 equally likely outcomes: $S = \{HHH, HHT, HTH, THH, ..., TTT\}$. Let $X$ be the number of heads.

a) What set of outcomes corresponds to $X = 3$?

b) What set of outcomes correspond to $X = 2$?

c) What set of outcomes correspond to $X = 1$?

d) What set of outcomes correspond to $X = 0$?

e) Use the theorem to calculate $P(X = 3)$, $P(X = 2)$, $P(X = 1)$, and $P(X = 0)$.

The theorem tells us that (at least some of the time) we can calculate probabilities by counting. The mathematics of counting is called **combinatorics**. Section 2.3 gives a quick introduction to some basic counting techniques, including the following.

**Method.** (Fundamental principle of counting) If a process occurs in two steps and there are $n_1$ options for the first step and $n_2$ options for the second, then there are $n_1 n_2$ total possibilities.

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**2.** A PIN consists of four of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, each of which may be used multiple times (selection with replacement, order matters).

a) How many PINs start with 123?

b) How many PINs start with 12?

c) How many total PINs are there?

d) What is the probability that a randomly selected PIN starts with 123?

e) What is the probability that a randomly selected PIN starts with 12?

**3.** Continue working with 4-digit PINs, but now suppose that each digit can be used at most once.

a) In this situation, how many different PINs use just the digits 1, 2, 3, and 4 (selection without replacement, order matters)?

b) How many different PINs are there if all the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 can be used (at most once)?

c) What is the probability that a randomly selected PIN in this scenario uses just the digits 1, 2, 3, and 4?

**4.** How many different PINs are there if each digit can be used at most once and the order doesn't matter (selection without replacement, order doesn't matter)? Hint: in the last problem you counted the different orderings of $1, 2, 3, 4$. Any four digits have the same number of orderings. And you counted all orderings of four digits from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

**5.** You're now ready to analyze the Match 4 lottery in which four numbers are chosen from 1 to 24 without replacement and order doesn't matter.

a) How many possible outcomes are there?

b) How many do not match any of my chosen numbers 01 02 03 04?

c) How many match at least one of my chosen numbers?

**Challenge.** Continue working with the Match 4 lottery.

a) How many outcomes match exactly one of my chosen numbers? Hint: think of this as a two-step process, with the first step being choosing one of my numbers to match.

b) What is the probability that I win any amount of money (by matching 2 or more numbers)?