CONDITIONAL PROBABILITIES

solutions

Theorem (The Law of Total Probability). *If $A_1, A_2, \ldots, A_n$ partition S and $P(A_i) > 0$ for each i, then* $P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$

Note that the law of total probability is often applied in the form: $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$.

1. An October, 2019 Pew Center survey of US adults asked respondents if they "think the federal government is doing too little to reduce of the effects of climate change." They report the following results:

1. Of those who preferred the Democratic Party, 90% said yes;
2. Of those who preferred the Republican Party, 39% said yes;
3. 67% of all respondents said yes.

What percent of respondents preferred the Democratic Party? Assume that everyone had to choose either Democratic or Republican. Name your events clearly and identify the conditional probabilities carefully.

D: preferred Democratic

D^c: preferred Republican (assumption for this problem)

Y: answered yes

Given: $P(Y|D) = 0.9$, $P(Y|D^c) = 0.39$, $P(Y) = 0.67$

LoTP: $P(Y) = P(Y|D)P(D) + P(Y|D^c)P(D^c)$

$= P(Y|D)P(D) + P(Y|D^c)[1 - P(D)]$

$0.67 = 0.9\,P(D) + 0.39[1 - P(D)]$ solve for $P(D) \approx 0.5490$

The remaining problems concern the **die-coin experiment**, which consists of rolling a fair, 4-sided die and then flipping a fair coin the number of times shown on the die. For example, if you roll a 1 you'll flip the coin once, but if you roll a 2 you'll flip the coin twice. A sample space for this experiment is

$$S = \{(1,H), (1,T), (2,HH), (2,HT), (2,TH), (2,TT), \ldots, (4,TTTT)\}.$$

Note that the outcomes are not equally likely.

We'll deal with two random variables in this experiment:

- R, the number you roll on the die;
- F, the number of times you flip heads.

Our goal is to find the **probability mass function** (pmf) for F, the number of heads you flip.

Example. The pmf for R is easy to find: this random variable has possible values $1, 2, 3, 4$ and it takes each of those values with probability $1/4$. The pmf for R is thus $m(x) = 1/4$ for $x = 1, 2, 3, 4$.

2. What are the possible values for F? (Don't calculate probabilities yet).

0, 1, 2, 3, 4 (don't forget 0)

Calculating the probabilities for F is difficult unless we are given information about the roll of the die. For example, if we know that $R = 1$, then we know that $F = 0$ with probability $1/2$. This is a **conditional probability**: $P(F = 0|R = 1) = 1/2$ ("the probability that $F = 0$ **given** $R = 1$ is $1/2$"). The table of conditional probabilities on the next page has the first row filled in for you with $P(F = 0|R = 1) = 1/2$, $P(F = 1|R = 1) = 1/2$, $P(F = 2|R = 1) = 0$ etc.

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$P(F=x\|R=y)$		$F=x$				
		0	1	2	3	4
$R=y$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
	2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0
	3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	0
	4	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$
$P(F=x)$		$\frac{15}{64}$	$\frac{26}{64}$	$\frac{16}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

← sum is 1 ✓

3. Fill in the rest of the conditional probabilities in the table.

4. Use the law of total probability to fill in the pmf of F (the bottom row of the table).

$$P(F=x) = P(F=x|R=1)P(R=1) + P(F=x|R=2)P(R=2) + P(F=x|R=3)P(R=3) + P(F=x|R=4)P(R=4)$$

(law of total prob)

This corresponds to adding columns in the table and multiplying by $\frac{1}{4}$.

5. Suppose you know that your friend ran the die-coin experiment and flipped 3 heads ($F=3$). Calculate the conditional probabilities of your friend having rolled 1, 2, 3, or 4 on the die. Which was most likely to have been her roll?

$P(R=1|F=3) = P(R=2|F=3) = 0$ (impossible to flip 3 on a roll of 1 or 2)

$$P(R=3|F=3) = \frac{P(R=3 \cap F=3)}{P(F=3)} = \frac{P(F=3|R=3)P(R=3)}{P(F=3)} = \frac{\frac{1}{8}\left(\frac{1}{4}\right)}{3/32} = \frac{1}{3}$$

$$P(R=4|F=3) = \frac{P(F=3|R=4)P(R=4)}{P(F=3)} = \frac{\frac{1}{4}\left(\frac{1}{4}\right)}{3/32} = \frac{2}{3}$$ ← more likely

Challenge. Repeat the last problem, but suppose your friend got $F=1$ instead of $F=3$.