Solutions

EARTHQUAKES!

Earthquakes in a given region occur at random, but at a rate that we can estimate. For example, the PNW region has an average of about 9.2 significant earthquakes (magnitude 4.0 or greater) each year (1070 such earthquakes since 1900 according to earthquake.usgs.gov, and according to my rough estimate for what the PNW region means). It also turns out that knowing when the last earthquake happened doesn't help with predicting the next earthquake. The (random) number of earthquakes over a fixed amount of time can be modeled as a Poisson random variable. A **Poisson random variable** with parameter $\lambda > 0$ has the PMF $\boxed{p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x = 0, 1, 2, 3, \ldots}$.

Proposition 1. *The mean of a Poisson random variable is* λ.

Challenge. Prove the proposition. $\sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda}\left[e^{\lambda}\right] = \lambda$

1. Let N_t be the number of significant earthquakes in the PNW over t years. N_t should have a Poisson distribution with a parameter that depends on t.[1] For example N_1 is the number of significant earthquakes in one year, so it should have a Poisson distribution with $\lambda = 9.2$.

a) What is the value of λ for N_2?

18.4

b) What is the probability that there will be at least one significant earthquake before the end of the school year (about 70 days)?

$\frac{70}{365} \approx 0.191781$ so $\lambda = 9.2\left(\frac{70}{365}\right) \approx 1.764384$

$1 - p(0) = 1 - \frac{(\lambda)^0 e^{-\lambda}}{0!} = 1 - e^{-9.2\left(\frac{70}{365}\right)} \approx 0.8287$

c) What is the value of λ for N_t (as a function of t)?

$9.2t$

2. Let T be the **time** to the next significant earthquake in the PNW (you can assume that time 0 is just after the last earthquake). Observe that T is a random variable that is related to the random variable N_t, but that T is continuous while N_t is discrete. Your goal in this problem is to use the known distribution of N_t to find the PDF for T.

a) Fill in the blank: $T > t_0$ if and only if $N_{t_0} =$ <u>0</u>

b) Use your last answer to find the CDF for T:

$$\begin{aligned} F(t) &= P(T \leq t) \\ &= 1 - P(T > t) \\ &= 1 - P\Big(N_t = \underline{\ 0\ }\Big) \\ &= 1 - \frac{(9.2t)^0 e^{-9.2t}}{0!} = 1 - e^{-9.2t} \end{aligned}$$

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[1]Technically, N_t is a family of random variables known as a **Poisson process**.

c) Differentiate your CDF to find a PDF for T.

$F(t) = 1 - e^{-9.2t}$ ← only for $t \geq 0$

$f(t) = F'(t) = 9.2e^{-9.2t}$ for $t \geq 0$

(and $f(t) = 0$ for $t < 0$.

$\boxed{\begin{array}{l}\text{General exponential with parameter } \lambda > 0 \\ \text{CDF: } F(x) = 1 - e^{-\lambda x} \\ \text{PDF: } f(x) = \lambda e^{-\lambda x} \\ \text{both for } x \geq 0\end{array}}$

d) Calculate the mean wait time to the next earthquake.

$E(T) = \int_{-\infty}^{\infty} t f(t)\, dt = \int_0^{\infty} t\left(9.2e^{-9.2t}\right) dt$ IBP $u = t$ $dv = 9.2e^{-9.2t}\,dt$, $du = dt$ $v = -e^{-9.2t}$

$= -te^{-9.2t}\Big|_0^{\infty} + \int_0^{\infty} e^{-9.2t}\, dt$

$= 0 + \left[-\frac{1}{9.2} e^{-9.2t}\right]_0^{\infty}$

$= \frac{1}{9.2}$ years ≈ 39.67 days

$\boxed{\text{General: } \mu = \frac{1}{\lambda}}$

d_2) Prob T is less than 70 days.

$P(T \leq 70) = F\left(\frac{70}{365}\right) = 1 - e^{-9.2\left(\frac{70}{365}\right)} \approx 0.8287$ (same as 1b)

e) How long until the probability of an earthquake occurring exceeds 0.5? (Suggestion: use the CDF, solve for t).

$0.5 = 1 - e^{-9.2t} \Rightarrow 0.5 = e^{-9.2t}$

$\Rightarrow -\ln 2 = -9.2t$

$\Rightarrow t = \frac{\ln 2}{9.2} \approx 0.07534$ years   (or about 27.5 days)

General: median of $\exp(\lambda)$ is $\frac{\ln 2}{\lambda}$.

Note that $\ln 2 < 1$ so the median is always less than the mean of $\frac{1}{\lambda}$.

Time independence: $P(T > a+b \mid T > a) = \frac{P(T > a+b)}{P(T > a)}$

Memorylessness

$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$

$= e^{-\lambda b} = P(T > b)$

Challenge. Let T_2 be the time to the second earthquake. Find the probability density function of T_2 (use the same process as for the last problem).